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Walsh's Conformal Map onto Lemniscatic Domains for Polynomial Pre-images I

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Abstract

We consider Walsh's conformal map from the exterior of a compact set $E \subseteq \mathbb{C}$ onto a lemniscatic domain. If E is simply connected, the lemniscatic domain is the exterior of a circle, while if E has several components, the lemniscatic domain is the exterior of a generalized lemniscate and is determined by the logarithmic capacity of E and by the *exponents* and *centers* of the generalized lemniscate. For general E, we characterize the exponents in terms of the Green's function of E^c. Under additional symmetry conditions on E, we also locate the centers of the lemniscatic domain. For polynomial pre-images $E = P^{-1}(\Omega)$ of a simply-connected infinite compact set Ω, we explicitly determine the exponents in the lemniscatic domain and derive a set of equations to determine the centers of the lemniscatic domain. Finally, we present several examples where we explicitly obtain the exponents and centers of the lemniscatic domain, as well as the conformal map.

Keywords Conformal map · Lemniscatic domain · Multiply connected domain · Polynomial pre-image · Green's function · Logarithmic capacity

Mathematics Subject Classification 30C35 · 30C20

1 Introduction

The famous Riemann mapping theorem says that for any simply connected, compact and infinite set E there exists a conformal map $\mathcal{R}_E : E^c := \widehat{\mathbb{C}} \backslash E \to \overline{\mathbb{D}}^c$, where

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$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane, $\mathbb{D}$ the open unit disk and $\overline{\mathbb{D}}$ the closed unit disk. By imposing the normalization $\mathcal{R}_E(z) = z/\operatorname{cap}(E) + \mathcal{O}(1)$ as $z \to \infty$, where $\operatorname{cap}(E)$ denotes the *logarithmic capacity* of E, this map is unique. In his 1956 article [13], Walsh found the following canonical generalization for multiply connected domains.

Theorem 1.1 *Let $E_1, \ldots, E_\ell \subseteq \mathbb{C}$ be disjoint simply connected, infinite compact sets and let*

$$E = \bigcup_{j=1}^{\ell} E_j. \tag{1.1}$$

In particular, $E^c = \widehat{\mathbb{C}} \backslash E$ is an ℓ-connected domain. Then there exists a unique compact set of the form

$$L := \{w \in \mathbb{C} : |U(w)| \leq \operatorname{cap}(E)\}, \quad U(w) := \prod_{j=1}^{\ell} (w - a_j)^{m_j}, \tag{1.2}$$

where $a_1, \ldots, a_\ell \in \mathbb{C}$ are distinct and $m_1, \ldots, m_\ell > 0$ are real numbers with $\sum_{j=1}^{\ell} m_j = 1$, and a unique conformal map

$$\Phi : E^c \to L^c \tag{1.3}$$

normalized by

$$\Phi(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{at } \infty. \tag{1.4}$$

If E is bounded by Jordan curves, then Φ extends to a homeomorphism from $\overline{E^c}$ to $\overline{L^c}$.

Remark 1.2 (i) By assumption, each E_j satisfies $\operatorname{cap}(E_j) > 0$ hence $\operatorname{cap}(E) > 0$.

(ii) The points $a_1, \ldots, a_\ell$ (sometimes called 'centers' of L) and also $m_1, \ldots, m_\ell$ in Theorem 1.1 are uniquely determined. The function U is analytic in $\mathbb{C} \backslash \{a_1, \ldots, a_\ell\}$ and in general not single-valued, but its absolute value is single-valued. Note that the compact set L, defined in (1.2), consists of ℓ disjoint compact components $L_1, \ldots, L_\ell$, with $a_j \in L_j$ for $j = 1, \ldots, \ell$. The components $L_1, \ldots, L_\ell$ are labeled such that a Jordan curve surrounding E_j is mapped by Φ onto a Jordan curve surrounding L_j.

(iii) If E is simply connected then the exterior Riemann map $\mathcal{R}_E : E^c \to \overline{\mathbb{D}}^c$ with $\mathcal{R}_E(z) = d_1 z + d_0 + \mathcal{O}(1/z)$ at ∞ and $d_1 = \mathcal{R}_E'(\infty) > 0$ and the Walsh map Φ are related by $\mathcal{R}_E(z) = d_1 \Phi(z) + d_0$, which follows from [13, Thm. 4]. The corresponding lemniscatic domain is the disk $L = \{w \in \mathbb{C} : |w - a_1| \leq \operatorname{cap}(E)\}$, where $a_1 = -d_0/d_1$ and $\operatorname{cap}(E) = 1/d_1$. This shows that Walsh's map onto lemniscatic domains is a canonical generalization of the Riemann map from simply to multiply connected domains.

(iv) The existence in Theorem 1.1 was first shown by Walsh; see [13, Thm. 3] and the discussion below. Other existence proofs were given by Grunsky [2, 3], and [4, Thm. 3.8.3], and also by Jenkins [5] and Landau [6]. However, these articles do not contain any analytic or numerical examples. The first analytic examples were constructed by Sète and Liesen in [12], and, subsequently, a numerical method for computing the Walsh map was derived in [8] for sets bounded by smooth Jordan curves.

(v) The domain L^c is usually called a *lemniscatic domain*. This term seems to originate in Grunsky [4, p. 106].

In this paper, we bring some light on the computation of the parameters m_j and a_j appearing in Theorem 1.1.

In Sect. 2, as a first main result, we derive a general formula (Theorem 2.3) for the exponents m_j in terms of the Green's function of E^c, denoted by g_E. Of special interest is of course the case where E is real or where E or some component E_j are symmetric with respect to the real line, i.e., $E^* = E$ or $E_j^* = E_j$, where

$$K^* := \{z \in \widehat{\mathbb{C}} : \overline{z} \in K\} \tag{1.5}$$

denotes the complex conjugate of a set $K \subseteq \widehat{\mathbb{C}}$. We prove that $E^* = E$ and $E_j^* = E_j$ implies that $a_j \in \mathbb{R}$ (Theorem 2.7). In the case that all components are symmetric, we give an interlacing property of the components E_j and the critical points of g_E (Theorem 2.8).

In Sect. 3, we consider the case when E is a polynomial pre-image of a simply connected compact infinite set Ω, that is, $E = P_n^{-1}(\Omega)$. In this case, we prove in Theorem 3.2 that the m_j are always rational of the form $m_j = n_j/n$, where n is the degree of the polynomial P_n and n_j is the number of zeros of $P_n(z) - \omega$ in E_j, where $\omega \in \Omega$. Moreover, the unknowns $a_1, \ldots, a_\ell$ are characterized by a system of equations which in particular can be solved explicitly in the case $\ell = 2$. With the help of these findings, we obtain an analytic expression for the map Φ if $P_n^{-1}(\Omega)$ is connected (Corrolary 3.7).

Finally, Sect. 4 contains several illustrative examples when $E = P_n^{-1}(\Omega)$ and when $\Omega = \overline{\mathbb{D}}$, $\Omega = [-1, 1]$ or when Ω is a Chebyshev ellipse. In particular, we determine the exponents and centers of the corresponding lemniscatic domain and visualize the conformal map Φ.

2 Results for General Compact Sets

Let the notation be as in Theorem 1.1. The Green's function (with pole at ∞) of L^c is

$$g_L(w) = \log|U(w)| - \log(\operatorname{cap}(E)) = \sum_{j=1}^{\ell} m_j \log\left|w - a_j\right| - \log(\operatorname{cap}(E)) \tag{2.1}$$

since g_L is harmonic in $\mathbb{C}\backslash\{a_1, \dots, a_\ell\}$, is zero on $\partial(L^c)$, and $g_L(w) - \log|w|$ is harmonic at ∞ with $\lim_{w\to\infty}(g_L(w) - \log|w|) = -\log(\operatorname{cap}(E))$. Then the Green's function of E^c is

$$g_E(z) = g_L(\Phi(z)), \quad z \in E^c, \tag{2.2}$$

since $\Phi : E^c \to L^c$ is conformal with $\Phi(z) = z + \mathcal{O}(1/z)$ at ∞. In particular, $\operatorname{cap}(E) = \operatorname{cap}(L)$. Denote for $R > 1$ the level curves of g_E and g_L by

$$\Gamma_R = \{z \in E^c : g(z) = \log(R)\}, \quad \Lambda_R = \{w \in L^c : g_L(w) = \log(R)\}.$$

Then $\Phi(\Gamma_R) = \Lambda_R$ and Φ maps the exterior of Γ_R onto the exterior of Λ_R. Let $R_* > 1$ be the largest number, such that g_E has no critical point interior to Γ_{R_*} (if $\ell = 1$, then $R_* = \infty$; see Theorem 2.5 below). Then Φ is the conformal map of $\operatorname{ext}(\Gamma_R)$ onto the lemniscatic domain $\operatorname{ext}(\Lambda_R)$ for all $1 < R < R_*$; see also [14, p. 31].

Here and in the following, we extensively use the *Wirtinger derivatives*

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

where $z = x+iy$ with $x, y \in \mathbb{R}$. We relate the exponents and centers of the lemniscatic domain to the Wirtinger derivatives $\partial_z g_E$ and $\partial_w g_L$ of the Green's functions. Note that $\partial_z g$ is analytic if g is a harmonic function, since then $\partial_{\overline{z}}(\partial_z g) = \frac{1}{4}\Delta g = 0$.

Lemma 2.1 *The Green's functions g_L and g_E from* (2.1) *and* (2.2) *satisfy*

$$\partial_z g_E(z) = \partial_w g_L(\Phi(z)) \cdot \Phi'(z). \tag{2.3}$$

Moreover, if $\gamma : [a, b] \to E^c$ is a smooth path, then

$$\int_\gamma \partial_z g_E(z)\, dz = \int_{\Phi\circ\gamma} \partial_w g_L(w)\, dw. \tag{2.4}$$

Proof Since Φ is analytic, we have $\partial_z \Phi = \Phi'$ and $\partial_{\overline{z}}\Phi = 0$. Moreover, $\partial_z \overline{\Phi} = \overline{\partial_{\overline{z}}\Phi} = 0$. With the chain rule for the Wirtinger derivatives and (2.2), we find

$$\frac{\partial g_E}{\partial z}(z) = \frac{\partial g_L}{\partial w}(\Phi(z)) \cdot \frac{\partial \Phi}{\partial z}(z) + \frac{\partial g_L}{\partial \overline{w}}(\Phi(z)) \cdot \frac{\partial \overline{\Phi}}{\partial z}(z) = \partial_w g_L(\Phi(z)) \cdot \Phi'(z), \tag{2.5}$$

which is (2.3). Integrating this expression over γ yields

$$\int_\gamma \partial_w g_L(\Phi(z))\Phi'(z)\, dz = \int_a^b \partial_w g_L(\Phi(\gamma(t)))\Phi'(\gamma(t))\gamma'(t)\, dt = \int_{\Phi\circ\gamma} \partial_w g_L(w)\, dw. \tag{2.6}$$

In combination with (2.3), this yields (2.4). □

Remark 2.2 Formally, Eq. (2.3) yields the relation between differentials

$$\partial_z g_E(z)\, dz = \partial_w g_L(\Phi(z)) \cdot \Phi'(z)\, dz = \partial_w g_L(w)\, dw, \tag{2.7}$$

which yields (2.4) upon integrating.

We are now ready to express the exponents m_j through the Wirtinger derivatives of the Green's function. For $j = 1, \ldots, \ell$, let γ_j be a closed curve in $\mathbb{C}\backslash E$ with $\operatorname{wind}(\gamma_j; z) = \delta_{jk}$ for $z \in E_k$ and $k = 1, \ldots, \ell$, where $\operatorname{wind}(\gamma; z_0)$ denotes the winding number of the curve γ about z_0, and δ_{jk} is the usual Kronecker delta. More informally, the curve γ_j contains E_j but no E_k, $k \neq j$, in its interior.

Theorem 2.3 *In the notation of Theorem 1.1, let g_E and g_L be the Green's functions of E^c and L^c, respectively. For each $j \in \{1, \ldots, \ell\}$, let γ_j be a closed curve in $\mathbb{C}\backslash E$ with* $\operatorname{wind}(\gamma_j; z) = \delta_{jk}$ *for $z \in E_k$ and $k = 1, \ldots, \ell$, and let $\lambda_j = \Phi \circ \gamma_j$. Then,*

$$m_j = \frac{1}{2\pi i} \int_{\lambda_j} 2\partial_w g_L(w)\, dw = \frac{1}{2\pi i} \int_{\gamma_j} 2\partial_z g_E(z)\, dz. \tag{2.8}$$

Moreover, if the function f is analytic interior to λ_j and continuous on $\operatorname{trace}(\lambda_j)$*, then*

$$m_j f(a_j) = \frac{1}{2\pi i} \int_{\lambda_j} f(w) 2\partial_w g_L(w)\, dw = \frac{1}{2\pi i} \int_{\gamma_j} f(\Phi(z)) 2\partial_z g_E(z)\, dz. \tag{2.9}$$

Proof Since $2\partial_w \log|w| = \partial_w \log(w\overline{w}) = 1/w$, we obtain from (2.1) that

$$2\partial_w g_L(w) = \sum_{j=1}^{\ell} \frac{m_j}{w - a_j}, \tag{2.10}$$

which is a rational function. By construction, λ_j is a closed curve in $\mathbb{C}\backslash L$ with $\operatorname{wind}(\lambda_j; a_k) = \delta_{jk}$. Integrating over λ_j yields the first equality in (2.8). The second equality follows by Lemma 2.1. Using (2.10) and the residue theorem, we obtain

$$\frac{1}{2\pi i} \int_{\lambda_j} f(w) 2\partial_w g_L(w)\, dw = \frac{1}{2\pi i} \int_{\lambda_j} \sum_{s=1}^{\ell} \frac{m_s f(w)}{w - a_s}\, dw = m_j f(a_j). \tag{2.11}$$

This proves the first equality in (2.9). Multiplying (2.3) by $f(\Phi(z))$ and integrating yields the second equality in (2.9). □

Remark 2.4 (i) By (2.8) in Theorem 2.3, the exponent m_j of the lemniscatic domain is the residue of $2\partial_w g_L$ at a_j. Moreover, m_j is (up to the factor $1/(2\pi i)$) the *module of periodicity* (or *period*) of the differential $2\partial_z g_E(z)\, dz$; see [1, p. 147]. The latter can be rewritten as

$$\int_{\gamma_j} 2\partial_z g_E(z)\, dz = \int_{\gamma_j} \left(-\frac{\partial g_E}{\partial y} dx + \frac{\partial g_E}{\partial x} dy \right) = \int_{\gamma_j} \frac{\partial g_E}{\partial n}(z)\ |dz|\,,$$

where the middle integral is over the *conjugate differential* of dg_E, and where $\partial G_E/\partial n$ is the derivative with respect to the normal pointing to the right of γ_j; see [1, pp. 162–164] for a detailed discussion.

(ii) Since $\partial_z g_E$ is analytic in E^c and $\partial_w g_L$ is analytic in $\widehat{\mathbb{C}}\backslash\{a_1, \dots, a_\ell\} \supseteq L^c$, the integrals in (2.8) have the same value for all positively oriented closed curves that contain only E_j or a_j in their interior.

The following well-known result due to Walsh [15] establishes a relationship between the critical points of the Green's function and the connectivity of E^c.

Theorem 2.5 [15, pp. 67–68] *Let $E \subseteq \mathbb{C}$ be compact such that $\mathcal{K} = E^c$ is connected and such that $\mathcal{K}$ possesses a Green's function g_E with pole at infinity. If $\mathcal{K}$ is of finite connectivity ℓ, then g_E has precisely $\ell - 1$ critical points in $\mathbb{C}\backslash E$, counted according to their multiplicity. If $\mathcal{K}$ is of infinite connectivity, g_E has a countably infinite number of critical points. Moreover, all critical points of g_E lie in the convex hull of E.*

As is typical for conformal maps with $\Phi(z) = z + \mathcal{O}(1/z)$ at ∞, symmetry of E (e.g., rotational symmetry or symmetry with respect to the real line) leads to the same symmetry of L, and to "symmetry" in the map Φ:

Lemma 2.6 [12, Lem. 2.2] *Let the notation be as in Theorem 1.1. Then the following symmetry relations hold.*

(i) *If $E = E^*$, then $L = L^*$ and $\Phi(z) = \overline{\Phi(\overline{z})}$.*
(ii) *If $E = e^{i\theta}E := \{e^{i\theta}z : z \in E\}$, then $L = e^{i\theta}L$ and $\Phi(z) = e^{-i\theta}\Phi(e^{i\theta}z)$.*
(iii) *In particular: If $E = -E = \{-z : z \in E\}$, then $L = -L$ and $\Phi(z) = -\Phi(-z)$.*

In the last two results of this section, we consider the case where E and one or all of its components E_j are symmetric with respect to the real line. This allows us to locate the points $a_1, \dots, a_\ell$ and the critical points of the Green's function g_E.

Theorem 2.7 *In the notation of Theorem 1.1, suppose that $E^* = E$. Let $j \in \{1, \dots, \ell\}$. If $E_j^* = E_j$ then $a_j \in \mathbb{R}$.*

Proof Since $E^* = E$, we have $\Phi(\overline{z}) = \overline{\Phi(z)}$ by Lemma 2.6 and $\partial_z g_E(\overline{z}) = \overline{\partial_z g_E(z)}$ by Lemma A.1. Next, if $E_j^* = E_j$ for some $j \in \{1, \dots, \ell\}$ then there exists a smooth Jordan curve γ_j in $\mathbb{C}\backslash E$ symmetric with respect to the real line which surrounds E_j in the positive sense, but no other component E_k, $k \neq j$, i.e., $\mathrm{wind}(\gamma_j; z) = \delta_{jk}$ for $z \in E_k$ and $k = 1, \dots, \ell$. By (2.9),

$$m_j a_j = \frac{1}{2\pi i} \int_{\gamma_j} \Phi(z) 2\partial_z g_E(z)\, dz,$$

where $\Phi(\overline{z})2\partial_z g_E(\overline{z}) = \overline{\Phi(z)2\partial_z g_E(z)}$ on γ_j. By Lemma A.3, we obtain $m_j a_j \in \mathbb{R}$, hence $a_j \in \mathbb{R}$ since $m_j > 0$. □

In Theorem 2.7, if a component E_j is not symmetric with respect to the real line, then the corresponding point a_j is in general not real, as the example of the star in [12, Cor. 3.3] shows.

If $E_j^* = E_j$ for all components of E then we order the components "from left to right": By Lemma A.2, each $E_j \cap \mathbb{R}$ is a point or an interval, and we label $E_1, \dots, E_\ell$ such that $x \in E_j \cap \mathbb{R}$ and $y \in E_{j+1} \cap \mathbb{R}$ implies $x < y$ for all $j = 1, \dots, \ell - 1$.

Theorem 2.8 *Let $E = E_1 \cup \ldots \cup E_\ell$ be as in Theorem 1.1 and suppose that $E_j^* = E_j$ for all $j = 1, \dots, \ell$. Then the following hold.*

(i) *The $\ell - 1$ critical points of the Green's function g_E are real. Moreover, each E_j intersects $\mathbb{R}$ in a point or an interval, and the critical points of g_E interlace the sets $E_j \cap \mathbb{R}$, $j = 1, \dots, \ell$.*
(ii) *If $E_1, \dots, E_\ell$ are ordered "from left to right" then $a_1 < a_2 < \ldots < a_\ell$.*

Proof (i) For each $j = 1, \dots, \ell$, the set $E_j \cap \mathbb{R}$ is a point or an interval by Lemma A.2. For $j = 1, \dots, \ell - 1$, denote the 'gap' on the real line between E_j and E_{j+1} by

$$I_j :=]\max(E_j \cap \mathbb{R}), \min(E_{j+1} \cap \mathbb{R})[, \quad j = 1, \dots, \ell - 1.$$

The Green's function g_E is positive on I_j and can be continuously extended to $\overline{I}_j$ with boundary values 0. Then g_E has a maximum on $\overline{I}_j$ at a point $x_j \in I_j$ at which $\partial_x g_E(x_j) = 0$. By (A.1), we have $\partial_y g_E(x_j) = -\partial_y g_E(x_j)$, i.e., $\partial_y g_E(x_j) = 0$. This shows that x_j is a critical point of g_E for $j = 1, \dots, \ell - 1$. These are the $\ell - 1$ critical points of g_E which are real and interlace the sets $E_j \cap \mathbb{R}$.

(ii) Since $E = E^*$, we have $\Phi(z) = \overline{\Phi(\overline{z})}$ by Lemma 2.6. In particular, Φ maps $\mathbb{R} \backslash E$ onto $\mathbb{R} \backslash L$. Since $\Phi(z) = z + \mathcal{O}(1/z)$ at infinity, Φ maps $I_0 :=]-\infty, \min(E_1 \cap \mathbb{R})[$ onto $J_0 :=]-\infty, \min(L \cap \mathbb{R})[$. Let γ_1 be a Jordan curve in $\mathbb{C} \backslash E$ which surrounds E_1 in the positive sense, but no other component E_k, $k \neq 1$. Then γ_1 intersects I_0 and I_1 (see (i)), hence the curve $\Phi(\gamma_1)$ intersects the images $J_0 = \Phi(I_0)$ and $J_1 := \Phi(I_1)$. This shows that L_1 is the leftmost component of L and a_1 is the minimum of $a_1, \dots, a_\ell$. Proceeding in a similar way gives that the components $L_1, \dots, L_\ell$ are ordered from left to right, and therefore $a_1 < a_2 < \ldots < a_\ell$. $\square$

3 Results for Polynomial Pre-images

Let $\Omega \subseteq \mathbb{C}$ be a compact infinite set such that Ω^c is a simply connected domain in $\widehat{\mathbb{C}}$ and let $\mathcal{R}_\Omega$ be the exterior Riemann map of Ω, i.e., the conformal map

$$\mathcal{R}_\Omega : \Omega^c \to \overline{\mathbb{D}}^c \quad \text{with } \mathcal{R}_\Omega(z) = d_1 z + d_0 + \sum_{k=1}^{\infty} \frac{d_{-k}}{z^k} \quad \text{for } |z| > R, \tag{3.1}$$

where

$$d_1 = \mathcal{R}'_\Omega(\infty) = \frac{1}{\operatorname{cap}(\Omega)} > 0, \tag{3.2}$$

$R := \max_{z\in\Omega} |z|$, and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. By [9, Thm. 4.4.4], the Green's function of Ω^c is

$$g_\Omega(z) = \log |\mathcal{R}_\Omega(z)| . \tag{3.3}$$

Let P_n be a polynomial of degree $n \geq 1$, more precisely,

$$P_n(z) = \sum_{j=0}^{n} p_j z^j \quad \text{with } p_n \in \mathbb{C}\backslash\{0\}, \tag{3.4}$$

and consider the pre-image of Ω under P_n, that is

$$E = P_n^{-1}(\Omega) = \{z \in \mathbb{C} : P_n(z) \in \Omega\}. \tag{3.5}$$

The set E is compact and, by Theorem A.4, the complement E^c is connected. Therefore, the Green's function of E^c is

$$g_E(z) = \frac{1}{n} g_\Omega(P_n(z)) = \frac{1}{n} \log |\mathcal{R}_\Omega(P_n(z))| , \tag{3.6}$$

see [9, p. 134]. Since $2\partial_z \log |f| = f'/f$ for an analytic function f, we have

$$2\partial_z g_E(z) = \frac{1}{n} \frac{\mathcal{R}_\Omega'(P_n(z)) P_n'(z)}{\mathcal{R}_\Omega(P_n(z))}. \tag{3.7}$$

The logarithmic capacity of E is

$$\operatorname{cap}(E) = \operatorname{cap}(P_n^{-1}(\Omega)) = \left(\frac{\operatorname{cap}(\Omega)}{|p_n|}\right)^{1/n} = \frac{1}{\sqrt[n]{d_1 \, |p_n|}}, \tag{3.8}$$

see [9, Thm. 5.2.5].

By Theorem 2.5, the number of components of E can be characterized as follows. For the case $\Omega = [-1, 1]$, see also [10, Thm. 4 and Thm. 5].

Theorem 3.1 *The set E in (3.5) consists of ℓ disjoint simply connected compact components $E_1, \ldots, E_\ell$, i.e.,*

$$E = P_n^{-1}(\Omega) = \bigcup_{j=1}^{\ell} E_j, \tag{3.9}$$

if and only if P_n has exactly $\ell - 1$ critical points $z_1, \ldots, z_{\ell-1}$ (counting multiplicities) for which $P_n(z_k) \notin \Omega$ for $k = 1, \ldots, \ell - 1$. Moreover, the number of zeros of $P_n(z) - \omega$ in E_j is the same for all $\omega \in \Omega$, and this number is denoted by n_j.

Proof By Theorem 2.5, E has ℓ components if and only if g_E has $\ell - 1$ critical points (in $\mathbb{C}\backslash E$). Since g_E is real-valued, $z_0 \in \mathbb{C}\backslash E$ is a critical point of g_E if and only if $\partial_z g_E(z_0) = 0$. By (3.7), the latter is equivalent to $P_n'(z_0) = 0$.

For $j = 1, \ldots, \ell$, let γ_j be a Jordan curve in $\mathbb{C}\backslash E$ with $\mathrm{wind}(\gamma_j; z) = \delta_{jk}$ for $z \in E_k$ and $k = 1, \ldots, \ell$. Let $z_0 \in E_j$ and $\omega_0 := P_n(z_0) \in \Omega$, then, by the argument principle,

$$n_j := \mathrm{wind}(P_n \circ \gamma_j; \omega_0) = \left|\{z \in E_j : P_n(z) = \omega_0\}\right| \geq 1. \tag{3.10}$$

Since $P_n \circ \gamma_j$ is a closed curve in $\mathbb{C}\backslash\Omega$, we have $\mathrm{wind}(P_n \circ \gamma_j; \omega) = \mathrm{wind}(P_n \circ \gamma_j; \omega_0)$ for all $\omega \in \Omega$, i.e., every point in Ω has exactly n_j pre-images under P_n in E_j. □

In the rest of this section, we assume that E has ℓ components $E_1, \ldots, E_\ell$, i.e., that P_n has exactly $\ell - 1$ critical points with critical values in $\mathbb{C}\backslash\Omega$.

Theorem 3.2 *Let $E = P_n^{-1}(\Omega)$ and the numbers $n_1, \ldots, n_\ell$ be defined as in Theorem 3.1. Then the exponents m_j in the lemniscatic domain in Theorem 1.1 are given by*

$$m_j = \frac{n_j}{n}, \quad j = 1, \ldots, \ell. \tag{3.11}$$

Proof For $j = 1, \ldots, \ell$, let γ_j be a positively oriented Jordan curve in $\mathbb{C}\backslash E$ with $\mathrm{wind}(\gamma_j; z) = \delta_{jk}$ for $z \in E_k$ and $k = 1, \ldots, \ell$. Using (2.8) and (3.7), we obtain

$$m_j = \frac{1}{2\pi i} \int_{\gamma_j} 2\partial_z g_E(z)\, dz = \frac{1}{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{\mathcal{R}_\Omega'(P_n(z)) P_n'(z)}{\mathcal{R}_\Omega(P_n(z))}\, dz. \tag{3.12}$$

Substituting $u = P_n(z)$ yields

$$m_j = \frac{1}{n} \frac{1}{2\pi i} \int_{P_n \circ \gamma_j} \frac{\mathcal{R}_\Omega'(u)}{\mathcal{R}_\Omega(u)}\, du. \tag{3.13}$$

Since $\mathrm{wind}(P_n \circ \gamma_j; u_0) = n_j$ for $u_0 \in \Omega$, the integral in (3.13) can be replaced by n_j times an integral over a positively oriented Jordan curve Γ in $\mathbb{C}\backslash\Omega$, i.e.,

$$m_j = \frac{n_j}{n} \frac{1}{2\pi i} \int_\Gamma \frac{\mathcal{R}_\Omega'(u)}{\mathcal{R}_\Omega(u)}\, du. \tag{3.14}$$

The integral is

$$\frac{1}{2\pi i} \int_\Gamma \frac{\mathcal{R}_\Omega'(u)}{\mathcal{R}_\Omega(u)}\, du = \frac{1}{2\pi i} \int_\Gamma \frac{\frac{(u-u_0)\,\mathcal{R}_\Omega'(u)}{\mathcal{R}_\Omega(u)}}{u - u_0}\, du = \lim_{u \to \infty} \frac{(u - u_0)\,\mathcal{R}_\Omega'(u)}{\mathcal{R}_\Omega(u)} = 1 \tag{3.15}$$

by Cauchy's integral formula for an infinite domain; see, e.g., [7, Problem 14.14]. □

Next, we derive a relationship between the Walsh map Φ and the Riemann map $\mathcal{R}_\Omega$. Let $E = P_n^{-1}(\Omega)$ be as in (3.5). Liesen and the second author proved in [12, Eqn. (3.2)] that the lemniscatic map Φ in Theorem 1.1 and the exterior Riemann map $\mathcal{R}_\Omega$ are related by

$$|U(\Phi(z))| = \operatorname{cap}(E)\, |\mathcal{R}_\Omega(P_n(z))|^{1/n}, \quad z \in E^c, \tag{3.16}$$

with U from (1.2). This follows by considering the identity (2.2) between the corresponding Green's functions. In Theorem 3.3, we establish a stronger result.

By Theorem 3.2, the exponents of U satisfy $m_j = n_j/n$. Together with (3.8), we see that

$$Q(w) := \frac{e^{i \arg(p_n)}}{\operatorname{cap}(E)^n} U(w)^n = d_1 p_n \prod_{j=1}^{\ell} (w - a_j)^{n_j} \tag{3.17}$$

is a polynomial of degree n. Note that $L = \{w \in \mathbb{C} : |Q(w)| \leq 1\}$, and $Q : L^c \to \overline{\mathbb{D}}^c$ is an n-to-1 map. Then, equation (3.16) is equivalent to

$$|Q(\Phi(z))| = |\mathcal{R}_\Omega(P_n(z))|, \quad z \in E^c. \tag{3.18}$$

Next, we show that equality is also valid without the absolute value. Moreover, we derive a relationship between the points a_j and the coefficients p_{n-1}, p_n of P_n for $n \geq 2$. The case $n = 1$ is discussed in Remark 3.4.

Theorem 3.3 *Let $E = P_n^{-1}(\Omega)$ be as in (3.5). We then have*

$$Q(\Phi(z)) = \mathcal{R}_\Omega(P_n(z)), \quad z \in E^c, \tag{3.19}$$

that is,

$$\Phi = Q^{-1} \circ \mathcal{R}_\Omega \circ P_n, \tag{3.20}$$

with that branch of Q^{-1} such that $\Phi(z) = z + \mathcal{O}(1/z)$ at ∞. Moreover, for $n \geq 2$,

$$\sum_{j=1}^{\ell} n_j a_j = -\frac{p_{n-1}}{p_n}. \tag{3.21}$$

Proof Consider the Laurent series at infinity of $\mathcal{R}_\Omega \circ P_n$ and $Q \circ \Phi$. By (3.1) and (3.4),

$$\mathcal{R}_\Omega(P_n(z)) = d_1 P_n(z) + d_0 + \sum_{k=1}^{\infty} \frac{d_{-k}}{P_n(z)^k} = d_1 p_n z^n + d_1 p_{n-1} z^{n-1} + \mathcal{O}(z^{n-2}). \tag{3.22}$$

Fig. 1 Commutative diagram of the maps in Theorem 3.3

$$\begin{tikzcd} z \in E^c \arrow[r, "P_n"] \arrow[d, "\Phi"'] & \Omega^c \arrow[d, "\mathcal{R}_\Omega"] \\ w \in L^c \arrow[r, "Q"'] & \overline{\mathbb{D}}^c \end{tikzcd}$$

Since $\Phi(z) = z + \mathcal{O}(1/z)$ at infinity, we have

$$(\Phi(z) - a_j)^{n_j} = (z - a_j)^{n_j} + \mathcal{O}(z^{n_j - 2}) = z^{n_j} - n_j a_j z^{n_j - 1} + \mathcal{O}(z^{n_j - 2})$$

and, by (3.17),

$$Q(\Phi(z)) = d_1 p_n \prod_{j=1}^{\ell} (\Phi(z) - a_j)^{n_j} = d_1 p_n z^n - d_1 p_n \sum_{j=1}^{\ell} n_j a_j z^{n-1} + \mathcal{O}(z^{n-2}). \tag{3.23}$$

The function $(Q \circ \Phi)/(\mathcal{R}_\Omega \circ P_n)$ is analytic in $\mathbb{C} \backslash E$ with constant modulus one, see (3.18), therefore constant (maximum modulus principle) and

$$Q(\Phi(z)) = c\, \mathcal{R}_\Omega(P_n(z)), \quad z \in E^c, \tag{3.24}$$

where $c \in \mathbb{C}$ with $|c| = 1$. By comparing the coefficients of z^n of the Laurent series at ∞, we see that $c = 1$, which shows (3.19). Comparing the coefficients of z^{n-1} then yields (3.21). □

Figure 1 illustrates Theorem 3.3.

Remark 3.4 In the case $n = 1$, i.e., $P_1(z) = p_1 z + p_0$ is a linear transformation, the conformal map and lemniscatic domain are given explicitly as follows. In this case, $E = P_1^{-1}(\Omega)$ consists of a single component, i.e., $\ell = 1$ and $m_1 = 1$, and $Q(w) = d_1 p_1 (w - a_1)$. Comparing the constant terms at infinity of $\mathcal{R}_\Omega(P_n(z)) = d_1 p_1 z + (d_1 p_0 + d_0) + \mathcal{O}(1/z)$ with $Q(\Phi(z))$ from (3.23) yields

$$a_1 = -\frac{d_1 p_0 + d_0}{d_1 p_1}. \tag{3.25}$$

By Theorem 3.3, the conformal map $\Phi : E^c \to L^c$ is

$$\Phi(z) = (Q^{-1} \circ \mathcal{R}_\Omega \circ P_1)(z) = \frac{1}{d_1 p_1} \mathcal{R}_\Omega(p_1 z + p_0) + a_1, \tag{3.26}$$

and

$$L = \left\{ w \in \mathbb{C} : |w - a_1| \leq \frac{1}{d_1 |p_1|} \right\}. \tag{3.27}$$

Formula (3.19) does not lead to *separate* expressions for Q and Φ, even if $\mathcal{R}_\Omega$ and P_n are known. However, if the polynomial $Q(w) = d_1 p_n \prod_{j=1}^{\ell}(w - a_j)^{n_j}$ is known, equation (3.20) yields an expression for Φ. Since the numbers n_j are already known (Theorem 3.1), our next aim is to determine $a_1, \ldots, a_\ell$.

Lemma 3.5 *Let $E = P_n^{-1}(\Omega)$ be as in (3.5) and with ℓ components.*

(i) *A point $z_* \in \mathbb{C}\backslash E$ is a critical point of P_n if and only if $w_* = \Phi(z_*)$ is a critical point of Q in $\mathbb{C}\backslash L$. Moreover, in that case*

$$Q(w_*) = (\mathcal{R}_\Omega \circ P_n)(z_*). \tag{3.28}$$

(ii) *The polynomial Q has $\ell - 1$ critical points in $\mathbb{C}\backslash L$ and these are the zeros of*

$$\sum_{k=1}^{\ell} n_k \prod_{j=1, j\neq k}^{\ell} (w - a_j). \tag{3.29}$$

Proof (i) By Theorem 3.1, P_n has $\ell - 1$ critical points in $\mathbb{C}\backslash E$. The functions P_n and $\mathcal{R}_\Omega \circ P_n$ have the same critical points in E^c since $\mathcal{R}_\Omega$ is conformal in Ω^c and $(\mathcal{R}_\Omega \circ P_n)'(z) = \mathcal{R}_\Omega'(P_n(z))P_n'(z)$. By Theorem 3.3, we have $Q \circ \Phi = \mathcal{R}_\Omega \circ P_n$ in E^c. Since $(Q \circ \Phi)'(z) = Q'(\Phi(z))\Phi'(z)$ and Φ is conformal, we conclude that z_* is a critical point of $Q \circ \Phi$ if and only if $w_* = \Phi(z_*)$ is a critical point of Q which gives (3.28).

(ii) By (i), Q has exactly $\ell - 1$ critical points in L^c. By (3.17),

$$Q'(w) = d_1 p_n \prod_{j=1}^{\ell}(w - a_j)^{n_j - 1} \cdot \Bigg(\sum_{k=1}^{\ell} n_k \prod_{j=1, j\neq k}^{\ell} (w - a_j) \Bigg), \tag{3.30}$$

hence $a_1, \ldots, a_\ell$ are critical points of Q with multiplicity $\sum_{j=1}^{\ell}(n_j - 1) = n - \ell$. The remaining $\ell - 1$ critical points of Q are the zeros of the polynomial in (3.29). □

In principle, the right hand side in (3.28) can be computed when P_n and $\mathcal{R}_\Omega$ are given. If also $Q(w_*)$ can be computed, (3.28) yields $\ell - 1$ equations for $a_1, \ldots, a_\ell$.

With the results that we have established, we obtain the conformal map onto a lemniscatic domain of polynomial pre-images under $P_n(z) = \alpha(z - \beta)^n + \gamma$, and of pre-images with one component ($\ell = 1$) and arbitrary polynomial.

Proposition 3.6 *Let $\Omega \subseteq \mathbb{C}$ be a simply connected infinite compact set. Let $P_n(z) = \alpha(z - \beta)^n + \gamma$ with $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha \neq 0$, and $n \geq 2$.*

(i) *If $\gamma \notin \Omega$ then $E = P_n^{-1}(\Omega)$ has n components, $m_j = 1/n$ for $j = 1, \ldots, n$, the points $a_1, \ldots, a_n$ are given by*

$$a_{1,\ldots,n} = \beta + \sqrt[n]{-\frac{\mathcal{R}_\Omega(\gamma)}{d_1 \alpha}}$$

with the n distinct values of the nth root and $d_1 = \mathcal{R}'_\Omega(\infty) > 0$,

$$L = \left\{ w \in \mathbb{C} : \prod_{j=1}^{n} \left| w - a_j \right|^{1/n} = \left| (w-\beta)^n + \frac{\mathcal{R}_\Omega(\gamma)}{d_1 \alpha} \right|^{1/n} \leq (d_1 \, |\alpha|)^{-1/n} \right\}, \tag{3.31}$$

and the Walsh map is

$$\Phi : E^c \to L^c, \quad \Phi(z) = \beta + \sqrt[n]{\frac{\mathcal{R}_\Omega(P_n(z)) - \mathcal{R}_\Omega(\gamma)}{d_1 \alpha}}, \tag{3.32}$$

with that branch of the nth root such that $\Phi(z) = z + \mathcal{O}(1/z)$ *at infinity.*

(ii) *If* $\gamma \in \Omega$ *then* $E = P_n^{-1}(\Omega)$ *has one component, L is the disk*

$$L = \{ w \in \mathbb{C} : |w - \beta| \leq \operatorname{cap}(E) = (d_1 \, |\alpha|)^{-1/n} \}, \tag{3.33}$$

and the conformal map of E^c *onto a lemniscatic domain is*

$$\Phi : E^c \to L^c, \quad \Phi(z) = \beta + \sqrt[n]{\frac{\mathcal{R}_\Omega(P_n(z))}{d_1 \alpha}}, \tag{3.34}$$

with that branch of the nth root such that $\Phi(z) = z + \mathcal{O}(1/z)$ *at infinity.*

Proof (i) Since $P_n(\beta) = \gamma$, the assumption $\gamma \notin \Omega$ is equivalent to $\beta \notin E$. The only critical point of P_n is $z_* = \beta$ with multiplicity $n - 1$, hence E has $\ell = n$ components by Theorem 3.1. The point $\beta_1 := \Phi(\beta) \in \mathbb{C}$ is then a critical point of Q of multiplicity $n - 1$ by Lemma 3.5 (i). Therefore, Q' is a constant multiple of $(w - \beta_1)^{n-1}$ and

$$Q(w) = \alpha_1 (w - \beta_1)^n + \gamma_1, \quad \alpha_1, \gamma_1 \in \mathbb{C}.$$

Next, let us determine $\alpha_1, \beta_1, \gamma_1$ in terms of α, β, γ. We have

$$\gamma_1 = Q(\beta_1) = Q(\Phi(\beta)) = \mathcal{R}_\Omega(P_n(\beta)) = \mathcal{R}_\Omega(\gamma).$$

By (3.17), the leading coefficient of Q is $\alpha_1 = d_1 \alpha \neq 0$. Since $\ell = n$, we have

$$Q(w) = \alpha_1 (w - \beta_1)^n + \gamma_1 = \alpha_1 \prod_{j=1}^{n} (w - a_j) \tag{3.35}$$

with distinct $a_1, \ldots, a_n \in \mathbb{C}$. In particular, $n_j = 1$ for $j = 1, \ldots, n$. Equating the coefficients of w^{n-1} in (3.35) and using Theorem 3.3, we obtain

$$n\beta_1 = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} n_j a_j = -\frac{p_{n-1}}{p_n} = -\frac{\alpha(-n\beta)}{\alpha} = n\beta,$$

i.e., $\beta_1 = \beta$. By (3.35),

$$a_{1,\dots,n} = \beta_1 + \sqrt[n]{-\frac{\gamma_1}{\alpha_1}} = \beta + \sqrt[n]{-\frac{\mathcal{R}_\Omega(\gamma)}{d_1\alpha}}$$

with the n distinct values of the nth root. By (3.17), we have $L = \{w \in \mathbb{C} : |Q(w)| \leq 1\}$, which is equivalent to (3.31). Then (3.32) follows from (3.20).

(ii) The assumption $\gamma \in \Omega$ is equivalent to $\beta \in E$, thus P_n has no critical point in $\mathbb{C}\setminus E$ and E is connected, i.e., $\ell = 1$. Then $m_1 = 1$ and $n_1 = n$. By Theorem 3.3, $na_1 = -p_{n-1}/p_n = n\beta$, hence $a_1 = \beta$. Together with (3.8), we obtain the expression (3.33) for L. In contrast to case (i), we have $Q(w) = d_1\alpha(w-\beta)^n$, which yields (3.34) by (3.20). □

In [12, Thm. 3.1], the lemniscatic domain and conformal map Φ were explicitly constructed under the additional assumptions that Ω is symmetric with respect to $\mathbb{R}$ (i.e., $\Omega^* = \Omega$), $\gamma \in \mathbb{R}$ is left of Ω, $\alpha > 0$ and $\beta = 0$. A shift $\beta \neq 0$ can be incorporated with [12, Lem. 2.3]. In Proposition 3.6 we can relax the assumptions on Ω and the coefficients α, β, γ.

The proof of Proposition 3.6 (ii) generalizes to arbitrary polynomials P_n of degree $n \geq 2$, which yields the following result for a connected polynomial pre-image.

Corollary 3.7 *Let $\Omega \subseteq \mathbb{C}$ be a simply connected infinite compact set. Let P_n be a polynomial of degree $n \geq 2$ as in (3.4) such that $E = P_n^{-1}(\Omega)$ is connected, i.e., $\ell = 1$. Then $L = \{w \in \mathbb{C} : |w - a_1| \leq (d_1 |p_n|)^{-1/n}\}$ with $m_1 = 1$ and $a_1 = -p_{n-1}/(np_n)$, and*

$$\Phi : E^c \to L^c, \quad \Phi(z) = a_1 + \sqrt[n]{\frac{\mathcal{R}_\Omega(P_n(z))}{d_1 p_n}},$$

with that branch of the nth root such that $\Phi(z) = z + \mathcal{O}(1/z)$ at infinity.

Proof The assumption $\ell = 1$ implies $m_1 = 1$ and $n_1 = n$. By Theorem 3.3, we have $a_1 = -p_{n-1}/(np_n)$, which yields the expressions for L, $Q(w) = d_1 p_n (w - a_1)^n$ and Φ. □

Let us consider the case $\ell = 2$ in more detail. In this case, P_n has exactly one critical point outside E.

Theorem 3.8 *Let $E = P_n^{-1}(\Omega)$ in (3.9) consist of two components, and let z_* be the critical point of P_n in $\mathbb{C}\setminus E$. Then a_1, a_2 satisfy*

$$\left(a_2 + \frac{p_{n-1}}{np_n}\right)^n = \frac{(-1)^{n_2} n_1^{n_2}}{d_1 p_n n_2^{n_2}} (\mathcal{R}_\Omega \circ P_n)(z_*), \tag{3.36}$$

$$a_1 = -\frac{1}{n_1}\left(\frac{p_{n-1}}{p_n} + n_2 a_2\right). \tag{3.37}$$

Proof By Theorem 3.3, the centers a_1, a_2 of L satisfy

$$n_1 a_1 + n_2 a_2 = -\frac{p_{n-1}}{p_n}, \tag{3.38}$$

or, equivalently,

$$a_1 = -\frac{1}{n_1}\left(\frac{p_{n-1}}{p_n} + n_2 a_2\right), \quad \text{and} \quad a_2 - a_1 = \frac{n}{n_1}\left(\frac{p_{n-1}}{n p_n} + a_2\right). \tag{3.39}$$

By Lemma 3.5 (ii), the only critical point w_* of Q in $\mathbb{C}\backslash L$ is the zero of $n_1(w - a_2) + n_2(w - a_1)$, i.e.,

$$w_* = \frac{n_2 a_1 + n_1 a_2}{n}.$$

The corresponding critical value is

$$\begin{aligned} Q(w_*) &= d_1 p_n (w_* - a_1)^{n_1} (w_* - a_2)^{n_2} = d_1 p_n \left(\frac{n_1}{n}(a_2 - a_1)\right)^{n_1} \left(\frac{n_2}{n}(a_1 - a_2)\right)^{n_2} \\ &= d_1 p_n (-1)^{n_2} \frac{n_1^{n_1} n_2^{n_2}}{n^n} (a_2 - a_1)^n = d_1 p_n (-1)^{n_2} \frac{n_2^{n_2}}{n_1^{n_2}} \left(a_2 + \frac{p_{n-1}}{n p_n}\right)^n, \end{aligned}$$

where we used (3.39) in the last step. Since $(\mathcal{R}_\Omega \circ P_n)(z_*) = Q(w_*)$ by Lemma 3.5 (i), formula (3.36) is established. □

In order to specify the branch of the nth root in (3.36), some additional information is needed. We show this for a set Ω which is symmetric with respect to the real axis and contains the origin, which covers the important examples $\Omega = \overline{\mathbb{D}}$ and $\Omega = [-1, 1]$.

Lemma 3.9 *Suppose that $\Omega^* = \Omega$ and $0 \in \Omega$. Let P_n be a polynomial of degree n as in (3.4) with real coefficients such that $P_n^{-1}(\Omega) = \cup_{j=1}^{\ell} E_j$ with $E_j^* = E_j$ for $j = 1, \ldots, \ell$. Denote the critical points of P_n in $\mathbb{C}\backslash E$ by $z_1, \ldots, z_{\ell-1}$.*

- (i) *Then $z_1, z_2, \ldots, z_{\ell-1} \in \mathbb{R}$ and z_j is between $E_j \cap \mathbb{R}$ and $E_{j+1} \cap \mathbb{R}$ for each $j = 1, \ldots, \ell - 1$, where we label $E_1, \ldots, E_\ell$ from left to right along the real line.*
- (ii) *For each $j \in \{1, \ldots, \ell - 1\}$ and each $z \in]\max(E_j \cap \mathbb{R}), \min(E_{j+1} \cap \mathbb{R})[$, we have*

$$\operatorname{sgn}(\mathcal{R}_\Omega(P_n(z))) = \operatorname{sgn}(P_n(z)) = (-1)^{n_{j+1} + \ldots + n_\ell} \operatorname{sgn}(p_n), \tag{3.40}$$

which holds in particular for $z = z_j$.

If Ω is additionally symmetric with respect to the imaginary axis, the assertions also hold if P_n has purely imaginary coefficients.

Proof (i) Note that P_n and g_E have the same critical points in E^c, compare the proof of Theorem 3.1. Then, since $E_j^* = E_j$, (i) is a special case of Theorem 2.8.

(ii) Since $\Omega^* = \Omega$, the Riemann map satisfies $\mathcal{R}_\Omega(z) = \overline{\mathcal{R}_\Omega(\overline{z})}$ for $z \in \Omega$. In particular, if $z \in \mathbb{R}\backslash\Omega$, also $\mathcal{R}_\Omega(z) \in \mathbb{R}$. Together with $\mathcal{R}'_\Omega(z) > 0$, we have that $\mathcal{R}_\Omega(]\max(\Omega\cap\mathbb{R}), \infty[) =]1, \infty[$ and $\mathcal{R}_\Omega(]-\infty, \min(\Omega\cap\mathbb{R})[) =]-\infty, 1[$. Since $0 \in \Omega$, we see that $\operatorname{sgn}(\mathcal{R}_\Omega(z)) = \operatorname{sgn}(z)$ for $z \in \mathbb{R}\backslash\Omega$.
Similarly, if Ω is additionally symmetric with respect to the imaginary axis, $\mathcal{R}_\Omega$ maps the imaginary axis onto itself and $\operatorname{sgn}(\mathcal{R}_\Omega(z)) = \operatorname{sgn}(z)$ for $z \in (i\mathbb{R})\backslash\Omega$.
We can treat the cases that the coefficients P_n are real or purely imaginary (provided that Ω is also symmetric with respect to the imaginary axis) together. If $z \in \mathbb{R}\backslash E$, we have $P_n(z) \in \mathbb{R}\backslash\Omega$ (or $P_n(z) \in (i\mathbb{R})\backslash\Omega$) and hence $\operatorname{sgn}(\mathcal{R}_\Omega(P_n(z))) = \operatorname{sgn}(P_n(z))$. It remains to compute $\operatorname{sgn}(P_n(z))$. Since $0 \in \Omega$, we have $\operatorname{sgn}(P_n(z)) = \operatorname{sgn}(p_n)$ for $z > \max(E_\ell\cap\mathbb{R})$. Moreover, P_n has n_ℓ zeros in E_ℓ which are either real or appear in complex conjugate pairs. Therefore $\operatorname{sgn}(P_n(z)) = (-1)^{n_\ell} \operatorname{sgn}(p_n)$ for z in the rightmost gap, i.e., $z \in]\max(E_{\ell-1} \cap \mathbb{R}), \min(E_\ell \cap \mathbb{R})[$. Similarly, we get the assertion for the next gap and so on. □

Corollary 3.10 *Suppose that $\Omega^* = \Omega$ and $0 \in \Omega$. Let P_n be a polynomial of degree n as in (3.4) with real coefficients such that $P_n^{-1}(\Omega) = E_1 \cup E_2$ with $E_1^* = E_1$ and $E_2^* = E_2$. Let n_1, n_2 be the number of zeros of P_n in E_1, E_2, respectively, and let z_* be the critical point of P_n in $\mathbb{C}\backslash E$. Then the points a_1, a_2 are real with $a_1 < a_2$ and are given by*

$$a_1 = -\frac{p_{n-1}}{np_n} - \left(\left(\frac{n_2}{n_1}\right)^{n_1} \frac{(-1)^{n_2}}{d_1 p_n} \mathcal{R}_\Omega(P_n(z_*))\right)^{1/n}, \tag{3.41}$$

$$a_2 = -\frac{p_{n-1}}{np_n} + \left(\left(\frac{n_1}{n_2}\right)^{n_2} \frac{(-1)^{n_2}}{d_1 p_n} \mathcal{R}_\Omega(P_n(z_*))\right)^{1/n}, \tag{3.42}$$

with the positive real nth root.

If Ω is additionally symmetric with respect to the imaginary axis, then P_n can also have purely imaginary coefficients.

Proof By Theorem 2.7 and Theorem 2.8, the points a_1, a_2 are real and $a_1 < a_2$. By Theorem 3.8, we have (3.36), which gives (3.42). Since

$$\frac{(-1)^{n_2}}{p_n} \mathcal{R}_\Omega(P_n(z_*)) > 0$$

by Lemma 3.9 (ii) and $d_1 > 0$, the right hand side in formula (3.36) is positive. By (3.37), $a_1 < a_2$ is equivalent to $a_2 > -p_{n-1}/(np_n)$, which shows that we have to take the positive real nth root in (3.42). Inserting (3.42) into (3.37) yields (3.41). □

4 Examples

In this section, we consider six examples of polynomial pre-images $E = P_n^{-1}(\Omega)$ for the cases $\Omega = [-1, 1]$, $\Omega = \overline{\mathbb{D}}$ and $\Omega = \mathcal{E}_R := \{(re^{it} + r^{-1}e^{-it})/2 : t \in [0, 2\pi[, 1 \leq$

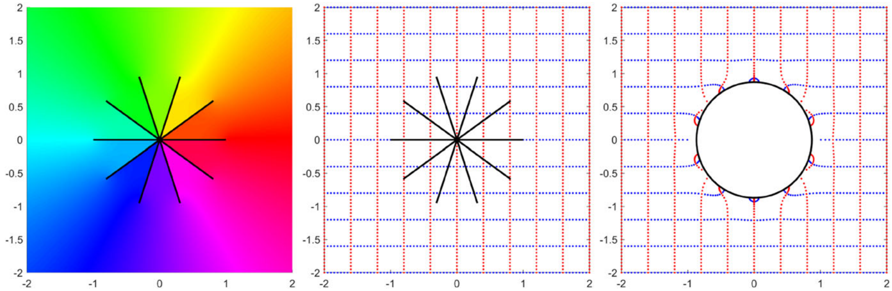

Fig. 2 Pre-image $E = P^{-1}([-1,1])$ with $P(z) = z^5$ in Example 4.1. Left: Phase plot of Φ, middle: E (black) and grid, right: ∂L (black) and image of the grid under Φ

$r \le R\}$ (Chebyshev ellipse), $R > 1$. We have the exterior Riemann maps

$$\mathcal{R}_{[-1,1]}(z) = z + \sqrt{z^2-1}, \quad \text{and} \quad \mathcal{R}_{\mathcal{E}_R}(z) = \frac{1}{R}\left(z + \sqrt{z^2-1}\right) = \frac{1}{R}\mathcal{R}_{[-1,1]}(z),$$

where the branch of the square root is chosen such that $\left|\mathcal{R}_{[-1,1]}(z)\right| > 1$. In particular, the coefficients of z at infinity are $\mathcal{R}'_{[-1,1]}(\infty) = 2$ and $\mathcal{R}'_{\mathcal{E}_R}(\infty) = 2/R$; see (3.1) and (3.2). We begin with three examples for Proposition 3.6.

Example 4.1 Let $\Omega = [-1,1]$ and $P_n(z) = z^n$. Since the critical value of P_n is $0 \in \Omega$, the set L and Walsh map Φ of the connected star $E = P_n^{-1}([-1,1]) = \bigcup_{k=1}^n e^{k2\pi i/n}[-1,1]$ are given by Proposition 3.6 (ii) as $L = \{w \in \mathbb{C} : |w| \le 2^{-1/n}\}$ and

$$\Phi : E^c \to L^c, \quad \Phi(z) = \sqrt[n]{\frac{z^n + \sqrt{z^{2n}-1}}{2}} = z\sqrt[n]{\frac{z^n + \sqrt{z^{2n}-1}}{2z^n}}.$$

We take the branch of the square root with $\left|z^n + \sqrt{z^{2n}-1}\right| > 1$. In the second representation of Φ we take the principal branch of the nth root; see [12, Thm. 3.1]. In particular, the logarithmic capacity of E is $2^{-1/n}$. Figure 2 illustrates the case $n = 5$. The left panel shows a phase plot of Φ. In a phase plot, the domain is colored according to the phase $f(z)/|f(z)|$ of the function f; see [16] for an introduction to phase plots. The middle and right panels show E and ∂L (in black) as well as a grid and its image under Φ.

Example 4.2 Let $\Omega = \mathcal{E}_{1.25}$ be the Chebyshev ellipse bounded by $\{\frac{1}{2}(\frac{5}{4}e^{it} + \frac{4}{5}e^{-it}) : t \in [0, 2\pi[\}$ and let $E = P^{-1}(\mathcal{E}_{1.25})$ with $P(z) = (z-1)^5 + \gamma$ for two different values of γ. For $\gamma = 0.3i \notin \Omega$, the set E consists of $n = 5$ components, while for $\gamma = 0.75 \in \Omega$, the set E has only one component; see Proposition 3.6. Figure 3 shows phase plots of Φ (left), the sets ∂E and ∂L in black and a grid and its image. The phase plots show Φ and an analytic continuation to the interior of E. The discontinuities in the phase (in the interior of E) are branch cuts of this analytic continuation.

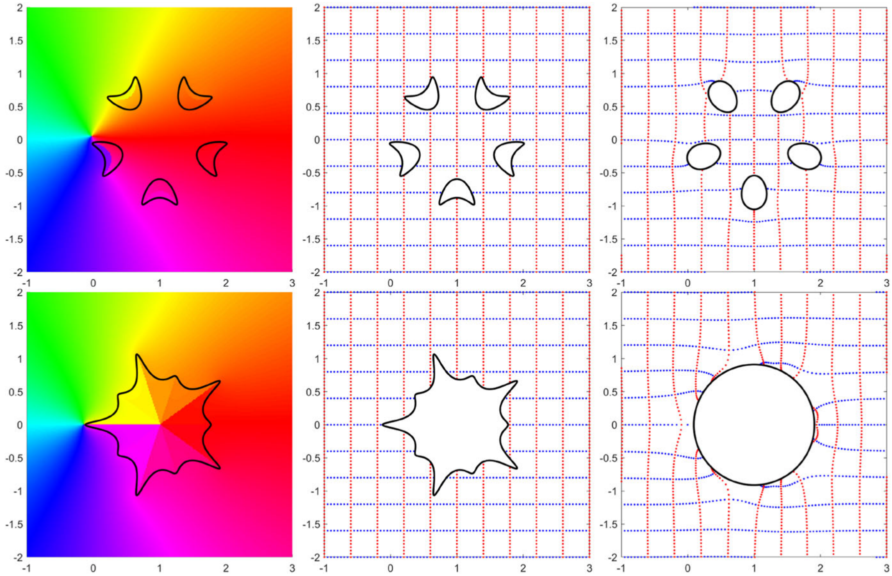

Fig. 3 Set $E = P_n^{-1}(\Omega)$ with a Chebyshev ellipse $\Omega = \mathcal{E}_{1.25}$ and $P_n(z) = (z-1)^5 + \gamma$, with $\gamma = 0.3i \notin \Omega$ (top row) and $\gamma = 0.75 \in \Omega$ (bottom row); see Example 4.2. Phase plot of Φ (left), original and image domains with ∂E and ∂L in black (middle and right)

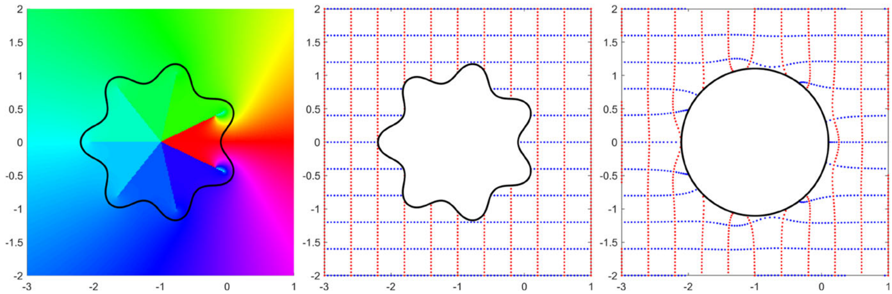

Fig. 4 The set $E = P_n^{-1}(\overline{\mathbb{D}})$ with $P_n(z) = \frac{1}{2}(z+1)^7 + \frac{3}{4}$. Phase plot of Φ (left), original and image domains with ∂E and ∂L in black (middle and right); see Example 4.3

Example 4.3 Let $P_n(z) = \alpha(z-\beta)^n + \gamma$ and $E = P_n^{-1}(\overline{\mathbb{D}})$.

(i) If $\gamma \notin \overline{\mathbb{D}}$ then $\Phi(z) = z$ by Proposition 3.6, hence $L = E = \{z \in \mathbb{C} : |P_n(z)| \leq 1\}$, i.e., E^c is a lemniscatic domain; see also Example 4.5 for pre-images of $\overline{\mathbb{D}}$ under general polynomials.

(ii) If $\gamma \in \overline{\mathbb{D}}$ then E has only one component. In this case $\Phi(z) = z$ if and only if $\gamma = 0$. Figure 4 shows an example with $\gamma \in \overline{\mathbb{D}} \backslash \{0\}$, where E is not a lemniscatic domain and $\Phi(z) \neq z$.

Next, we present an example for Corollary 3.7.

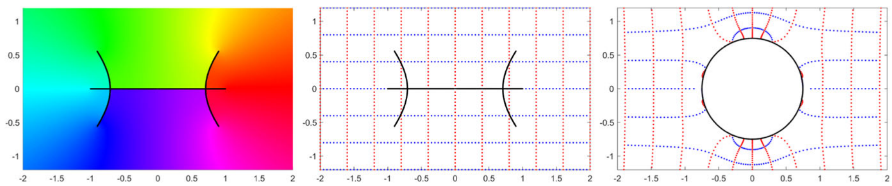

Fig. 5 The set $E = P_4^{-1}([-1, 1])$ with $\alpha = 5$ in Example 4.4. Phase plot of Φ (left), original and image domains with E and ∂L in black (middle and right)

Example 4.4 Consider the polynomial

$$P_4(z) = \frac{8z^4 - 8z^2 + \alpha}{\alpha}$$

with $\alpha \geq 1$ from [11, Ex. (iv)]. Then $E = P_4^{-1}([-1, 1])$ is connected, since the critical points of P_4 are $0, \pm 1/\sqrt{2}$ with corresponding critical values $P_4(0) = 1 \in [-1, 1]$ and $P_4(\pm 1/\sqrt{2}) = 1 - \frac{2}{\alpha} \in [-1, 1]$; see Theorem 3.1. By Corollary 3.7,

$$L = \left\{ w \in \mathbb{C} : |w| \leq \operatorname{cap}(E) = \frac{\alpha^{1/4}}{2} \right\},$$

and the conformal map is

$$\Phi : E^c \to L^c, \quad \Phi(z) = \frac{\sqrt[4]{\alpha}}{2} \sqrt[4]{P_4(z) + \sqrt{P_4(z)^2 - 1}},$$

see Fig. 5. Since $E^* = E$, we have that $\Phi(z) = \overline{\Phi(\overline{z})}$ and, since $\Phi(z) = z + \mathcal{O}(1/z)$, we have in particular $\Phi(]1, \infty[) =]\operatorname{cap}(E), \infty[$ and $\Phi(]-\infty, -1[) =]-\infty, -\operatorname{cap}(E)[$. Since E is also symmetric with respect to the imaginary axis, we similarly have $\Phi(]0, i\infty[) =]i \operatorname{cap}(E), i\infty[$ and $\Phi(] - i\infty, 0[) =] - i\infty, -i \operatorname{cap}(E)[$. Hence, Φ maps each quadrant to itself. We use this to determine the correct branch of the fourth root.

Example 4.5 Let $P_n(z) = p_n \prod_{j=1}^{n}(z - b_j)$ be a polynomial of degree n. If $E = P_n^{-1}(\overline{\mathbb{D}})$ consists of n components then E^c is a lemniscatic domain, i.e., $L = E$ with $a_j = b_j$, $m_j = 1/n$, $\operatorname{cap}(E) = |p_n|^{-1/n}$, and $\Phi(z) = z$. Similarly, if $P_n(z) = p_n \prod_{j=1}^{\ell}(z - b_j)^{n_j}$ with distinct $b_1, \ldots, b_\ell \in \mathbb{C}$ and if E has ℓ components, then E^c is a lemniscatic domain, $L = E$ with $a_j = b_j$, $m_j = n_j/n$, and $\Phi(z) = z$.

Finally, we consider an example for Theorem 3.8.

Example 4.6 For $\alpha, \beta \in \mathbb{C}$, consider the polynomial

$$P_3(z) = (z - \alpha)(z^2 - \beta^2) = z^3 - \alpha z^2 - \beta^2 z + \alpha\beta^2$$

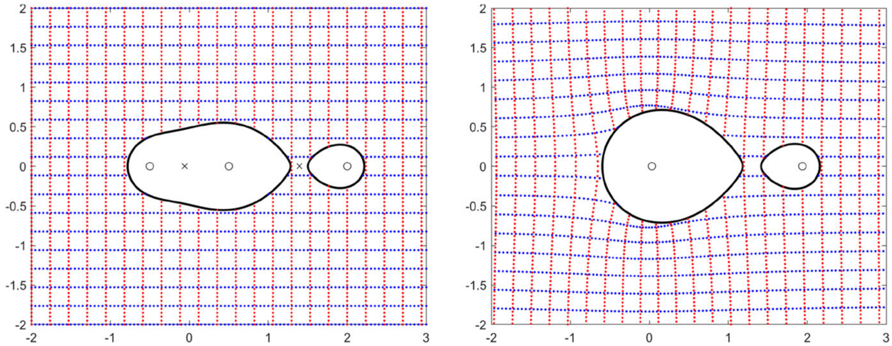

Fig. 6 Pre-image $E = P_3^{-1}(\overline{\mathbb{D}})$ in Example 4.6. Left: ∂E (black line), zeros of P_3 (circles) and P_3' (crosses), and a cartesian grid. Right: ∂L (black line), a_1, a_2 (circles) and the image of the grid under Φ

of degree $n = 3$. The critical points of P_3 are

$$z_\pm = \frac{\alpha \pm \sqrt{\alpha^2 + 3\beta^2}}{3}.$$

In the case $\alpha = 2$ and $\beta = 1/2$, we have $P_3(z_-) \approx 0.5076 \in \overline{\mathbb{D}}$ and $P_3(z_+) \approx 1.9375 \in \mathbb{C}\backslash\overline{\mathbb{D}}$, hence $E = P_3^{-1}(\overline{\mathbb{D}})$ has $\ell = 2$ components by Theorem 3.1; see Fig. 6 (left). Note that E^c is not a lemniscatic domain (in contrast to the case considered in Example 4.5). Write $E = E_1 \cup E_2$, where E_1 is the component on the left (with $\pm\beta \in E_1$). Then $m_1 = 2/3$ and $m_2 = 1/3$ by Theorem 3.2. Moreover, $E_1^* = E_1$ and $E_2^* = E_2$, since P_n is real and $\overline{\mathbb{D}}$ is symmetric with respect to the real line, which implies that $a_1, a_2 \in \mathbb{R}$ by Theorem 2.7. Then, by Theorem 3.8,

$$\left(a_2 - \frac{\alpha}{3}\right)^3 = -2P_3(z_+) \in \mathbb{R}.$$

Since $a_2 - \alpha/3$ is real, taking the real third root yields

$$a_2 = \frac{\alpha}{3} + \sqrt[3]{-2P_3(z_+)} \approx 1.9375 \quad \text{and} \quad a_1 = \frac{1}{3}\alpha - \frac{1}{2}\sqrt[3]{-2P_3(z_+)} \approx 0.0313.$$

Moreover, $\operatorname{cap}(E) = 1$ by (3.8), hence

$$L = \left\{ w \in \mathbb{C} : |w - a_1|^{2/3} \, |w - a_2|^{1/3} \leq 1 \right\}.$$

Here, $Q(w) = (w - a_1)^2 (w - a_2)^1$, hence

$$\Phi(z) = Q^{-1}(P_3(z)),$$

with a branch of Q^{-1} such that $\Phi(z) = z + \mathcal{O}(1/z)$ at infinity. Here, we can obtain the boundary values of Φ for $z \in \partial E$ by solving $Q(w) = P_3(z)$ and identifying the boundary points in the correct way. Then, since $\Phi(z) - z$ is analytic in E^c and zero at infinity, we have

$$\Phi(z) = z + \frac{1}{2\pi i} \int_{\partial E} \frac{\Phi(\zeta) - \zeta}{\zeta - z} \, d\zeta, \quad z \in \mathbb{C} \backslash E, \tag{4.1}$$

where ∂E is negatively oriented, such that E^c lies to the left of ∂E. Figure 6 also shows a cartesian grid (left) and its image under Φ (right). For the computation, we numerically approximate the integral in (4.1) with the trapezoidal rule.

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Appendix A

Lemma A.1 *Let $E \subseteq \mathbb{C}$ be compact, such that E^c has a Green's function g_E. If $E^* = E$, then $g_E(\overline{z}) = g_E(z)$ and $\partial_z g_E(\overline{z}) = \overline{\partial_z g_E(z)}$. Moreover, the critical points of g_E are real or appear in complex conjugate pairs.*

Proof Since $E^* = E$, the function $z \mapsto g_E(\overline{z})$ is also a Green's function with pole at infinity of E^c, hence $g_E(\overline{z}) = g_E(z)$ for all $z \in E^c$ by the uniqueness of the Green's function. Write $g(x, y) = g_E(z)$, then $g(x, y) = g(x, -y)$ and

$$\frac{\partial g}{\partial x}(x, y) = \frac{\partial g}{\partial x}(x, -y), \quad \frac{\partial g}{\partial y}(x, y) = -\frac{\partial g}{\partial y}(x, -y), \tag{A.1}$$

hence

$$2\partial_z g_E(z) = \frac{\partial g}{\partial x}(x, y) - i \frac{\partial g}{\partial y}(x, y) = \frac{\partial g}{\partial x}(x, -y) + i \frac{\partial g}{\partial y}(x, -y) = \overline{2(\partial_z g_E)(\overline{z})}.$$

The critical points of g_E are the zeros of the analytic function $\partial_z g_E$. Since $\partial_z g_E(\overline{z}) = \overline{\partial_z g_E(z)}$, if z_* is a zero of $\partial_z g_E$ then also $\overline{z}_*$ is a zero. □

Lemma A.2 *Let $K \subseteq \mathbb{C}$ be a non-empty compact, simply connected set with $K^* = K$, then $K \cap \mathbb{R}$ is either an interval or a single point.*

Proof Since $K^* = K$ and K is connected, $K \cap \mathbb{R}$ is not empty. Since K^c is connected, $K \cap \mathbb{R}$ must be connected (otherwise the symmetry and simply-connectedness of K would imply that K^c is not connected). Thus, $K \cap \mathbb{R}$ is a point or an interval. □

Lemma A.3 *Let γ be a smooth Jordan curve symmetric with respect to the real line and let f be integrable with $f(\overline{z}) = \overline{f(z)}$ on γ. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz \in \mathbb{R}.$$

Proof Since γ is symmetric with respect to the real line, we can write $\gamma = \gamma_1 + \gamma_2$ with $\gamma_2 := -\overline{\gamma}_1$. Then

$$\begin{aligned}\int_\gamma f(z)\, dz = \int_{\gamma_1} f(z)\, dz - \int_{\overline{\gamma}_1} f(z)\, dz &= \int_a^b f(\gamma_1(t)) \cdot \gamma_1'(t)\, dt \\ &\quad - \int_a^b f(\overline{\gamma_1(t)}) \cdot \overline{\gamma_1'(t)}\, dt \\ &= 2i \int_a^b \operatorname{Im}(f(\gamma_1(t)) \cdot \gamma_1'(t))\, dt,\end{aligned}$$

which yields the result. □

Though the following theorem must be known, we did not find it in the literature. For completeness, we include a proof.

Theorem A.4 *Let P be a non-constant polynomial and $\Omega \subseteq \mathbb{C}$ be a simply connected compact set. Then $\widehat{\mathbb{C}} \setminus P^{-1}(\Omega)$ is open and connected, i.e., a region.*

Proof Clearly, $G := \widehat{\mathbb{C}} \setminus P^{-1}(\Omega) = P^{-1}(\widehat{\mathbb{C}} \setminus \Omega)$ is open and contains ∞. Let $G_\infty \subseteq G$ be that component of G that contains ∞. Suppose that G is not connected, i.e., $G \neq G_\infty$. Then there exists another component $G_1 \subseteq G$, and G_1 is a bounded region. Then $P(G_1)$ is a bounded region with $P(G_1) \subseteq \widehat{\mathbb{C}} \setminus \Omega$.

Next, we show that $\partial P(G_1) \subseteq \partial \Omega$. Let $w \in \partial P(G_1)$. Then there exists $w_k \in P(G_1)$ with $w_k \to w$. For each k, there exists $z_k \in G_1$ with $P(z_k) = w_k$. Since G_1 is bounded, the sequence $(z_k)_k$ has a convergent subsequence $(z_{k_j})_j$ with $z_{k_j} \to z \in \overline{G}_1$. This implies that $P(z) = w$. Since $w \in \partial P(G_1)$, we have $z \in \partial G_1$ (otherwise, $z \in G_1$ would imply $P(z) \in P(G_1)$ and, since $P(G_1)$ is open, $w = P(z) \notin \partial P(G_1)$). Since G is open, this implies that $z \notin G$ and hence that $z \in P^{-1}(\Omega)$ and $w = P(z) \in \Omega$. Since $w_k \in P(G_1) \subseteq \widehat{\mathbb{C}} \setminus \Omega$ and $w_k \to w$, we obtain that $w \in \partial \Omega$.

We have shown that $P(G_1) \subseteq \widehat{\mathbb{C}} \setminus \Omega$ is a region with $\partial P(G_1) \subseteq \partial \Omega = \partial(\widehat{\mathbb{C}} \setminus \Omega)$. Since $\widehat{\mathbb{C}} \setminus \Omega$ is connected, this implies that $P(G_1) = \widehat{\mathbb{C}} \setminus \Omega$, which contradicts that $P(G_1)$ is bounded. This shows that $G = \widehat{\mathbb{C}} \setminus P^{-1}(\Omega)$ is connected. □

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