Studies on Positive and Symmetric Two-Faced Universal Products

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Introduction

This thesis considers, in a broad sense, the notion of noncommutative stochastic independence in the realm of quantum probability theory. First, we notice that the notion of stochastic independence is necessary in order to define stochastic processes with independent increments (Lévy processes) or to achieve the formulation of central limit theorems. In other words, independence is one of the main concepts in classical probability theory. But why do we actually bother about noncommutative probability, as quantum probability is also called? For a motivation coming from physics we refer to [FS16, Sec. 1.3]. From a mathematical point of view, we can think of quantum probability theory as a generalization of classical probability theory in a noncommutative way. The basic idea is to replace the commutative algebra of random variables by a possibly noncommutative algebra of operators. Investigations of this kind are already present in the work of von Neumann ([Neu32]), and they were systematically pursued at least since the late 70's for instance by Accardi ([Acc76]). For a nice overview of the development of quantum probability theory we refer to [Bel00].

There exist several different aspects of quantum probability theory. We want to focus on a "natural" definition of stochastic independence in quantum probability which in turn allows to define quantum Lévy processes or quantum central limit theorems for a certain independence. Naturality here means that we want to establish a possible transition from the classical world to the quantum world. There is no master plan for doing something like this. Thus, our concept of stochastic independence which we would like to extend to the noncommutative case needs to be justified by answering the question: "Is this a good definition?". Let us try to briefly clarify what we could mean by stochastic independence in quantum probability which also covers the classical case. For this, we quickly review classical stochastic independence. Let

$$X\colon \Omega \longrightarrow E_1, \quad Y\colon \Omega \longrightarrow E_2 \tag{1}$$

be two random variables over the same probability space $(\Omega, \mathcal{F}, P)$ with values in measurable spaces $(E_i, \mathcal{E}_i)$. We say that X and Y are independent if and only if for all measurable sets $B_1 \in \mathcal{E}_1$ and $B_2 \in \mathcal{E}_2$

$$P\big(\{ X \in B_1 \} \cap \{ Y \in B_2\}\big) = P\big(\{ X \in B_1 \}\big)P\big(\{ Y \in B_2 \}\big)$$

holds. Using the pushforward measure (i. e. the distribution) of the random variable (X, Y), this is equivalent to

$$P_{(X,Y)} = P_X \otimes P_Y. \tag{2}$$

Now, we dualize the picture of classical probability theory. The probability measure P on Ω induces a linear functional Φ on $L^\infty(\Omega)$ if we put

$$\Phi(F) = \int_\Omega F(\omega)\,\mathrm{d}P(\omega).$$

In this dualized setting, for each classical random variable $X\colon \Omega \longrightarrow E$ we can assign a map

$$j_X\colon \begin{cases} L^\infty(E) \longrightarrow L^\infty(\Omega) \\ f \longmapsto f\circ X. \end{cases}$$

We can turn the L^∞-spaces into $*$-algebras and we can see that j_X is a homomorphism of $*$-algebras. Now, we apply the dualized setting to the random variables of equation (1) ([BS02, p. 532]). Thus, we obtain homomorphisms of algebras

$$j_X\colon L^\infty(E_1) \longrightarrow L^\infty(\Omega), \quad j_Y\colon L^\infty(E_2) \longrightarrow L^\infty(\Omega) \tag{3}$$

and equation (2) is equivalent to

$$\forall f \in L^\infty(E_1),\ \forall g \in L^\infty(E_2)\colon \Phi\big(j_X(f)j_Y(g)\big) = \Phi\big(j_X(f)\big)\Phi\big(j_Y(g)\big)$$

or

$$\Phi\circ\mu\circ(j_X\otimes j_Y) = (\Phi\circ j_X)\otimes(\Phi\circ j_Y), \tag{4}$$

where $\mu\colon L^\infty(\Omega)\otimes L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$ denotes the multiplication of the algebra $L^\infty(\Omega)$.

The above described dualized picture of stochastic independence, leads to the following translation from classical probability theory as a commutative realization on $L^\infty(\Omega)$ to quantum probability theory as a noncommutative version:

- a classical probability space $(\Omega, \mathcal{F}, P)$ is replaced by $(\mathscr{A}, \Phi)$, where $\mathscr{A}$ a unital involutive algebra not necessarily commutative and Φ is a positive and normalized linear functional on $\mathscr{A}$,
- measurable functions $X\colon \Omega \longrightarrow E$ are replaced by homomorphisms $j\colon \mathscr{B} \longrightarrow \mathscr{A}$ of involutive unital algebras,
- the distribution $P_X = P\circ X^{-1}$ is replaced by $\phi = \Phi\circ j$, since $\phi_X = \Phi\circ j_X$ induces a linear functional on $L^\infty(E)$ for each measurable function $X\colon \Omega \longrightarrow E$,
- independence of two random variables X and Y given by equation (2) is replaced by the following definition: two homomorphisms of unital involutive algebras $j_1\colon \mathscr{B}_1 \longrightarrow \mathscr{A}$, $j_2\colon \mathscr{B}_2 \longrightarrow \mathscr{A}$ are independent if and only if

$$\Phi\circ\mu_{\mathscr{A}}\circ(j_1\otimes j_2) = (\Phi\circ j_1)\otimes(\Phi\circ j_2). \tag{5}$$

Now, we may ask if this is a good generalization of independence? The answer is no. If we take a look at the map $j_1 \uplus j_2 := \mu_{\mathscr{A}}\circ(j_1\otimes j_2)\colon \mathscr{B}_1\otimes\mathscr{B}_2 \longrightarrow \mathscr{A}$, then for commutative unital algebras $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{A}$ the map $j_1\uplus j_2$ is a homomorphism. But this is not true for all algebras which are possibly not commutative. Without having that $j_1\uplus j_2$ is homomorphic, our theory would not be consistent. This stems from the transition of equation (2) to the noncommutative case in the sense of equation (5). According to our dualized setting (equation (3)) the classical random variable $(X, Y)\colon \Omega \longrightarrow E_1\times E_2$ should correspond to a homomorphism of algebras $j_1\uplus j_2\colon \mathscr{B}_1\otimes\mathscr{B}_2 \longrightarrow \mathscr{A}$ for any algebras $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{A}$. To resolve this dilemma, we can put further restrictions on the maps j_1, j_2 such that $j_1\uplus j_2$ defines a homomorphism of arbitrary algebras. In this way we obtain the notion of *tensor independence* ([CH71], [Hud73]). This kind of stochastic independence plays a prominent role in quantum probability and is closest to classical stochastic independence.

Instead of imposing assumptions on the homomorphisms j_1 and j_2, we can also ask for another suitable "product" between the algebras $\mathscr{A}_1$ and $\mathscr{A}_2$. The idea ([BS02, p. 532]) is to recognize the map $\mu_{\mathscr{A}} \circ (j_1 \otimes j_2)$ as the unique homomorphism of algebras defined on the coproduct of unital, commutative algebras. The coproduct in the category of unital, commutative algebras is given by the tensor product $\otimes$. Thus, if we want to extend the notion of independence to the noncommutative case, then we need to replace the tensor product $\otimes$ on the left hand side of equation (5) by the coproduct $\sqcup$ in the category of unital algebras.

Furthermore, we need a replacement for the tensor product between linear functionals on the right hand side of equation (5). Ben Ghorbal and Schürmann proposed in [BS02] that it needs to be some kind of product which maps each pair of normalized linear functionals $\big(\varphi_1 \colon \mathscr{B}_1 \longrightarrow \mathbb{C}, \varphi_2 \colon \mathscr{B}_2 \longrightarrow \mathbb{C}\big)$ to a normalized linear functional $\varphi_1 \odot \varphi_2 \colon \mathscr{B}_1 \sqcup \mathscr{B}_2 \longrightarrow \mathbb{C}$. Given such a product, following [BS02, p. 538], we say two homomorphisms of unital algebras $j_1 \colon \mathscr{B}_1 \longrightarrow \mathscr{A}$, $j_2 \colon \mathscr{B}_2 \longrightarrow \mathscr{A}$ are $\odot$-independent (with respect to a normalized linear functional $\Phi \colon \mathscr{A} \longrightarrow \mathbb{C}$ on the unital algebra $\mathscr{A}$) if and only if

$$\Phi \circ (j_1 \sqcup j_2) = (\Phi \circ j_1) \odot (\Phi \circ j_2). \tag{6}$$

We want to point out that the above equation might formally look the same as equation (5) but the subtle differences are:

- to replace the (ordinary) tensor product $\otimes$ between homomorphisms by the coproduct $\sqcup$ in the category of unital algebras (*free product of unital algebras*),
- to replace the (ordinary) tensor product $\otimes$ of linear functionals by a yet to be defined product $\odot$ of linear functionals.

At this point, we need to axiomize the product $\odot$. We follow [BS02] for this. First, let us free ourselves from only allowing unital algebras and let us more generally consider arbitrary algebras $\mathscr{A}$ with some linear functionals $\varphi \colon \mathscr{A} \longrightarrow \mathbb{C}$ defined on them. Then, we view $\odot$ as a map which assigns to each pair $\big((\mathscr{A}_1, \varphi_1), (\mathscr{A}_2, \varphi_2)\big)$ a linear functional $\varphi_1 \odot \varphi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathbb{C}$. The following defining axioms for $\odot$ seem to be appropriate

(a) *unitality/restriction property:* $(\varphi_1 \odot \varphi_2) \circ \iota_{1/2} = \varphi_{1/2}$,

(b) *associativity:* $(\varphi_1 \odot \varphi_2) \odot \varphi_3 = \varphi_1 \odot (\varphi_2 \odot \varphi_3)$,

(c) *universality:* $(\varphi_1 \circ j_1) \odot (\varphi_1 \circ j_2) = (\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2)$,

(d) *symmetry:* $\varphi_1 \odot \varphi_2 = \varphi_2 \odot \varphi_1$

(e) *positivity:* if $\varphi_i \colon \mathscr{A}_i \longrightarrow \mathbb{C}$ are strongly positive on the $*$-algebra $\mathscr{A}_i$, then $\varphi_1 \odot \varphi_2$ is also strongly positive

Here, $\iota_{1/2} \colon \mathscr{A}_{1/2} \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ denote the canonical homomorphisms to the coproduct of two algebras. Moreover, we say that $\varphi \colon \mathscr{A} \longrightarrow \mathbb{C}$ is strongly positive on a $*$-algebra if and only if the normalized unital extension $\varphi^{\mathbb{1}} \colon \mathscr{A}^{\mathbb{1}} \longrightarrow \mathbb{C}$ is positive on $\mathscr{A}^{\mathbb{1}}$, the unitization of the algebra $\mathscr{A}$. Let us give a motivation for the above axioms. The first condition is motivated by the fact $(j_1 \sqcup j_2) \circ \iota_{1/2} = \iota_{1/2}$ and equation (6). We demand associativity because we want to define independence for more than two homomorphisms (of unital $*$-algebras). The universality condition is descended from the fact that if two classical random variables X and Y are independent, then functions $f(X)$ and $g(Y)$ of these random variables are independent too. The second last condition can be demanded if we want to recover the symmetry of classical independence, which says that X and

Y are independent if and only if Y and X are independent. Positivity is motivated by the fact that the product measure of two probability measures is a probability measure. We abbreviate unital associative universal product by *u.a.u.-product*.

All of the above heuristic for an axiomatic approach towards a universal product, which in the end leads to a "good" definition of stochastic independence by equation (6), would be worthless if it did not cover at least any known example. At its time of publishing in 2002, Schürmann and Ben Ghorbal showed in their work [BS02] that their axiomatic approach is sufficiently general to include the notions of

(a) tensor independence,

(b) free independence ([Voi85]),

(c) boolean independence ([Wal73], [Wal75] and [SW97]).

These are very prominent examples for noncommutative stochastic independence in quantum probability since in particular for each of these independences a quantum stochastic calculus had been developed. One might ask if there exist more such universal products. Using a similar axiomatic setting as in [BS02], where universality is replaced by a so-called "universal calculation rule for mixed moments", Speicher could show that the tensor, free and boolean product are the only (nondegenerate) symmetric u.a.u.-products ([Spe97]). Later, in 2002, Schürmann and Ben Ghorbal confirmed this result by their work [BS02]. Their strategy was to find a so-called universal coefficient theorem ([BS02, Thm. 5]) for a product $\odot$ between linear functionals on associative algebras satisfying **(a)–(d)**. By this result, they could show that any such product $\odot$ satisfies Speicher's universal calculation rule for mixed moments and thus could use Speicher's classification result ([BS02, Thm. 8])). Dropping the condition of commutativity but imposing a normalization condition

$$(\varphi_1 \odot \varphi_2)(a_1 a_2) = \varphi_{\varepsilon_1}(a_1)\varphi_{\varepsilon_2}(a_2)$$

for all $(\varepsilon_1, \varepsilon_2) \in \{(1,2),(2,1)\}$, $a_1 \in \mathscr{A}_{\varepsilon_1}$, $a_2 \in \mathscr{A}_{\varepsilon_2}$, Muraki ([Mur03]) has shown that there are exactly five instances of u.a.u.-products. In addition to the three symmetric ones, he has found the monotone and anti-monotone product. These five are also the only positive products as Muraki showed in [Mur13a]. We also refer to these independences by *Muraki's five*. In recent years questions concerning classification of u.a.u.-products have been studied by Lachs and Gerhold in [GL15], [Lac15] and [Ger15]. They do not impose the normalization condition from above and get five one- or two parameter families.

Besides the notion of an independence coming from a positive u.a.u.-product, there exist many other notions of independence not satisfying these conditions. Some are not associative like Muraki's q-deformation of free independence [Mur13b] or Wysoczański's bm-independence ([Wys07]). But there also exist other noncommutative stochastic independences like *braided independence* ([FS99, Def. 4.2.1]) which generalizes a definition of independence originally introduced by Schürmann ([Sch93]). Although braided independence does not fit into the above defined framework of Schürmann's u.a.u.-products, there exist symmetrization theorems ([FSSV] and [Mal17]) which provide a *reduction of independence* in the sense of Franz [Fra06, Def. 3.40] to tensor independence.

Voiculescu's discovery of *bifree independence* ([Voi14]) opened a new chapter in quantum probability theory and in turn for other possible realizations of noncommutative stochastic independence. For bifree independence we need *two-faced algebras* $\mathscr{A}$ which can have a "left face" $\mathscr{A}^{(l)} \subseteq \mathscr{A}$ and a "right face" $\mathscr{A}^{(r)} \subseteq \mathscr{A}$ such that $\mathscr{A} \cong \mathscr{A}^{(l)} \sqcup \mathscr{A}^{(r)}$, where $\sqcup$ denotes the free product of algebras. Schürmann quickly adapted his axiomatic framework to this then new kind of independence. In [MS17] Manzel and Schürmann introduced a product suitable for linear functionals living on so-called *m-faced algebras* $\mathcal{A}$ for $m \in \mathbb{N}$, i. e. on algebras $\mathcal{A}$ such that $\mathcal{A} \cong \mathcal{A}^{(1)} \sqcup \cdots \sqcup \mathcal{A}^{(m)}$. We can think of this product as a product again with the properties **(a)–(c)** but with morphisms j_i replaced by morphisms which respect the face of an algebra, i. e., $j_i(\mathcal{B}_i^{(j)}) \subseteq \mathcal{A}_i^{(j)}$. Manzel and Schürmann also allowed the linear functionals to have d-components ($d \in \mathbb{N}$), i. e., they have a product between linear functionals $\varphi \colon \mathcal{A} \longrightarrow \mathbb{C}^{\times d}$, where $\mathcal{A}$ is an m-faced algebra. Thus, they arrive at so-called *(d,m)*-universal products. This approach turns out to be very fruitful since it covers many prominent examples of noncommutative stochastic independence, like

- Voiculescu's bifreeness (corresponding to $m = 2, d = 1$),
- Bożejko and Speicher's c-freeness ([BLS96]) (corresponding to $m = 1, d = 2$),
- Hasebe's indented independence ([Has10]) (corresponding to $m = 1, d = 2$ and $d = 3$),
- Liu's free-boolean independence ([Liu19]) (corresponding to $m = 2, d = 1$),
- Gerhold's bimonotone independence ([Ger17]) (corresponding to $m = 2, d = 1$),
- Lachs' (r, s)-products ([Lac15]) (corresponding to $m = 1, d = 1$).

In the introduction of Gerhold's paper [Ger21]) we can find even more examples. Among these examples positive (d, m)-universal products are very attractive. They have the striking property that the Schoenberg correspondence holds ([Ger21]). This in turn opens the door to quantum Lévy processes via *Schürmann triples* which in particular are in one-to-one correspondence to quantum Lévy processes on involutive bialgebras ([Fra06, Thm. 1.9]) and on involutive dual semigroups ([Fra06, Sec. 4]).

By an m-faced universal product we mean a (d, m)-universal product for $d = 1$ and $m \in \mathbb{N}$. This thesis is a contribution to the classification of positive and symmetric two-faced u.a.u.-products. We view universal products as bifunctors in the category of algebraic quantum probability spaces. A proper definition is given in Definition 2.1.9 of this work. In the following we shall outline our approach and present our achievements towards a classification of such universal products.

The foundation of our work is [MS17]. There a version of the universal coefficient theorem for (positive) (d, m)-universal products [MS17, Thm. 4.2, Rem. 4.4] is proven by Manzel and Schürmann. Roughly speaking, this universal coefficient theorem makes a statement about the expression

$$(\varphi_1 \odot \cdots \odot \varphi_k)(\underbrace{a_1 \cdot \cdots \cdot a_n}_{\in \bigsqcup_{i=1}^k \mathcal{A}_i}) \in \mathbb{C}^d \tag{7}$$

for associative m-faced algebras $\mathcal{A}_i$ and linear functionals $\varphi_i \colon \mathcal{A}_i \longrightarrow \mathbb{C}^d$. From [MS17, Thm. 4.2] we can conclude that the expression (7) is a linear combination of products of the form $\varphi_j(a_{i_1} \cdot \cdots \cdot a_{i_\ell})$ such that

- $a_{i_1} \cdot \cdots \cdot a_{i_\ell} \in \mathcal{A}_j$,
- every $i \in \{1, \ldots, n\}$ appears as index i_j exactly once for each summand,
- the coefficients for a given product are *universal*, i. .e., do not depend on the choice of $\mathcal{A}_j$, φ_j or the arguments a_i.

Among all the nonzero universal coefficients, which can appear in the calculation of expression (7) using the the universal coefficient theorem, so-called *highest coefficients* are of great importance for us, in particular when $\odot$ is positive and $d = 1$. In this case we have for any m-faced algebras $(\mathscr{A}_i, (\mathscr{A}_i^{(j)}))_{j\in\{1,\ldots,k\}}$, any linear functionals $(\varphi_i\colon \mathscr{A}_i \longrightarrow \mathbb{C})_{i\in\{1,\ldots,k\}}$, any $\sigma = \big(\varepsilon_i, \delta_i\big)_{i\in[n]} \in \big(\{1,\ldots,k\}\times\{1,\ldots,m\}\big)^{\times n}$, any $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_i}^{(\delta_i)}$ and any $(t_i)_{i\in[n]} \in \mathbb{R}^{\times n}$

$$\big((t_1\varphi_1)\odot\cdots\odot(t_k\varphi_k)\big)(a_1\cdot\cdots\cdot a_n) = \underbrace{\alpha_{\max}^{(\sigma)}}_{\in\mathbb{C}}\prod_{i=1}^{k}(t_i\varphi_i)(M_i) + \mathcal{O}(t_i^2),$$

where M_i is a "sub-monomial" of $a_1\cdot\cdots\cdot a_n$ with elements in $\mathscr{A}_i$ which are multiplied in the order as they appear in $a_1\cdot\cdots\cdot a_n$ (*right-ordered monomials property*). We call the complex coefficient $\alpha_{\max}^{(\sigma)}$ *highest coefficient* with respect to σ. Apart from proving the universal coefficient theorem, in [MS17] Manzel and Schürmann also prove existence of cumulants with respect to a given (d,m)-universal product and certain "cumulant Lie algebras". Cumulants in quantum probability theory have also been investigated by Ebrahimi-Fard and Patras ([EP18]), Lehner and Hasebe ([HL19]) and Anshelevich ([Ans01]). We use the definition of cumulants from [MS17]. Having the notion of a cumulant with respect to a given (d,m)-universal product, Manzel and Schürmann can prove a moment-cumulant formula for (d,m)-universal products ([MS17, eq. (7.3), eq. (7.4)]). This is the foundation from where we develop our investigations. We would like to enrich it by a terminology using partitions.

A moment-cumulant formula for a positive associative universal product in the single-faced setting using the partition structure is given by Hasebe and Saigo ([HS11]). Their axiomatic approach makes it very clear how a partition structure appears in a moment-cumulant formula for the single-faced setting. Thanks to their work, we can see how to use partitions to formulate a moment-cumulant formula for a universal product. In ([HS11, Thm. 4.3, Thm. 5.3]) Hasebe and Saigo show that each independence of Muraki's five is uniquely characterized by its so-called highest coefficients and its cumulants. For each independence the nonzero highest coefficients define a certain type of partitions. For the symmetric universal products of Muraki's five they can reproduce the result from Speicher ([Spe97]) that the following partitions appear in the moment-cumulant formula of a given product:

- interval partitions for the boolean product,
- noncrossing partitions for the free product,
- all partitions for the tensor product.

For the nonsymmetric universal products in the single-faced case they obtain the result from Muraki ([Mur02]) that the following partitions appear in the moment-cumulant of a given product:

- monotone partitions for the monotone product,
- anti-monotone partitions for the anti-monotone product.

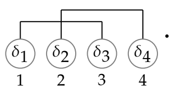
We also have examples in the two-faced case. The following partitions appear in the moment-cumulant formula of a given product:

- binoncrossing partitions for the bifree product ([MN15] and [CNS15]),
- interval-noncrossing partitions for the free-boolean product ([Liu19]).

The notion of a cumulant can be axiomized and thus cumulants become combinatorial objects which are connected to partitions ([Spe83]). A natural question arises: can we use partitions for a classification of universal products? The answer to this question is twofold as we shall see. We can say that all of our combinatorics is encoded in the highest coefficients for a given positive u.a.u.-product. This in turn leads to a certain picture of partitions. A priori it is not clear what such partitions satisfy among themselves. It was one task to investigate this problem in a systematic way for the symmetric case and has lead to our discovery of what we call *m-colored universal classes of partitions*. The starting point of our investigations is:

> **Result 1 (Theorem 2.5.13).** Any positive m-faced u.a.u.-product is uniquely determined by its highest coefficients.

This is an implication of the work of Manzel and Schürmann [MS17]. We use this result to define a *partition induced universal product*. For this, we mimic the moment-cumulant formula of Manzel and Schürmann, where a sum over partitions coming from *m-colored universal class of partitions* is introduced (Definition 3.4.9). These partition induced universal products are modeled in such a way that their nonzero highest coefficients are 1. Let us briefly present what we mean by an m-colored partition. To grasp a better understanding, we want to draw partitions. Let $m \in \mathbb{N}$, $\delta \in [m]^{\times 4}$, then we draw the partition $\{\{(1, \delta_1), (3, \delta_3)\}, \{(2, \delta_2), (4, \delta_4)\}\}$ as



To avoid these big circles around δ_i, we just use • instead to indicate any specific color. In this picture, *m-colored partitions* can be seen as an ordinary set partition with a color label attached to their legs. The color labels are determined by the "decoration tuple" $(\delta_i)_i$.

> **Result 2 (Theorem 3.3.9, Theorem 3.4.32).** Any m-colored universal class of partitions (Definition 3.4.9) induces a symmetric m-faced u.a.u.-product $\odot_{\mathcal{P}}$ (Definition 3.4.30).

The defining properties for an m-colored universal class of partitions are nice enough to obtain a classification in the case $m = 1$ and $m = 2$.

> **Result 3 (Theorem 4.1.17, Theorem 4.2.44).** For $m \in \{1, 2\}$ the set of all m-colored universal class of partitions is nonempty and finite and each member in this set has a construction in terms of certain "smallest generating partitions".

The above classification result for two-colored universal classes also contains concrete new realizations beyond the known partitions like binoncrossing or interval-noncrossing. This brings us back to Result 2 and we obtain the following result.

> **Result 4.** We find three new universal products, not yet known in the literature:
>
> **(a)** a universal product induced by what we call *pure noncrossing* partitions (Definition 4.2.13 **(e)**),
>
> **(b)** a universal product induced by what we call *noncrossing-crossing* resp. *crossing-noncrossing* partitions (Definition 4.2.13 **(i)** resp. Definition 4.2.13 **(j)**),
>
> **(c)** a universal product induced by what we call *pure crossing* partitions (Definition 4.2.13 **(l)**).

We want to use Result 3 for a classification of positive and symmetric m-faced u.a.u.-products for $m \in \{1, 2\}$. As a first step we have:

> **Result 5 (Theorem 5.1.15, Theorem 5.2.17).** Any positive and symmetric m-faced u.a.u.-product for $m \in \mathbb{N}$ induces an m-colored universal class of partitions.

As we have mentioned above, the single-faced case has already been classified by Speicher and Ben Ghorbal & Schürmann. We also retrieve this result. Although, it is not new, we list it here since we have used an alternative proof.

> **Result 6 (Corollary 5.1.25).** The only positive and symmetric u.a.u.-products in the single-faced case are the boolean, free and tensor product.

We do obtain at first sight a less nice but somehow remarkable result in the two-faced case, because we only know that the nonzero coefficients constitute a two-colored universal class of partitions. In order to have a one-to-one correspondence between positive and symmetric two-faced u.a.u.-products $\odot$ and two-colored universal classes of partitions, we would need that the nonzero highest coefficients of $\odot$ are 1. But this is not the case, as we will discuss. Instead, we have a "partial" classification result due to Result 5 and Result 3.

> **Result 7 (Theorem 5.2.25, Proposition 5.2.26).** Any positive and symmetric two-faced u.a.u.-product must be one of our determined types. The nonzero highest coefficients of a positive and symmetric two-faced u.a.u.-product are necessarily contained in the pointed complex unit disk. We have found the known examples of:
>
> **(a)** Voiculescu's bifree product
>
> **(b)** Liu's free-boolean product
>
> **(c)** two-faced incarnations of tensor, free and boolean product
>
> Moreover, we obtain the possibility of three *q-deformed* positive and symmetric two-faced u.a.u.-products for $q \in \mathbb{C}$ and $0 < |q| \leq 1$.

The above result stimulated research of Hasebe, Gerhold and Ulrich, looking for possible Hilbert space realizations of two-faced u.a.u.-products, thereby establishing positivity of those products. In an unpublished draft of their paper ([HGU21]), among other results, positive universal products are constructed which seem to coincide with the q-deformed two-faced universal products mentioned in Result 7 as long as $|q| = 1$.

Let us discuss open questions which arise from the above results or we do not answer in this thesis:

- According to Result 2 we may speak of partition induced universal products. Such products are also uniquely determined by their highest coefficients. We have designed these products in such a way that their nonzero highest coefficients are all one. Is it possible to relax the definition of partition induced universal products such that their highest coefficients are not necessarily one? At the end of Remark 5.2.28 we discuss this question.
- Concerning Result 7, which products in the list of all candidates of positive and symmetric two-faced u.a.u.-products are actually positive? For some of these products we can prove positivity but others remain unclear. Once this question has been answered, the classification of such products would be completed.
- Can we directly encode the positivity of a partition induced u.a.u.-product into the defining axioms for a certain class of partitions? In other words, can we find sufficient conditions

for a certain class of partitions which induces a positive (symmetric) u.a.u.-product?

- In Remark 5.2.18 we discuss why we cannot drop the assumption of positivity from Result 5. In particular, we can see that the two-colored universal classes of partitions are not enough in order to classify all the symmetric two-faced u.a.u.-products. This leads to the natural question if there might exist weaker axioms for a class of partitions which correspond to all the symmetric u.a.u.-products (without assuming positivity).
- Is it possible to classify m-colored universal classes of partitions for an arbitrary $m \in \mathbb{N}$? In particular, answering this question would lead to many more examples of symmetric m-faced u.a.u.-products.
- We could show that single-colored and two-colored universal classes of partitions are complete lattices. Is it possible to extend this result for arbitrary m-colored universal classes of partitions? Identifying this kind of structure or any higher combinatorial structure could help us to answer the next question.
- How would a "universal class of m-colored ordered partitions" look like for the non-symmetric case? In other words, can we find sufficient conditions for a class of ordered partitions which lead to a well-defined m-faced u.a.u.-product in the nonsymmetric case? If this can be done, do these axioms allow a classification of such classes of partitions?

The plan of this dissertation is as follows:

In Chapter 1 we introduce some notations which we will use throughout the work. Furthermore, we gather some tools to be able to formulate in the needed generality the famous Baker–Campbell–Hausdorff formula (abbreviated by *BCH-formula*). This is done in two steps. First, we consider a more general setting of an algebra with a so-called *topologically admissible family* which turns it into a topological algebra and discuss the existence of a BCH-formula there. Second, we consider the convolution algebra $\operatorname{Lin}(\mathscr{H}, \mathscr{A})$ and discuss the existence of a BCH-formula there. Herein, $\mathscr{H}$ denotes a connected filtered bialgebra and $\mathscr{A}$ denotes a unital algebra. We close this chapter by a brief review of derivations.

In Chapter 2 we introduce our main object of consideration; universal products and its properties. To derive a moment-cumulant formula for a given universal product, we need the machinery which goes under the name of "reduction of convolution". For this, we need to introduce the notion of *dual semigroups*. Then, we quickly review how the universal coefficient theorem leads to the definition of the *Lachs functor*. Roughly speaking, the Lachs functor assigns to each dual semigroup a bialgebra. Reduction of convolution then means, to express the convolution between elements in the dual space of a dual semigroup by the convolution between elements in the dual space of the bialgebra, coming from the Lachs functor. After a description of this procedure, we can prove a moment-cumulant formula by the results provided in Chapter 1. Later, we introduce the notion of highest coefficients of a universal product with *right-ordered monomials property* and prove that such products are uniquely determined by their highest coefficients.

In Chapter 3 we lay the groundwork for our partition induced universal products. For this, we define m-colored universal classes of partitions, *partition induced logarithms (cumulants)* and their inverses as *partition induced exponentials*. Then, we show that such products are actually symmetric m-faced u.a.u.-products.

In Chapter 4 a classification of m-colored universal classes of partitions is done for $m \in \{1, 2\}$.

In Chapter 5 we show how any positive and symmetric unital associative (d, m)-universal product induces an m-colored universal class of partitions by its highest coefficients. We use this result to obtain a full classification of such universal products in the single-faced case and a "partial" classification of such universal products in the two-faced case.

In Chapter 6 we show positivity of the partition induced universal products which stem from what we call *interval-crossing* resp. *crossing-interval* partitions. This is done by a suitable product of representations inspired by a previous work of Gerhold for bimonotone independence ([Ger17]). Since we could not show positivity for certain universal products, which we obtain by our "partial" classification result, we performed some computational tests in order to exclude nonpositive products. We have considered two different test methods, where both start from a result concerning Schoenberg correspondence for m-faced u.a.u.-products, recently proven by Gerhold in [Ger21], leading to a Gaussian functional. We present the results of our calculations which involve pair partitions for a given two-colored universal class of partitions.

Chapter 1

Exponential and logarithm for the convolution algebra

Statements and definitions in this chapter can be seen as decoupled from considerations of quantum probability theory but serve as a foundation to prove a moment-cumulant formula for a universal product in Chapter 2. We first gather some notations and algebraic preliminaries and then formulate the BCH-formula in two steps: first for certain topological algebras and then for so-called convolution algebras. In the end of this chapter review results on derivations which will be needed in Chapter 2.

1.1 Notations and algebraic preliminaries

In this section we will review basic definitions and introduce notations which are used throughout this work. Most of these definitions and stated results can be found in standard textbooks and therefore we omit any proofs. We provide the references to works from where we have actually taken the definitions or results, including their proofs.

1.1.1 Convention. Within a calculation or a proof we put small justifications into *proof brackets* $[\![\dots]\!]$. The abbreviation "TFAE" means "The Following Are Equivalent".

1.1.2 Convention.

(a) In this document, the symbol $\mathbb{N}$ always denotes the set $\{1, 2, \dots\}$ and the symbol $\mathbb{N}_0$ denotes the set $\{0, 1, 2, \dots\}$. For any $n \in \mathbb{N}$ we define $[n] := \{1, 2, \dots, n\}$ and $[0] = \emptyset$.

(b) For any set X and for any $n \in \mathbb{N}$ we denote its n-fold Cartesian product by $X^{\times n}$, i. e.,

$$X^{\times n} := \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}. \tag{1.1.1}$$

(c) In this thesis vector spaces only will be considered over the complex numbers $\mathbb{C}$. In particular, the notion *vector space* will stand for a $\mathbb{C}$-vector space.

(d) Let U and V be two vector spaces. Then, $\mathrm{Lin}(U, V)$ denotes the vector space of all linear maps $U \longrightarrow V$.

1.1.3 Convention. By an *algebra* we always mean a not necessarily unital, associative algebra over $\mathbb{C}$. There exist at least two equivalent definitions for an algebra $\mathscr{A}$. One is to say that a set $\mathscr{A}$ is an algebra if and only if $(\mathscr{A}, \cdot)$ is a magma, $\mathscr{A}$ is a vector space and $\cdot$ is bilinear and associative.

Another definition says that $\mathscr{A}$ is an algebra if and only if $(\mathscr{A}, \mu_{\mathscr{A}})$ for $\mu_{\mathscr{A}} \colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ satisfies $\mu_{\mathscr{A}} \circ (\mu_{\mathscr{A}} \otimes \mathsf{id}_{\mathscr{A}}) = \mu_{\mathscr{A}} \circ (\mathsf{id}_{\mathscr{A}} \otimes \mu_{\mathscr{A}})$. It is a standard task to show that both definitions are equivalent (e.g. [Grin18, p. 30]). The latter one is more suitable to recognize that the concept of a coalgebra is meant to be something dual to an algebra, i. e., the concept of reversing arrows. We use both definitions for an algebra interchangeably and it should be clear from the context which one we actually use.

1.1.4 Remark. There are several *universal mapping properties* for certain objects in a category which we will abbreviate by "UMP" for certain objects. Here we provide two of them which become important later and are used throughout this work.

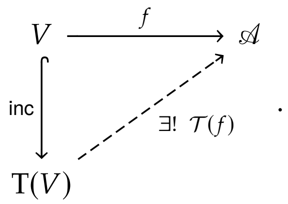
(a) *UMP for Tensor Algebras.*([BF12, Thm. 2.38]) Let V be a vector space and $\mathscr{A}$ an algebra. The *tensor algebra* over V is an algebra and as a vector space given by

$$\mathrm{T}(V) := \bigoplus_{n \in \mathbb{N}} V^{\otimes n}. \tag{1.1.2}$$

Be aware that using the sign $\otimes$ as the multiplication in the algebra $\mathrm{T}(V)$ is bad notation and we prefer $\cdot$ to avoid any possible conflicts ([Grin17, Rem. 14]). Nonetheless, we still have

$$\forall n, m \in \mathbb{N}, \forall a \in V^{\otimes n}, \forall b \in V^{\otimes m} \colon a \cdot b = a \otimes b. \tag{1.1.3}$$

Furthermore, we use the following notation which shall reflect the universal mapping property of the tensor algebra $\mathrm{T}(V)$ given $f \in \mathrm{Lin}(V, \mathscr{A})$



(1.1.4)

If W is another vector space, $f \in \mathrm{Lin}(V, W)$ and $\mathsf{inc} \colon W \hookrightarrow \mathrm{T}(W)$ is the canonical inclusion map, then we set

$$\mathrm{T}(f) := \mathcal{T}(\mathsf{inc} \circ f) \colon \mathrm{T}(V) \longrightarrow \mathrm{T}(W). \tag{1.1.5}$$

For the unital tensor algebra over V we use the symbol

$$\mathrm{T}_0(V) := \bigoplus_{n \in \mathbb{N}_0} V^{\otimes n}. \tag{1.1.6}$$

The unital tensor algebra T_0 satisfies a similar commutative diagram as in equation (1.1.4) but for unital algebras .

(b) *UMP for Symmetric Algebras.*([BF12, Thm. 10.7]) Let V be a vector space and $\mathscr{A}$ a unital algebra. The symmetric tensor algebra $\mathrm{Sym}(V)$ over V is defined by

$$\mathrm{Sym}(V) := \bigoplus_{n \in \mathbb{N}_0} \mathrm{Sym}^n(V) \cong \mathrm{T}(V) \big/ K(V), \tag{1.1.7}$$

where

$$\mathrm{Sym}^n(V) = V^{\otimes n} \big/ K_n(V), \tag{1.1.8}$$

$$K_n(V) = \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (K_2(V)) \cdot V^{\otimes(n-1-i)}, \tag{1.1.9}$$

$$K_2 := \langle v_1 \otimes v_2 - v_2 \otimes v_1 \colon (v_1, v_2) \in V^{\times 2} \rangle, \tag{1.1.10}$$

$$K(V) = \bigoplus_{n \in \mathbb{N}_0} K_n(V). \tag{1.1.11}$$

A proof of these statements can be found in [Grin17, Cor. 56]. Given $f \in \mathrm{Lin}(V, \mathscr{A})$ with the property $\forall v,\, w \in V \colon f(v) \cdot f(w) = f(w) \cdot f(v)$, then the UMP of the commutative unital algebra $\mathrm{Sym}(V)$ is characterized by

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & \mathscr{A} \\ {\scriptstyle \mathrm{i_s}}\big\downarrow & \nearrow{\scriptstyle \exists !\ \mathcal{S}(f)} & \\ \mathrm{Sym}(V) & & \end{array} \quad . \tag{1.1.12}$$

If W is another vector space, $f \in \mathrm{Lin}(V, W)$ and $\mathsf{inc}\colon W \hookrightarrow \mathrm{Sym}(W)$ is the canonical inclusion map, then we set

$$\mathrm{Sym}(f) := \mathcal{S}(\mathsf{inc} \circ f)\colon \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(W). \tag{1.1.13}$$

1.1.5 Convention. Let V_1 and V_2 be two $\mathbb{C}$-vector spaces. Then, there exists a canonical isomorphism of algebras such that

$$\mathrm{T}(V_1 \oplus V_2) \cong \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{(\varepsilon_i)_{i \in [n]} \\ \in \{1,2\}^{\times n}}} V_{\varepsilon_1} \otimes \cdots \otimes V_{\varepsilon_n} \tag{1.1.14}$$

For the idea of a proof of the above statement we refer to [Gre78, Sec. 3.7] or [Bou98, Ch. III, § 5.5]. In this work we will identify both sides of equation (1.1.14) without explicitly mentioning the canonical isomorphism. Also compare the above result to the one in Remark 2.2.7 **(a)**.

1.1.6 Lemma ([Gre78, Sec. 9.8]). Let V and W be vector spaces, then there exists a canonical isomorphism of unital algebras such that

$$\mathrm{Sym}(V \oplus W) \cong \mathrm{Sym}(V) \otimes \mathrm{Sym}(W). \tag{1.1.15}$$

1.1.7 Remark (Coproduct in a category [Rot09, Def. on p. 220]). The coproduct of a family of objects $(A_i)_{i \in I}$ in a category $\mathfrak{C}$ is an ordered pair $\big(C, (\alpha_i \colon A_i \longrightarrow C)_{i \in I}\big)$, where $C \in \mathsf{Obj}(\mathfrak{C})$ and $(\alpha_i \colon A_i \longrightarrow C)_{i \in I}$ is a family of morphisms (called *inclusions*), which satisfies the following universal property: for every object $X \in \mathsf{Obj}(\mathfrak{C})$ and for a family of morphisms $(f_i \colon X_i \longrightarrow X)_{i \in I}$, there exists a unique morphism $\theta\colon C \longrightarrow X$ making the diagram commute for each $i \in I$

$$\begin{array}{ccc} A_i & \xrightarrow{\quad f_i \quad} & X \\ {\scriptstyle \alpha_i}\big\downarrow & \nearrow{\scriptstyle \exists !\ \theta} & \\ C & & \end{array} \quad . \tag{1.1.16}$$

Typically, should the coproduct exist, one denotes the object C by $\bigsqcup_{i\in I} A_i$ (the inclusions α_i are not mentioned) which is unique up to isomorphism and fulfills ([Bor94, Prop. 2.2.3])

$$\bigsqcup_{i\in I} A_i \cong \bigsqcup_{k\in K}\Big(\bigsqcup_{j\in J_k} A_j\Big), \tag{1.1.17}$$

wherein the index set I is equal to a partition $\bigcup_{k\in K} J_k$ of disjoint subsets J_k and it is assumed that all coproducts exist. The unique isomorphism θ usually has the symbol

$$\bigsqcup_{i\in I} f_i \coloneqq \theta. \tag{1.1.18}$$

The above mentioned universal property of the coproduct has the implication (similar to the proof of [Fra06, Prop. 3.20 (b)]): for a family of morphisms $(f_i\colon X_i \longrightarrow Y_i)_{i\in I}$ there exists a unique morphism $\vartheta \in \mathsf{Morph}_{\mathfrak{C}}(\bigsqcup_{i\in I} X_i, \bigsqcup_{i\in I} Y_i)$ making the diagram commute for each $j \in I$

$$\begin{array}{ccc} X_j & \xrightarrow{\quad f_j \quad} & Y_j \\ {\scriptstyle \alpha_j}\big\downarrow & & \big\downarrow{\scriptstyle \beta_j} \\ \bigsqcup_{i\in I} X_i & \overset{}{\underset{\exists!\ \vartheta}{\dashrightarrow}} & \bigsqcup_{i\in I} Y_i \end{array}, \tag{1.1.19}$$

with inclusions $(\alpha_j\colon X_j \longrightarrow \bigsqcup_{i\in I} X_i)_{j\in I}$ and $(\beta_j\colon Y_j \longrightarrow \bigsqcup_{i\in I} Y_i)_{j\in I}$. Usually, this unique morphism ϑ is also denoted by

$$\bigsqcup_{i\in I} f_i \coloneqq \vartheta. \tag{1.1.20}$$

1.1.8 Convention.

(a) Let $\mathscr{A}$ be a unital algebra. Then, $\mu_{\mathscr{A}}$ denotes the multiplication map $\mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ of the algebra $\mathscr{A}$, and $\eta_{\mathscr{A}}$ denotes the unity map $\mathbb{C} \longrightarrow \mathscr{A}$ of the algebra $\mathscr{A}$. ([Grin18, eq.(1.1)–(1.3)]).

(b) Let $\mathbb{K}$ be a field and $\mathscr{C}$ be a $\mathbb{K}$-coalgebra. Then, $\Delta_{\mathscr{C}}$ denotes the comultiplication map $\mathscr{C} \otimes \mathscr{C} \longrightarrow \mathscr{C}$ of the $\mathbb{K}$-coalgebra $\mathscr{C}$, and $\varepsilon_{\mathscr{C}}$ denotes the counit map $\mathscr{C} \longrightarrow \mathbb{K}$ of the $\mathbb{K}$-coalgebra $\mathscr{C}$. When we say that $\mathscr{C}$ is a coalgebra, we mean that $\mathscr{C}$ is a $\mathbb{K}$-coalgebra for $\mathbb{K} = \mathbb{C}$.

1.1.9 Convention (Convolution algebra [Grin21, Def. 1.9]). Let $\mathscr{A}$ be a unital algebra and let $\mathscr{C}$ be a coalgebra. Then, the vector space $\mathrm{Lin}(\mathscr{C}, \mathscr{A})$ becomes an algebra $(\mathrm{Lin}(\mathscr{C}, \mathscr{A}), \star)$ by setting

$$f \star g = \mu_{\mathscr{A}} \circ (f \otimes g) \circ \Delta_{\mathscr{C}} \tag{1.1.21}$$

for any $f \in \mathrm{Lin}(\mathscr{C}, \mathscr{A})$ and $g \in \mathrm{Lin}(\mathscr{C}, \mathscr{A})$. The algebra $(\mathrm{Lin}(\mathscr{C}, \mathscr{A}), \star)$ is called the *convolution algebra* of $\mathscr{C}$ and $\mathscr{A}$. In the following, we will simply refer to this algebra as $\mathrm{Lin}(\mathscr{C}, \mathscr{A})$. The binary operation $\star$ defined in equation (1.1.21) is called *convolution*. In particular, for any $f \in \mathrm{Lin}(\mathscr{C}, \mathscr{A})$ and $g \in \mathrm{Lin}(\mathscr{C}, \mathscr{A})$, we will refer to $f \star g$ as the *convolution of the maps* f and g. The expression $e_{\mathscr{C},\mathscr{A}}$ shall denote the map $\eta_{\mathscr{A}} \circ \varepsilon_{\mathscr{C}}\colon \mathscr{C} \longrightarrow \mathscr{A}$. This map $e_{\mathscr{C},\mathscr{A}}$ is the unity of the convolution algebra $\mathrm{Lin}(\mathscr{C}, \mathscr{A})$.

1.1.10 Definition (Bialgebra [Grin18, Def. on p. 102]). Let $\mathbb{K}$ be a field. We call a $\mathbb{K}$-vector space $\mathscr{B}$ $\mathbb{K}$-*bialgebra* if and only if it is a unital $\mathbb{K}$-algebra and a $\mathbb{K}$-coalgebra such that $\Delta_{\mathscr{B}}$ and $\varepsilon_{\mathscr{B}}$ are morphisms for the unital algebra structure $(\mathscr{B}, \eta_{\mathscr{B}}, \mu_{\mathscr{B}})$. By *bialgebra* we mean a $\mathbb{K}$-bialgebra for $\mathbb{K} = \mathbb{C}$.

1.1.11 Definition (Hopfalgebra [Grin18, Def. on p. 104]). A bialgebra $\mathscr{H}$ is called a *Hopf algebra* if and only if there is an element $\mathsf{S} \in \mathrm{Lin}(\mathscr{H}, \mathscr{H})$, called the *antipode* for $\mathscr{H}$, which is a two-sided inverse for the map $\mathsf{id}_{\mathscr{H}}$ under the convolution $\star$.

1.1.12 Definition (Filtration of vector space [Grin21, Def. 1.13]). A *filtered* vector space means a vector space V equipped with a family $(V_{\leq \ell})_{\ell \in \mathbb{N}_0}$ of vector subspaces of V satisfying $V_{\leq 0} \subseteq V_{\leq 1} \subseteq V_{\leq 2} \subseteq \cdots$ and $V = \bigcup_{\ell \in \mathbb{N}_0} V_{\leq \ell}$. The family $(V_{\leq \ell})_{\ell \in \mathbb{N}_0}$ is called the *filtration* of the filtered vector space V. For each $m \in \mathbb{N}_0$ the vector subspace $V_{\leq m}$ of V is called the *m-th part of the filtration* $(V_{\leq \ell})_{\ell \in \mathbb{N}_0}$.

1.1.13 Convention. Whenever V is a filtered vector space and ℓ is a negative integer, we define $V_{\leq \ell}$ to mean the vector subspace $\{0\}$ of V.

1.1.14 Definition (Θ-graded, graded vector space [Lac15, Sec. 2.4]). Let Θ be a commutative monoid with the commutative operation "+" and neutral element 0. A Θ-*graded* vector space means a vector space V equipped with a family $(V_\ell)_{\ell \in \Theta}$ of vector subspaces of V satisfying $V = \bigoplus_{\ell \in \Theta} V_\ell$. The family $(V_\ell)_{\ell \in \Theta}$ is called the Θ-*grading* of the graded vector space V. If $\Theta = (\mathbb{N}_0, +)$, then we call the Θ-graded vector space $\mathbb{N}_0$-*graded* or just *graded* and the Θ-grading a $\mathbb{N}_0$-*grading* or just a *grading*.

1.1.15 Proposition (Graded vector space is filtered [Grin21, Prop. 16.2]). Let V be a graded vector space. For every $n \in \mathbb{N}_0$ we define a vector subspace $V_{\leq n}$ of V by $V := \bigoplus_{l=0}^{n} V_l$. Then, $(V, (V_{\leq n})_{n \in \mathbb{N}_0})$ is a filtered vector space.

1.1.16 Lemma ([Gre78, Sec. 9.9]). Let $V = \bigoplus_{n \in \mathbb{N}_0} V_n$ be a graded vector space. If we set

$$\big(\mathrm{Sym}(V)\big)_0 := \mathbb{C} \oplus \bigoplus_{\ell \in \mathbb{N}} \Big((V_0)^{\otimes_{\mathrm{Sym}} \ell} \Big), \tag{1.1.22a}$$

$$\forall k \in \mathbb{N}\colon \big(\mathrm{Sym}(V)\big)_k := \bigoplus_{\substack{(p_\ell)_{\ell \in [n]} \in (\mathbb{N}_0)^n : \\ n \in \mathbb{N}_0,\, \sum_{\ell=1}^n \ell p_\ell = k}} \left(\bigotimes_{\ell \in [n]} (V_\ell)^{\otimes_{\mathrm{Sym}} p_\ell} \right), \tag{1.1.22b}$$

then this defines the unique grading for the unital algebra $\mathrm{Sym}(V)$ such that the canonical injection $V \hookrightarrow \mathrm{Sym}(V)$ is homogeneous of degree 0.

1.1.17 Definition (Filtered algebra/coalgebra/bialgebra/Hopf algebra [Grin18, Def. on p. 299 & p. 311]).

(a) Let $\mathscr{A}$ be a unital algebra and assume that $\mathscr{A}$ is filtered as a vector space. Then, $(\mathscr{A}, (\mathscr{A}_{\leq \ell})_{\ell \in \mathbb{N}_0})$ is called a *filtered unital algebra*, if and only if the following two conditions are satisfied

- $\mathbb{1}_{\mathscr{A}} \in \mathscr{A}_{\leq 0}$,
- for all $n, m \in \mathbb{N}_0$ we have $\mathscr{A}_{\leq n} \cdot \mathscr{A}_{\leq m} \subseteq \mathscr{A}_{\leq n+m}$.

(b) Let $\mathscr{C}$ be a coalgebra and assume that $\mathscr{C}$ is a filtered as a vector space. Then, $(\mathscr{C}, (\mathscr{C}_{\leq \ell})_{\ell \in \mathbb{N}_0})$ is called a *filtered coalgebra* if and only if for all $n \in \mathbb{N}_0$ and $x \in \mathscr{C}_n$ we have

$$\Delta(x) \in \sum_{i=0}^{n} \mathscr{C}_i \otimes \mathscr{C}_{n-i}. \tag{1.1.23}$$

(c) Let $\mathscr{H}$ be a bialgebra and assume that $\mathscr{H}$ is a filtered vector space. Then, $(\mathscr{H}, (\mathscr{H}_{\leq \ell})_{\ell \in \mathbb{N}_0})$ is called a *filtered bialgebra* if and only if $(\mathscr{H}, (\mathscr{H}_{\leq \ell})_{\ell \in \mathbb{N}_0})$ is a filtered algebra and a filtered coalgebra.

(d) Let $\mathscr{H}$ be a Hopf algebra and assume that $\mathscr{H}$ is a filtered vector space. Then, $(\mathscr{H}, (\mathscr{H}_{\leq \ell})_{\ell \in \mathbb{N}_0})$ is called a *filtered Hopf algebra* if and only if $(\mathscr{H}, (\mathscr{H}_{\leq \ell})_{\ell \in \mathbb{N}_0})$ is a filtered bialgebra and $\mathsf{S}(\mathscr{H}_{\leq n}) \subseteq \mathscr{H}_{\leq n}$.

1.1.18 Definition (Homogeneous map, graded map [Lac15, Sec. 2.4]).

(a) Let $(V, (V_\ell)_{\ell \in \Theta})$ and $(W, (W_\ell)_{\ell \in \Theta})$ be two Θ-graded vector spaces and let $f \colon V \longrightarrow W$ be a linear map. The map f is called *homogeneous of degree* α_0 if and only if there exists a fixed element $\alpha_0 \in \Theta$ such that

$$\forall \alpha \in \Theta \colon f(V_\alpha) \subseteq W_{\alpha + \alpha_0}. \tag{1.1.24}$$

Herein, α_0 is called the *degree* of the homogeneous mapping f. By a *homogeneous* map we just mean a map which is homogeneous of degree 0.

(b) Let $(V, (V_n)_{n \in \mathbb{N}_0})$ and $(W, (W_n)_{n \in \mathbb{N}_0})$ be two graded vector spaces. A linear map $f \colon V \longrightarrow W$ is called *graded* if and only if $\forall n \in \mathbb{N}_0 \colon f(V_n) \subseteq W_n$.

1.1.19 Definition (Θ-graded (unital) algebra [Lac15, Def. 2.4.1]). Let $(\mathscr{A}, \mu_{\mathscr{A}}, \eta_{\mathscr{A}})$ be an algebra with multiplication map $\mu_{\mathscr{A}} \colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$. Then, $\mathscr{A}$ is called a Θ-graded algebra if and only if the multiplication map is homogeneous. A unital algebra $(\mathscr{A}, \mu_{\mathscr{A}}, \eta_{\mathscr{A}})$ is called a Θ-graded unital algebra if and only if additionally the unit map $\eta_{\mathscr{A}} \colon \mathbb{C} \longrightarrow \mathscr{A}$ is homogeneous.

1.1.20 Definition (Graded algebra/coalgebra/bialgebra/Hopf algebra [Grin18]).

(a) Let $\mathscr{A}$ be a unital algebra and assume that $\mathscr{A}$ is graded as a vector space. Then, $(\mathscr{A}, (\mathscr{A}_n)_{n \in \mathbb{N}_0})$ is called a *graded unital algebra* if and only if the following two conditions are satisfied

- $\mathbb{C} \cdot \mathbb{1}_{\mathscr{A}} \subseteq \mathscr{A}_0$
- for all $n, m \in \mathbb{N}_0$ we have $\mathscr{A}_n \mathscr{A}_m \subseteq \mathscr{A}_{n+m}$

It is equivalent to say that the multiplication $\mu_{\mathscr{A}} \colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ and unit map $\eta_{\mathscr{A}} \colon \mathbb{K} \longrightarrow \mathscr{A}$ are graded.

(b) Let $\mathscr{C}$ be a coalgebra and assume that $\mathscr{C}$ is graded as a vector space. Then, $(\mathscr{C}, (\mathscr{C}_n)_{n \in \mathbb{N}_0})$ is called a *graded coalgebra* if and only if the comultiplication $\Delta_{\mathscr{C}} \colon \mathscr{C} \longrightarrow \mathscr{C} \otimes \mathscr{C}$ and counit $\varepsilon_{\mathscr{C}} \colon \mathscr{C} \longrightarrow \mathbb{K}$ are graded. It is equivalent to say that

$$\forall n \in \mathbb{N}_0 \colon \Delta_{\mathscr{C}}(\mathscr{C}_n) \subseteq \sum_{\ell=0}^{n} \mathscr{C}_\ell \otimes \mathscr{C}_{n-\ell} \tag{1.1.25}$$

and $\varepsilon_{\mathscr{C}}(\mathscr{C}_n) = 0$ for all $n \geq 1$.

(c) Let $\mathscr{H}$ be a bialgebra and assume that $\mathscr{H}$ is a graded vector space. Then, $(\mathscr{H}, (\mathscr{H}_n)_{n\in\mathbb{N}_0})$ is called a *graded bialgebra* if and only if $(\mathscr{H}, (\mathscr{H}_n)_{n\in\mathbb{N}_0})$ is a graded unital algebra and a graded coalgebra.

(d) Let $\mathscr{H}$ be a Hopf algebra and assume that $\mathscr{H}$ is a graded vector space. Then, $(\mathscr{H}, (\mathscr{H}_n)_{n\in\mathbb{N}_0})$ is called a *graded Hopf algebra* if and only if $(\mathscr{H}, (\mathscr{H}_n)_{n\in\mathbb{N}_0})$ is a graded bialgebra and the antipode $\mathsf{S}\colon \mathscr{H} \longrightarrow \mathscr{H}$ is graded.

1.1.21 Lemma ([Grin21, Prop. 16.8]). Let $(\mathscr{H}, (\mathscr{H}_n)_{n\in\mathbb{N}_0})$ be a graded bialgebra. For every $n \in \mathbb{N}_0$ define a vector subspace $\mathscr{H}_{\leq n} \subseteq \mathscr{H}$ by

$$\mathscr{H}_{\leq n} := \bigoplus_{\ell=0}^{n} \mathscr{H}_\ell. \tag{1.1.26}$$

Then, $\big(\mathscr{H}, (\mathscr{H}_{\leq n})_{n\in\mathbb{N}_0}\big)$ is a filtered bialgebra.

1.1.22 Definition (Connected filtered bialgebra [Grin21, Rem. 2.12]). Let $(\mathscr{H}, (\mathscr{H}_{\leq n})_{n\in\mathbb{N}_0})$ be a filtered bialgebra. We say that the filtered bialgebra $\mathscr{H}$ is *connected* if and only if $\mathscr{H}_{\leq 0} = \mathbb{K} \cdot \mathbb{1}_{\mathscr{H}}$, where $\mathbb{1}_{\mathscr{H}}$ denotes the unity of the unital algebra $\mathscr{H}$.

1.1.23 Definition (Connected graded $\mathbb{K}$-bialgebra [Grin18, p. 156]). Let $(\mathscr{H}, (\mathscr{H}_n)_{n\in\mathbb{N}_0})$ be a graded bialgebra. We say that the graded bialgebra $\mathscr{H}$ is *connected* if and only if $\mathscr{H}_0 = \mathbb{K} \cdot \mathbb{1}_{\mathscr{H}}$, where $\mathbb{1}_{\mathscr{H}}$ denotes the unity of the unital algebra $\mathscr{H}$.

1.1.24 Convention. Let $\mathscr{A}$ be an algebra. Consider the linear space $\mathscr{A}^{\mathbb{1}} := \mathbb{C} \oplus \mathscr{A}$ with multiplication defined by

$$\forall \lambda,\ \mu \in \mathbb{C},\ \forall a,\ b \in \mathscr{A}\colon (\lambda \oplus a)(\mu \oplus b) = \lambda\mu \oplus (\lambda b + \mu a + ab). \tag{1.1.27}$$

The map $a \longmapsto 0 \oplus a$ embeds $\mathscr{A}$ isomorphically into this algebra. Furthermore, $1 \oplus 0$ is a unital element of $\mathscr{A}^{\mathbb{1}}$. We will call $\mathscr{A}^{\mathbb{1}}$ the *unitization* of $\mathscr{A}$.
Denote by $\varphi^{\mathbb{1}}\colon \mathscr{A}^{\mathbb{1}} \longrightarrow \mathbb{C}$ the unique linear extension of φ with $\varphi^{\mathbb{1}}(\mathbb{1}) = 1$ and $\varphi^{\mathbb{1}}\restriction_{\mathscr{A}} = \varphi$. We will call $\varphi^{\mathbb{1}}$ the *unital extension* of φ.

1.1.25 Definition (Hermitian, positive and extensible linear functional [Pal01, Def. 9.4.2]). A linear functional φ on a $*$-algebra $\mathscr{A}$ is called

(a) *hermitian* if and only if it satisfies $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathscr{A}$;

(b) *positive* if and only if it satisfies $\varphi(a^*a) \geq 0$ for all $a \in \mathscr{A}$;

(c) *extensible* if and only if there exists a positive linear functional $\omega\colon \mathscr{A}^{\mathbb{1}} \longrightarrow \mathbb{C}$, such that $\omega\restriction_{\mathscr{A}} = \varphi$. Herein, $\mathscr{A}^{\mathbb{1}} = \mathbb{C}\mathbb{1} \oplus \mathscr{A}$ is the unitization of $\mathscr{A}$.

1.1.26 Lemma ([Pal01, Lem. 9.4.3]). Let $\mathscr{A}$ be a $*$-algebra. A positive linear functional $\varphi\colon \mathscr{A} \longrightarrow \mathbb{C}$ satisfies

$$\forall a,\ b \in \mathscr{A}\colon \quad \varphi(a^*b) = \varphi(b^*a)^*, \tag{1.1.28}$$

$$\forall a,\ b \in \mathscr{A}\colon |\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b). \tag{1.1.29}$$

An extensible positive linear functional is hermitian.

1.1.27 Definition (Strongly positive linear functional [Lac15, p. 101]). Let $\mathscr{A}$ be a $*$-algebra and $\varphi\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a linear functional. The linear functional φ is said to be *strongly positive* if $\varphi^{\mathbb{1}}$ is positive.

1.1.28 Remark. An extensible linear functional is obviously positive. By Definition 1.1.25 any strongly positive linear functional is extensible and therefore positive, i. e., it is an extensible, positive linear functional. By equation (1.1.28) any strongly positive linear functional is hermitian.

1.2 BCH-formula: topological algebra case

In this section we want to gather prerequisites which allow us to formulate the famous Baker–Campbell–Hausdorff formula. Thus, we need to establish some topological notions. The guidance for this section are the statements found in [EG07, Sec. 3.1.2] and [Bou89, Ch. II. § 6.1]. We shall review the main steps and our main tools are provided by [BF12]. In this section we adapt the notation for an algebra $(\mathscr{A}, *)$ with multiplication map $*\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ from [BF12] for compatibility reasons to this book. We emphasize that in [BF12] an algebra does not necessarily need to be associative which is in contrast to our definition

1.2.1 Definition (Convergent sequence in a metric space [Hei11, Ch. 1.2]). Let (X, d) be a metric space and $(x_j)_{j\in\mathbb{N}} \in X^{\mathbb{N}}$. We say

$$\begin{aligned} &(x_j)_{j\in\mathbb{N}} \text{ is } d\text{-convergent in } (X,d) \text{ if and only if} \\ &\qquad \exists a \in X,\ \forall \varepsilon > 0,\ \exists j_\varepsilon \in \mathbb{N},\ \forall j \geq j_\varepsilon\colon d(x_j, a) < \varepsilon. \end{aligned} \tag{1.2.1}$$

The element $a \in X$ is called *limit* of the sequence $(x_j)_{j\in\mathbb{N}}$ and we also say $(x_j)_{j\in\mathbb{N}}$ is *convergent towards a* (in (X, d)), in formal language

$$d\text{-}\lim_{j\to\infty} := a \quad \text{or} \quad (x_j)_{j\in\mathbb{N}} \to_d a. \tag{1.2.2}$$

1.2.2 Definition (Convergent sequence in a topological space [Hei11, Ch. 1.2]). Let (X, τ) be a topological space, $x \in X$ and $U \subseteq X$.

$$U \text{ is } \textit{neighborhood of } x \text{ (w. r. t. } \tau) \text{ if and only if } \exists O \in \tau\colon x \in O \subseteq U. \tag{1.2.3}$$

The set

$$\mathcal{U}_\tau(x) := \{\, U \subseteq X \mid U \text{ is neighborhood of } x \text{ w. r. t. } \tau \,\} \tag{1.2.4}$$

is called *neighborhood system* of x w. r. t. τ. Then, we define for $(x_j)_{j\in\mathbb{N}} \in X^{\mathbb{N}}$

$$\begin{aligned} &(x_j)_{j\in\mathbb{N}} \textit{ convergent} \text{ in } (X,\tau) \text{ if and only if} \\ &\qquad \exists a \in X,\ \forall U \in \mathcal{U}_\tau(a),\ \exists j_U \in \mathbb{N},\ \forall j \geq j_u\colon x_j \in U. \end{aligned} \tag{1.2.5}$$

The element $a \in X$ is called *limit* of the sequence $(x_j)_{j\in\mathbb{N}}$ and we also say $(x_j)_{j\in\mathbb{N}}$ is *convergent towards a* (in (X, τ)), in formal language

$$\tau\text{-}\lim_{j\to\infty} := a \quad \text{or} \quad (x_j)_{j\in\mathbb{N}} \to_\tau a. \tag{1.2.6}$$

1.2.3 Remark. In a metrizable space both meanings of convergence are the same, i. e.,

$$(x_j)_{j\in\mathbb{N}} \to_{\tau_d} a \quad \Longleftrightarrow \quad (x_j)_{j\in\mathbb{N}} \to_d a. \tag{1.2.7}$$

1.2.4 Definition (Topologically admissible family [BF12, Def. 2.57]). Let $(\mathscr{A}, *)$ be an algebra. We say that $(\Omega_k)_{k\in\mathbb{N}_0}$ is a *topologically admissible family* in $\mathscr{A}$ if and only if the sets Ω_k are subsets of $\mathscr{A}$ satisfying the properties

(a) $\forall\, k \in \mathbb{N}_0 \colon \Omega_k$ is an ideal of $\mathscr{A}$,

(b) $\Omega_0 = \mathscr{A}$ and $\forall\, k \in \mathbb{N}_0 \colon \Omega_k \supseteq \Omega_{k+1}$,

(c) $\forall\, h,\, k \in \mathbb{N}_0 \colon \Omega_h * \Omega_k \subseteq \Omega_{h+k}$,

(d) $\bigcap_{k\in\mathbb{N}_0} \Omega_k = \{0\}$.

1.2.5 Theorem ([BF12, Thm. 2.58]). Let $(\mathscr{A}, *)$ be an algebra and suppose that $(\Omega_k)_{k\in\mathbb{N}_0}$ is a topologically admissible family of subsets of $\mathscr{A}$. Then, the family

$$\emptyset \cup \big(a + \Omega_k\big)_{a\in\mathscr{A},k\in\mathbb{N}_0} \tag{1.2.8}$$

is a basis for a topology Ω on $\mathscr{A}$ endowing $\mathscr{A}$ with the structure of a topological algebra. Even more, the topology Ω is induced by the metric $d \colon \mathscr{A} \times \mathscr{A} \longrightarrow [0, \infty)$ defined as follows (we set $\exp(-\infty) := 0$)

$$\forall\, x,\, y \in \mathscr{A} \colon d(x, y) := \exp(-\nu(x - y)), \tag{1.2.9}$$

where

$$\nu \colon \begin{cases} \mathscr{A} \longrightarrow \mathbb{N} \cup \{0, \infty\} \\ z \longmapsto \begin{cases} \max\{\, n \geq 1 \mid z \in \Omega_n \,\} & \text{for } z \neq 0 \\ \infty & \text{for } z = 0. \end{cases} \end{cases} \tag{1.2.10}$$

The triangle inequality for d holds in the stronger form

$$\forall\, x,\, y,\, z \in \mathscr{A} \colon d(x, y) \leq \max\{d(x, z), d(z, y)\}. \tag{1.2.11}$$

1.2.6 Lemma ([BF12, Rem. 2.62]). Let $(\mathscr{A}, *)$ be an algebra and denote by d the metric on $\mathscr{A}$ induced by a topologically admissible family $(\Omega_k)_{k\in\mathbb{N}_0}$. Then, we have

TFAE: **(a)** $(a_n)_{n\in\mathbb{N}_0} \in \mathscr{A}^{\mathbb{N}_0}$ is a Cauchy sequence in $(\mathscr{A}, d)$

(b) $d\text{-}\lim_{n\to\infty}(a_{n+1} - a_n) = 0$

In particular, a series $\sum_{n=0}^{\infty} a_n$ consisting of elements in $\mathscr{A}$ is a Cauchy sequence in $(\mathscr{A}, d)$ if and only if $d\text{-}\lim_{n\to\infty} a_n = 0$.

1.2.7 Lemma ([EG07, Sec. 3.1.2]). Let $(\mathscr{A}, *)$ be a unital algebra and suppose that $(\Omega_k)_{k\in\mathbb{N}_0}$ is a topologically admissible family of subsets of $\mathscr{A}$ such that $\mathbb{1}_{\mathscr{A}} \in \Omega_0 \setminus \Omega_1$. If $\mathscr{A}$ is complete w. r. t. the metric induced by the family $(\Omega_k)_{k\in\mathbb{N}_0}$, then we can set

$$\exp \colon \begin{cases} \Omega_1 \longrightarrow \mathbb{1}_{\mathscr{A}} + \Omega_1 \\ u \longmapsto \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} u^{*k}, \end{cases} \tag{1.2.12a}$$

$$\log_1 : \begin{cases} \Omega_1 \longrightarrow \Omega_1 \\ u \longmapsto \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} u^{*k}. \end{cases} \tag{1.2.12b}$$

PROOF: We have to prove that the above defined maps are well defined. We set

$$\forall u \in \Omega \colon u^{*0} := \mathbb{1}_{\mathscr{A}}.$$

Assume that both series converge. Then, $\forall n \in \mathbb{N}$, $\forall u \in \Omega_1 \colon u^{*n} \in \Omega_n$ by Definition 1.2.4 **(b)** and **(c)**. Therefore, $\exp(u) \in \mathbb{1}_{\mathscr{A}} + \Omega_1$ and $\log_1(u) \in \Omega_1$ for any $u \in \Omega_1$, since Ω_1 is an ideal in $\mathscr{A}$ and thus a linear subspace of $\mathscr{A}$.

Now, let us show that $\exp(u)$ converges for any $u \in \Omega_1$ with respect to the metric d induced by $(\Omega_k)_{k\in\mathbb{N}_0}$. We define for any $N \in \mathbb{N}$

$$S_N := \sum_{k=0}^{N} \frac{1}{k!} u^{*k} \in \Omega_1.$$

We need to show that $d\text{-}\lim_{N\to\infty} S_N$ exists in Ω_1. We shall show that $(S_N)_{n\geq 1}$ is a Cauchy-sequence with respect to d. From Lemma 1.2.6 we can see that this equivalent to show that $d\text{-}\lim_{N\to\infty} \frac{1}{n!}u^{*n} = 0$. According to Theorem 1.2.5 $\{\emptyset\} \cup \{\, \Omega_k \mid k \in \mathbb{N}_0 \,\}$ is a neighborhood basis for $0 \in \mathscr{A}$. By Remark 1.2.3 and equation (1.2.5) we need to show

$$\forall k \in \mathbb{N},\ \exists j_k \in \mathbb{N},\ \forall j \geq j_k \colon \frac{1}{j!} u^{*j} \in \Omega_k, \tag{I}$$

in order to conclude that $(S_N)_{n\geq 1}$ is a Cauchy-sequence with respect to d. Let $k \in \mathbb{N}_0$. Since $u \in \Omega_1$, we have $u^{*(k+n)} \in (\Omega_1)^{(k+n)} \subseteq \Omega_k$ for all $n \in \mathbb{N}_0$ by Definition 1.2.4 **(c)**. By Definition 1.2.4 **(b)** we have $\Omega_{k+n} \subseteq \Omega_k$ for all $n \in \mathbb{N}_0$ and therefore we have shown equation (I), if we set $j_k := k$. Since $(\mathscr{A}, d)$ is assumed to be complete and we have shown that $(S_N)_{n\geq 1}$ is a Cauchy-sequence with respect to d, we can conclude that $d\text{-}\lim_{N\to\infty} S_N$ exists in Ω_1. This is what we needed to show.

An analogous proof holds for the convergence of $\log_1(u)$ for any $u \in \Omega_1$. □

1.2.8 Remark. We notice that [BF12, Lem. 3.3] provide a similar result to the following Lemma 1.2.7 under the assumptions that the algebra $\mathscr{A}$ is graded which we do not demand. We will somehow imitate the proof of [BF12, Lem. 3.3], but adapted to our situation where $(\mathscr{A}, d)$ is complete and thus there is no need to consider the metric completion.

1.2.9 Lemma. By assumptions and definitions from Lemma 1.2.7, the functions $\exp, \log_1$ are continuous on their corresponding domains.

PROOF: To prove the assertion we use the following standard fact from analysis, known as the uniform limit theorem, which states that the uniform limit of any sequence of continuous functions is continuous. We set

$$\forall n \in \mathbb{N}_0 \colon f_n \colon \begin{cases} \Omega_1 \longrightarrow \mathbb{1}_{\mathscr{A}} + \Omega_1 \\ u \longmapsto \displaystyle\sum_{k=0}^{n} \frac{1}{k!} (u)^{*k}. \end{cases}$$

All functions f_n are continuous, because they are compositions of continuous maps in the topological algebra $\mathscr{A}$. We are left to show that $(f_n)_{n\in\mathbb{N}_0}$ uniformly converges to exp. Since

$(\mathscr{A}, d)$ is complete, using Cauchy's criteria of convergence we can say that $(f_n)_{n\in\mathbb{N}_0}$ is uniformly convergent on Ω_1 with limit exp if and only if $\forall\, \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$\sup_{u\in\Omega_1} d(f_n(u), f_m(u)) < \varepsilon \quad \text{for all } n, m \geq N_\varepsilon. \tag{I}$$

Now, we can calculate

$$\begin{aligned}
&\sup_{u\in\Omega_1} d(f_n(u), f_m(u))\\
&= \sup_{u\in\Omega_1} d\left(\sum_{k=0}^{n} \frac{1}{k!} u^{*k}, \sum_{k=0}^{m} \frac{1}{k!} u^{*k}\right)\\
&= \sup_{u\in\Omega_1} d\left(\sum_{k=n}^{n+m} \frac{1}{k!} u^{*k}, 0\right) \quad [\![\, \forall\, x, y, z \in \mathscr{A} : d(x, y) = d(x+z, y+z) \text{ by [BF12, Rem. 2.61]} \,]\!]\\
&= \sup_{u\in\Omega_1} \exp\left(-\max\left\{ \ell \geq 1 \,\middle|\, \sum_{k=n}^{n+m} \frac{1}{k!} u^{*k} \in \Omega_\ell \right\}\right) \quad [\![\, \text{def. of } d(\,\cdot\,,\,\cdot\,) \text{ in eq. (1.2.9)} \,]\!]\\
&= \sup_{u\in\Omega_1} \exp(-n) \quad [\![\, \text{Definition 1.2.4 \textbf{(b)} and \textbf{(c)}} \,]\!]\\
&= \exp(-n) \longrightarrow 0 \text{ for } n \longrightarrow \infty.
\end{aligned}$$

Thus, we have shown equation (I). A similar proof holds for $\log_1$. □

1.2.10 Definition. By assumptions from Lemma 1.2.7 and by the definition of the map $\log_1 : \Omega_1 \longrightarrow \Omega_1$, we define

$$\log : \begin{cases} \mathbb{1}_{\mathscr{A}} + \Omega_1 \longrightarrow \Omega_1 \\ \tilde{u} \longmapsto \log_1(\tilde{u} - \mathbb{1}_{\mathscr{A}}). \end{cases} \tag{1.2.13}$$

Although, the following two propositions get cited by [BF12], we mention that they have different prerequisites in its original source, in the sense of Remark 1.2.8. But the proofs still remain formally the same when we replace [BF12, Def. 3.2] by our Lemma 1.2.7.

1.2.11 Proposition ([BF12, Prop. 3.4]). The functions exp and log introduced in Lemma 1.2.7 are inverse to each other, so that

$$\exp(\log(\mathbb{1}_{\mathscr{A}} + u)) = \mathbb{1}_{\mathscr{A}} + u, \qquad \log(\exp(u)) = u \qquad \text{for every } u \in \Omega_1. \tag{1.2.14}$$

1.2.12 Proposition ([BF12, Thm. 3.6]). Let $(\mathscr{A}, *_{\mathscr{A}})$ and $(\mathscr{B}, *_{\mathscr{B}})$ be two unital algebras which posses a topologically admissible family, $(\Omega_n)_{n\in\mathbb{N}_0}$ respectively $(\tilde{\Omega}_n)_{n\in\mathbb{N}_0}$, and the algebras $\mathscr{A}, \mathscr{B}$ are assumed to be complete regarding the metric induced by $(\Omega_n)_{n\in\mathbb{N}_0}$ resp. $(\tilde{\Omega}_n)_{n\in\mathbb{N}_0}$. Furthermore, assume $\mathbb{1}_{\mathscr{A}} \in \Omega_0 \setminus \Omega_1$ and $\mathbb{1}_{\mathscr{B}} \in \tilde{\Omega}_0 \setminus \tilde{\Omega}_1$. Let $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$ be a continuous unital algebra homomorphism such that

$$\varphi(\Omega_1) \subseteq \tilde{\Omega}_1. \tag{1.2.15}$$

Then, we have

$$\varphi \circ \exp_{*_{\mathscr{A}}} = \exp_{*_{\mathscr{B}}} \circ\, \varphi \restriction_{\Omega_1} \tag{1.2.16a}$$

$$\varphi \circ \log_{*_{\mathscr{A}}} = \log_{*_{\mathscr{B}}} \circ\, \varphi \restriction_{\mathbb{1}_{\mathscr{A}}+\Omega_1} \tag{1.2.16b}$$

We now give a statement similar to [Bou89, Prop. 1 in Ch. II. § 5.1] which is less general in our version, but more suited for our needs.

1.2.13 Proposition. Let $\mathscr{A}$ be a unital algebra which possesses a topologically admissible family $(\mathcal{T}_n)_{n\in\mathbb{N}_0}$ and which is complete w. r. t. the induced metric $d_{\mathscr{A}}$ from Theorem 1.2.5. Let $f: X \longrightarrow \mathscr{A}$ be a map such that there exists some $\ell \in \mathbb{N}$ for which

$$f(X) \subseteq \mathcal{T}_\ell. \tag{1.2.17}$$

Then, there exists a unique continuous unital homomorphism $\hat{f}: \hat{\mathrm{T}}_0(\mathbb{C}X) \longrightarrow \mathscr{A}$ such that $\hat{f}\upharpoonright_X = f$, whereby $\mathbb{C}X$ denotes the free vector space over X, $\mathrm{T}_0(\mathbb{C}X)$ is the unital tensor algebra of $\mathbb{C}X$ and $\hat{\mathrm{T}}_0(\mathbb{C}X)$ denotes its isometric completion of the metric space $\mathrm{T}_0(\mathbb{C}X)$ as a naturally graded algebra.

Proof: By the UMP of the tensor algebra there exists a unique unital algebra homomorphism

$$f'\colon\ \mathrm{T}_0(\mathbb{C}X) \longrightarrow \mathscr{A}$$

such that $f'\upharpoonright_X = f$. We need to show that f' is continuous. Since the tensor algebra $\mathrm{T}_0(\mathbb{C}X) = \bigoplus_{n\in\mathbb{N}_0}(\mathbb{C}X)^{\otimes n}$ is naturally graded by $((\mathbb{C}X)^{\otimes n})_{n\in\mathbb{N}_0}$, we can regard this algebra as a topological one ⟦ [BF12, Prop. 2.65] ⟧. In particular a topologically admissible family $(\Omega_k)_{k\in\mathbb{N}_0}$ is given by

$$\Omega_k := \bigoplus_{j\geq k}(\mathbb{C}X)^{\otimes j} \tag{I}$$

⟦ [BF12, Exa. 2.64, 3.] ⟧. The isometric completion of $\mathrm{T}_0(\mathbb{C}X)$ is given by the space of formal power series on $\mathrm{T}_0(\mathbb{C}X)$, i. e., $\hat{\mathrm{T}}_0(\mathbb{C}X) \cong \prod_{j\in\mathbb{N}_0}(\mathbb{C}X)^{\otimes j}$ ⟦ [BF12, Thm. 2.75] ⟧. A topologically admissible family $(\hat{\Omega}_k)_{k\in\mathbb{N}_0}$ on $\hat{\mathrm{T}}(\mathbb{C}X)$ is given by $\hat{\Omega}_k := \prod_{j\geq k}(\mathbb{C}X)^{\otimes j}$ ⟦ [BF12, Rem. 2.74] ⟧. We note that

$$\forall\, k \in \mathbb{N}_0\colon \Omega_k = \hat{\Omega}_k \cap \mathrm{T}(\mathbb{C}X) \tag{II}$$

whence the subspace topology on $\mathrm{T}(\mathbb{C}X)$ coincides with the one induced by $(\Omega_k)_{k\in\mathbb{N}_0}$. By the fact that the tensor algebra $\mathrm{T}_0(\mathbb{C}X)$ as an algebra is generated by elements of $\mathbb{C}X$, the map $f'\colon\ \mathrm{T}_0(\mathbb{C}X) \longrightarrow \mathscr{A}$ is a homomorphism of unital algebras and from the fact that $(\mathcal{T}_n)_{n\in\mathbb{N}_0}$ is a topologically admissible family for the algebra $\mathscr{A}$ we obtain

$$\forall\, k \in \mathbb{N}_0\colon f'\big((\mathbb{C}X)^{\otimes k}\big) \subseteq \mathcal{T}_{k\ell}.$$

By linearity of f', property of Definition 1.2.4 **(b)** and equation (I), we can conclude from the above equation that

$$f'(\Omega_k) \subseteq \mathcal{T}_{k\ell} \quad \text{for all } k \in \mathbb{N}_0. \tag{III}$$

By equation (II) this is equivalent to $f'(\hat{\Omega}_k \cap \mathrm{T}_0(\mathbb{C}X)) \subseteq \mathcal{T}_{k\ell}$ for all $k \in \mathbb{N}_0$. Hence, we have shown that the map f' is continuous w. r. t. the subspace topology for $\mathrm{T}(\mathbb{C}X)$ as a dense subset of $\hat{\mathrm{T}}(\mathbb{C}X)$.

Next, we show that $f'\colon\ \mathrm{T}(\mathbb{C}X) \longrightarrow \mathscr{A}$ maps Cauchy sequences to Cauchy sequences. Assume $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathrm{T}(\mathbb{C}X), d)$. We show that $(f'(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathscr{A}, d_{\mathscr{A}})$ and according to Lemma 1.2.6 this is equivalent to $d_{\mathscr{A}}\text{-}\lim_{n\to\infty}(f'(a_{n+1}) - f'(a_n)) = 0$, i. e., we have to show that

$$\forall\, k \in \mathbb{N},\ \exists\, j_k \in \mathbb{N},\ \forall\, j \geq j_k\colon f'(a_{j+1}) - f'(a_j) \in \mathcal{T}_k. \tag{IV}$$

Since $(a_n)_{n\in\mathbb{N}}$ is assumed to be a Cauchy sequence, we know that

$$\forall \tilde{k} \in \mathbb{N},\ \exists \tilde{j}_k \in \mathbb{N},\ \forall j \geq \tilde{j}_k : a_{j+1} - a_j \in \Omega_{\tilde{k}}.$$

By equation (III) we obtain

$$\forall \tilde{k} \in \mathbb{N}\ \exists \tilde{j}_k \in \mathbb{N},\ \forall j \geq \tilde{j}_k : f'(a_{j+1}) - f'(a_j) \in \mathcal{T}_{\tilde{k}\ell}.$$

The property $\mathcal{T}_{\tilde{k}\ell+n} \subseteq \mathcal{T}_{\tilde{k}\ell}$ for all $\tilde{k}, n \in \mathbb{N}$ shows equation (IV). Considering the proof of the well known extension theorem of a uniform continuous function on a dense subset ([AE08, Thm. 2.1]), we can see that the continuity of f' on $\mathrm{T}(\mathbb{C}X)$) and the property that f' maps Cauchy sequences to Cauchy sequences suffices to conclude that there exists a unique continuous extension $\hat{f}\colon \hat{\mathrm{T}}_0(\mathbb{C}X) \longrightarrow \mathscr{A}$ of $f'\colon\ \mathrm{T}_0(\mathbb{C}X) \longrightarrow \mathscr{A}$. □

1.2.14 Convention (Right-nested brackets [BF12, p. 125]). Let $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}})$ be a Lie algebra and $u, v \in \mathfrak{g}$. According to [BF12, p. 125] we introduce the following notation for so-called *right-nested* brackets of u and v (recall that $(\operatorname{ad} u)(v) = [u, v]_{\mathfrak{g}}$)

$$\forall n \in \mathbb{N},\ \forall (h_i)_{i\in[n]} \in \mathbb{N}_0^{\times n},\ \forall (k_i)_{i\in[n]} \in \mathbb{N}_0^{\times n}:$$
$$[u^{h_1} v^{k_1} \dots u^{h_n} v^{k_n}]_{\mathfrak{g}} := (\operatorname{ad} u)^{h_1} \circ (\operatorname{ad} v)^{k_1} \circ \cdots \circ (\operatorname{ad} u)^{h_n} \circ (\operatorname{ad} v)^{k_n - 1}(v). \tag{1.2.18}$$

Indeed, another visualization of equation (1.2.18) is given by

$$[u^{h_1} v^{k_1} \dots u^{h_n} v^{k_n}]_{\mathfrak{g}} = [\underbrace{u, \dots [u,}_{h_1 \text{ times}} [\underbrace{v, \dots [v,}_{k_1 \text{ times}} \dots [\underbrace{u, \dots [u,}_{k_n \text{ times}} [\underbrace{v, \dots v}_{k_1 \text{ times}}]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} \tag{1.2.19}$$

Let V be a vector space. Let $\hat{\mathrm{T}}(V) = \prod_{k=0}^{\infty} V^{\otimes k}$ be the usual completion of the tensor algebra $\mathrm{T}_0(V)$ (i. e., $\hat{\mathrm{T}}_0(V)$ is the algebra of the formal power series of the tensor algebra of V). We denote the multiplication of the tensor algebra $\mathrm{T}_0(V)$ by a dot $\cdot$. For the completion $\hat{\mathrm{T}}_0(V)$ we put $\hat{\cdot}$ as the multiplication in this algebra. By $[\,\cdot\,,\,\cdot\,]_{\hat{\cdot}}$ we mean the Lie bracket canonically induced by $\hat{\cdot}$. Similar to [BF12, eq. (3.15)] we define for all $u, v \in \hat{\mathrm{T}}_+(V) := \prod_{k=1}^{\infty} V^{\otimes k}$

$$\mathsf{BCH}_j(u, v) = \sum_{n=1}^{j} \frac{(-1)^{n+1}}{n} \sum_{\substack{(h_1,k_1),\dots,(h_n,k_n)\neq 0 \\ h_1+k_1+\dots+h_n+k_n=j}} \frac{\left[u^{h_1} v^{k_1} \dots u^{h_n} v^{k_n}\right]_{\hat{\cdot}}}{h_1! \dots h_n!\, k_1! \dots k_n! (\sum_{i=1}^{n} (h_i + k_i))}. \tag{1.2.20}$$

Herein, $[u^{h_1} \dots v^{k_n}]_{\hat{\cdot}}$ is a right-nested bracket in the Lie algebra associated to $\hat{\mathrm{T}}_0(V)$. Then, set

$$\mathsf{BCH}_{\hat{\cdot}}(u, v) := \sum_{j=1}^{\infty} \mathsf{BCH}_j(u, v). \tag{1.2.21}$$

The expression $\mathsf{BCH}_{\hat{\cdot}}(u, v)$ appears in the formulation of the famous Campbell-Baker-Hausdorff-Dynkin theorem according to [BF12, Thm. 3.8], where in particular it is proven that equation (1.2.21) actually makes sense. The next theorem makes use of the BCH-formula stated in [BF12, Thm. 3.8], describes substitutions in the BCH-series and is inspired by the statement of [Bou89, Prop. 3 in Ch. II. §6.5], but once again less general and more suited to our needs.

1.2.15 Theorem (Campbell-Baker-Hausdorff-Dynkin [Bou89, Prop. 3 in Ch. II. §6.5]). Given an unital algebra $(\mathscr{A}, *)$ with a topologically admissible family $(\Omega_n)_{n\in\mathbb{N}_0}$, where $\mathscr{A}$ is assumed to be complete w. r. t. the induced metric and $\mathbb{1}_{\mathscr{A}} \in \Omega_0 \setminus \Omega_1$. For arbitrary elements $u, v \in \Omega_1$ holds the so-called Campbell-Baker-Hausdorff-Dynkin formula (abbreviated by BCH-formula)

$$\exp_*(u) * \exp_*(v) = \exp_*(\mathsf{BCH}_*(u, v)), \tag{1.2.22}$$

where the definition of $\mathsf{BCH}_*(\,\cdot\,,\,\cdot\,)$ is to be understood in the sense of equations (1.2.20) and (1.2.21), i. e., as a series of right nested brackets $[u^{h_1} \dots v^{k_n}]_*$ with Lie bracket $[\,\cdot\,,\,\cdot\,]_*$ canonically induced by $*$.

Proof: Assume $u, v \in \Omega_1$. Let $X := \{x, y\}$ be a set and define a map $f\colon X \longrightarrow \mathscr{A}$ by $x \longmapsto u$ and $y \longmapsto v$. Then the assertion of Proposition 1.2.13 applies and we obtain the existence of a continuous unital algebra homomorphism $\hat{f}\colon \hat{\mathrm{T}}_0(\mathbb{C}X) \longrightarrow \mathscr{A}$, where $\hat{\mathrm{T}}_0(\mathbb{C}X)$ denotes the isometric completion of the metric space $\mathrm{T}_0(\mathbb{C}X)$ as a naturally graded algebra. According to [BF12, Thm. 2.69] this completion can be equipped with a structure of a unital algebra $(\hat{\mathrm{T}}_0(\mathbb{C}X), \hat{\cdot})$ which is also a topological algebra containing a dense subalgebra isomorphic to $\mathrm{T}_0(\mathbb{C}X)$. Now, we can calculate

$$\begin{aligned}
&\exp_*(u) * \exp_*(v)\\
&\quad= \exp_*\big(\hat{f}(u)\big) * \exp_*\big(\hat{f}(v)\big)\big)\\
&\quad= \hat{f}\big(\exp_{\hat{\cdot}}(x) \mathbin{\hat{\cdot}} \exp_{\hat{\cdot}}(y)\big) \quad [\![\text{ Proposition 1.2.13 and 1.2.12 }]\!]\\
&\quad= \hat{f}\big(\exp_{\hat{\cdot}}\big(\mathsf{BCH}_{\hat{\cdot}}(x, y)\big)\big)\\
&\qquad\quad [\![\text{ BCH-formula stated in [BF12, Thm. 3.8] for } (\hat{\mathrm{T}}_0(\mathbb{C}\langle X\rangle), \hat{\cdot})]\!]\\
&\quad= \exp_*\Big(\hat{f}\big(\mathsf{BCH}_{\hat{\cdot}}(x, y)\big)\Big) \quad [\![\text{ Proposition 1.2.13 }]\!]\\
&\quad= \exp_*\big(\mathsf{BCH}_*(u, v)\big) \quad [\![\ \hat{f} \text{ is unital algebra hom. and def. of } \mathsf{BCH}_{\hat{\cdot}}(\,\cdot\,,\,\cdot\,)]\!].
\end{aligned}$$

This proves the statement of equation (1.2.22) since $u, v \in \Omega_1$ were arbitrarily chosen. □

1.3 BCH-formula: convolution algebra case

In this section we want to review the framework which allows to us to formulate a BCH-formula on the convolution algebra. For this purpose we continue to follow the path presented in [EG07, Sec. 3.3] or in [EM09, Sec. 2.3].

1.3.1 Definition ([Grin21, Def. 3.2]). Let $\mathscr{A}$ be a unital algebra and let $\mathscr{H}$ be a connected filtered bialgebra.

(a) For every $n \in \mathbb{N}_0$ we denote by

$$\mathrm{Lin}^n(\mathscr{H}, \mathscr{A}) := \{\, f \in \mathrm{Lin}(\mathscr{H}, \mathscr{A}) \mid f\restriction_{\mathscr{H}_{\leq n-1}} = 0 \,\} \subseteq \mathrm{Lin}(\mathscr{H}, \mathscr{A}). \tag{1.3.1}$$

(b) Let $\mathbb{1}_{\mathscr{H}}$ denote the unity of the underlying algebra structure of $\mathscr{H}$ then we put

$$\mathfrak{g}(\mathscr{H}, \mathscr{A}) := \{\, f \in \mathrm{Lin}(\mathscr{H}, \mathscr{A}) \mid f(\mathbb{1}_{\mathscr{H}}) = 0 \,\} \subseteq \mathrm{Lin}(\mathscr{H}, \mathscr{A}). \tag{1.3.2}$$

(c) Let $\mathbb{1}_{\mathscr{A}}$ denote the unity of the unital algebra $\mathscr{A}$ then we put

$$G(\mathscr{H}, \mathscr{A}) := \{ f \in \operatorname{Lin}(\mathscr{H}, \mathscr{A}) \mid f(1_{\mathscr{H}}) = 1_{\mathscr{A}}) \}. \tag{1.3.3}$$

1.3.2 Lemma ([Grin21, Def. 3.1 (b)]). Let $\mathscr{A}$ be a unital algebra and let $\mathscr{H}$ be a connected filtered bialgebra. Then,

$$\operatorname{Lin}^1(\mathscr{H}, \mathscr{A}) = \mathfrak{g}(\mathscr{H}, \mathscr{A}) \tag{1.3.4}$$

1.3.3 Lemma ([Grin21, Rem. 3.3, Rem 3.5]).

(a) Let $\mathscr{A}$ be a unital algebra and let H be a connected filtered bialgebra. Recall from Convention 1.1.9 that $e_{\mathscr{H},\mathscr{A}}$ denotes the map $\eta_{\mathscr{A}} \circ \varepsilon_{\mathscr{H}} \colon \mathscr{H} \longrightarrow \mathscr{A}$, then

- $e_{\mathscr{H},\mathscr{A}} \in G(\mathscr{H}, \mathscr{A})$.
- $G(\mathscr{H}, \mathscr{A}) = e_{\mathscr{H},\mathscr{A}} + \mathfrak{g}(\mathscr{H}, \mathscr{A})$.

(b) $\forall f \in \mathfrak{g}(\mathscr{H}, \mathscr{A}),\ \forall n \in \mathbb{N}_0,\ \forall i \in \mathbb{N}_0 \colon (i > n \implies f^{\star i}\restriction_{\mathscr{H}_{\leq n}} = 0)$.

In particular the statement of Lemma 1.3.3 **(b)** allows a direct definition of the well-known *exponential* $\exp_\star f$ and *logarithm* $\log_\star f$ for $f \in \mathfrak{g}(\mathscr{H}, \mathscr{A})$ as is done in [Grin21, Def. 3.6, Def. 3.7]. In each point of the connected filtered bialgebra $\mathscr{H}$ the series for $\exp_\star f$ and $\log_\star f$ turn out to be actually finite sums. But we want more than that and we want to obtain a BCH-formula for the convolution algebra. Therefore, we shall show that we can establish the prerequisite of Theorem 1.2.15 in the case of the convolution algebra $\operatorname{Lin}(\mathscr{H}, \mathscr{A})$. The next lemma builds the foundation for this approach which somehow resembles the statement of [EM09, Prop. 2.5].

1.3.4 Remark. There exists a different approach for the "convolution exponential" if we replace the algebra $\mathscr{A}$ by the complex numbers $\mathbb{C}$ and replace the connected filtered bialgebra $\mathscr{B}$ by a coalgebra $\mathscr{C}$ in Proposition 1.3.6. In order to define a convolution exponential on $\operatorname{Lin}(\mathscr{C}, \mathbb{C})$ we can use the fundamental theorem of coalgebras and show convergence of the series from equation (1.3.9a). This has been done in [Sch93, Sec. 1.7]. Here, we follow [SSV10, Sec. 4], [Voß13, Satz 3.3.1] and [Ger21, Lem. 2.1]. Let $(\mathscr{C}, \Delta, \varepsilon)$ be a coalgebra and let $\varphi \in \operatorname{Lin}(\mathscr{C}, \mathbb{C})$. The map $T\colon \operatorname{Lin}(\mathscr{C}, \mathbb{C}) \longrightarrow \operatorname{Lin}(C, C), \psi \longmapsto (\mathsf{id}_{\mathscr{C}} \otimes \psi) \circ \Delta = \mathsf{id}_{\mathscr{C}} \star \psi$ defines an injective unital algebra homomorphism with $\varepsilon \circ T = \mathsf{id}$. Moreover, each $T(\psi)$ leaves every subcoalgebra of $\mathscr{C}$ invariant. On an arbitrary finite-dimensional subcoalgebra $\mathscr{C}_c \ni c$ of $\mathscr{C}$ the series

$$\exp\big(T(\varphi)\big)\restriction_{\mathscr{C}_c} := \sum_{n=0}^{\infty} \frac{T(\phi)\restriction_{\mathscr{C}_c}}{n!} \tag{1.3.5}$$

converges in any norm. The fundamental theorem of coalgebras yields that for every $c \in \mathscr{C}$ there exists a subcoalgebra $\mathscr{D} \subseteq \mathscr{C}$ such that $c \in \mathscr{D}$ and $\dim \mathscr{D} < \infty$ ([DNR01, Thm. 1.4.7]). We can deduce that the series

$$\big(\exp_\star \varphi\big)(c) := \sum_{n=0}^{\infty} \frac{\varphi^{\star n}}{n!} = \varepsilon \circ \exp\big(T(\varphi)\big)(c) \tag{1.3.6}$$

converges for all $\varphi \in \operatorname{Lin}(\mathscr{C}, \mathbb{C})$ and all $c \in \mathscr{C}$. This limit of complex numbers does not depend on the choice of $\mathscr{C}_c$. For two elements $\varphi_1,\ \varphi_2 \in \operatorname{Lin}(\mathscr{C}, \mathbb{C})$ it can be shown that

$$\big(\exp_\star \varphi_1\big) \star \big(\exp_\star \varphi_2\big) = \exp_\star(\varphi_1 + \varphi_2). \tag{1.3.7}$$

Furthermore, there is another equivalent characterization of the convolution exponential of equation (1.3.6) given by the following formula

$$\big(\exp_\star \varphi\big)(c) = \lim_{n\to\infty}\Big(\varepsilon + \frac{\varphi}{n}\Big)^{\star n}(c). \tag{1.3.8}$$

For a proof of this statement we refer to [SV14, Lem. 3.1].
We do not use this approach since we want certain topological features which can be "easily" derived from the used approach. One of our goals is to find a moment-cumulant formula for a given u.a.u.-product (Proposition 2.4.12). Such a realization as a moment-cumulant formula relies on a BCH-formula for the convolution algebra $\mathrm{Lin}(\mathscr{B}, \mathbb{C})$, where $\mathscr{B}$ is a certain connected filtered bialgebra. Thus, we are seeking for a framework which already suits the needs to formulate a "BCH-like formula". This framework, described by the prerequisites of Theorem 1.2.15, seems to be appropriate. The statement of Theorem 1.3.10 is then a special case of Theorem 1.2.15.

1.3.5 Lemma. Let $\mathscr{A}$ be a unital algebra and $(\mathscr{H}, (\mathscr{H}_{\le\ell})_{\ell\in\mathbb{N}_0})$ a connected filtered $\mathbb{K}$-bialgebra. Let $\mathrm{Lin}^n(\mathscr{H},\mathscr{A})$ denote the subspace of $\mathrm{Lin}(\mathscr{H},\mathscr{A})$ defined in equation (1.3.1). Then, $\mathrm{Lin}^0(\mathscr{H},\mathscr{A}) = \mathrm{Lin}(\mathscr{H},\mathscr{A})$ and moreover the family $(\mathrm{Lin}^n(\mathscr{H},\mathscr{A}))_{n\in\mathbb{N}_0}$ is a topologically admissible family in $(\mathrm{Lin}(\mathscr{H},\mathscr{A}),\star)$.

Proof: Obviously, every $f \in \mathrm{Lin}^0(\mathscr{H},\mathscr{A})$ satisfies $f\restriction_{\mathscr{H}_{0-1}} = 0$, since $\mathscr{H}_{0-1} = \mathscr{H}_{-1} = \{0\}$ (Conv. 1.1.13) and therefore $\mathrm{Lin}^0(\mathscr{H},\mathscr{A}) = \mathrm{Lin}(\mathscr{H},\mathscr{A})$. For the other part we need to prove the properties from Definition 1.2.4 **(a)–(d)**.
Ad **(a)**: We show that $\mathrm{Lin}^n(\mathscr{H},\mathscr{A})$ is a vector subspace of $\mathrm{Lin}(\mathscr{H},\mathscr{A})$. Let $\alpha,\beta\in\mathbb{K}$ and $g,h \in \mathrm{Lin}^n(\mathscr{H},\mathscr{A})$. The assumption $g\in\mathrm{Lin}^n(\mathscr{H},\mathscr{A})$ is equivalent to $g\restriction_{\mathscr{H}_{\le n-1}} = 0$ and analogously for $h \in \mathrm{Lin}(\mathscr{H},\mathscr{A})$. Now

$$(\alpha g + \beta h)\restriction_{\mathscr{H}_{\le n-1}} = \alpha \underbrace{g\restriction_{\mathscr{H}_{\le n-1}}}_{=0} + \beta \underbrace{h\restriction_{\mathscr{H}_{\le n-1}}}_{=0} = \alpha 0 + \beta 0 = 0,$$

hence we obtain $\alpha g + \beta h \in \{\, f \in \mathrm{Lin}(\mathscr{H},\mathscr{A}) \mid f\restriction_{\mathscr{H}_{\le n-1}} = 0\,\} = \mathrm{Lin}^n(\mathscr{H},\mathscr{A})$. This proves that $\mathrm{Lin}^n(\mathscr{H},\mathscr{A})$ is a vector subspace of $\mathrm{Lin}(\mathscr{H},\mathscr{A})$. In [Grin21, Prop. 14.2] it has been proven that $\mathrm{Lin}^n(\mathscr{H},\mathscr{A})$ is an ideal of $\mathrm{Lin}(\mathscr{H},\mathscr{A})$.
Ad **(b)**: We have already shown that $\mathrm{Lin}^0(\mathscr{H},\mathscr{A}) = \mathrm{Lin}(\mathscr{H},\mathscr{A})$. It is true that $\mathrm{Lin}^n(\mathscr{H},\mathscr{A}) \supseteq \mathrm{Lin}^{n+1}(\mathscr{H},\mathscr{A})$ for every $n\in\mathbb{N}_0$. Because, for $g\in\mathrm{Lin}^n(\mathscr{H},\mathscr{A}) = \{\, f\in\mathrm{Lin}(\mathscr{H},\mathscr{A}) \mid f\restriction_{\mathscr{H}_{\le n-1}} = 0\,\}$, we have $g(\mathscr{H}_{\le n-1}) = 0$. Since $(\mathscr{H}_{\le n})_{n\in\mathbb{N}_0}$ is a filtration for $\mathscr{H}$, this implies in particular $\mathscr{H}_{\le n} \subseteq \mathscr{H}_{\le n+1}$ for every $n\in\mathbb{N}_0$. Therefore, $g(\mathscr{H}_{\le n-1}) \subseteq g(\mathscr{H}_{\le n}) = 0$.
Ad **(c)**: We need to show that every $(f,g) \in \big(\mathrm{Lin}^h(\mathscr{H},\mathscr{A}) \times \mathrm{Lin}^k(\mathscr{H},\mathscr{A})\big)$ satisfies $f\star g \in \mathrm{Lin}^{h+k}(\mathscr{H},\mathscr{A})$. This has been proven in [Grin21, Prop. 21.4 (a)].
Ad **(d)**: Obviously $0 \in \mathrm{Lin}(\mathscr{H},\mathscr{A})$. On the other hand $f \in \bigcap_{n\in\mathbb{N}_0}\mathrm{Lin}^n(\mathscr{H},\mathscr{A})$ is equivalent to $f\restriction_{\mathscr{H}_{\le n-1}} = 0$ for all $n\in\mathbb{N}_0$. Let $x\in\mathscr{H}$, then there exists some $n\in\mathbb{N}_0$ such that $x\in\mathscr{H}_{\le n}$, since $(\mathscr{H}_{\le n})_{n\in\mathbb{N}_0}$ is a filtration for $\mathscr{H}$. Then, $f(x) = f\restriction_{\mathscr{H}_{\le n}}(x) = 0(x) = 0$ and therefore $f = 0$. □

Once again, we formulate a statement which resembles statements of [EM09, Prop. 2.5] (in particular Proposition 1.3.6 **(b)**).

1.3.6 Proposition. Let $\mathscr{A}$ be a unital algebra and $(\mathscr{H}, (\mathscr{H}_{\le\ell})_{\ell\in\mathbb{N}_0})$ a connected filtered bialgebra.

(a) The family $(\mathrm{Lin}^n(\mathscr{H},\mathscr{A}))_{n\in\mathbb{N}_0}$ defined in equation (1.3.1) induces a metrizable topology on $\mathrm{Lin}(\mathscr{H},\mathscr{A})$ with metric d in the sense of Theorem 1.2.5 and thus making $(\mathrm{Lin}(\mathscr{H},\mathscr{A}),\star)$

a topological algebra.

(b) The topological space $(\operatorname{Lin}(\mathscr{H}, \mathscr{A}), \tau_d)$ endowed with the metrizable topology τ_d defined in equation (1.2.8) is complete.

(c) The mappings

$$\exp_\star : \begin{cases} \mathfrak{g}(\mathscr{H}, \mathscr{A}) \longrightarrow e_{\mathscr{H},\mathscr{A}} + \mathfrak{g}(\mathscr{H}, \mathscr{A}) \\ \ni u \longmapsto \sum_{k=0}^{\infty} \frac{1}{k!} u^{\star k}, \end{cases} \tag{1.3.9a}$$

$$(\log_\star)_1 : \begin{cases} \mathfrak{g}(\mathscr{H}, \mathscr{A}) \longrightarrow \mathfrak{g}(\mathscr{H}, \mathscr{A}) \\ u \longmapsto \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} u^{\star k} \end{cases} \tag{1.3.9b}$$

are well defined and continuous on their domains.

PROOF: AD **(a)**: By Lemma 1.3.5 the family $(\operatorname{Lin}^n(\mathscr{H}, \mathscr{A}))_{n \in \mathbb{N}_0}$ is a topologically admissible family in $(\operatorname{Lin}(\mathscr{H}, \mathscr{A}), \star)$. The assertion follows, if we apply Theorem 1.2.5 to this family of subsets in $\operatorname{Lin}(\mathscr{H}, \mathscr{A})$.

AD **(b)**: A proof of this assertion has been given in [EM09, Prop. 2.5]. We want to give here more details of the proof. First of all, let us define a shorthand notation and set for all $n \in \mathbb{N}_0$

$$\Omega_n := \operatorname{Lin}^n(\mathscr{H}, \mathscr{A}). \tag{I}$$

If $(\psi_n)_{n \in \mathbb{N}_0}$ is a Cauchy sequence, then due to Lemma 1.2.6 this is equivalent to $(\psi_{n+1} - \psi_n)_{n \in \mathbb{N}_0} \to_{\tau_d} 0$. From Theorem 1.2.5 we know that $(\Omega_n)_{n \in \mathbb{N}_0}$ is a neighborhood basis of $0 \in \operatorname{Lin}(\mathscr{H}, \mathscr{A})$ and by equation (1.2.7) we can calculate

$$\begin{aligned} &(\psi_{n+1} - \psi_n)_{n \in \mathbb{N}_0} \to_{\tau_d} 0 \\ &\implies \forall k \in \mathbb{N}_0,\ \exists j_k \in \mathbb{N}_0,\ \forall j \geq j_k : \psi_{j+1} - \psi_j \in \Omega_k \\ &\implies \forall k \in \mathbb{N}_0,\ \exists j_k \in \mathbb{N}_0,\ \forall j \geq j_k : (\psi_{j+1} - \psi_j) \restriction_{\mathscr{H}_{\leq k-1}} = 0 \\ &\qquad [\![\text{ Def. of } \Omega_k \text{ in eq. (I) }]\!] \\ &\implies \forall k \in \mathbb{N}_0,\ \exists j_k \in \mathbb{N}_0,\ \forall j \geq j_k : \psi_{j+1} \restriction_{\mathscr{H}_{\leq k-1}} = \psi_j \restriction_{\mathscr{H}_{\leq k-1}} \\ &\implies \forall k \in \mathbb{N}_0,\ \exists j_k \in \mathbb{N}_0\ \forall j \geq j_k : \psi_{j_k} \restriction_{\mathscr{H}_{\leq k-1}} = \psi_{j_k + j} \restriction_{\mathscr{H}_{\leq k-1}}. \end{aligned} \tag{II}$$

For any $x \in \mathscr{H}$ there exists $k_x \in \mathbb{N}_0$ such that $x \in \mathscr{H}_{\leq (k_x - 1)}$, since $\mathscr{H} = \bigcup_{k \in \mathbb{N}_0} \mathscr{H}_{\leq k}$. Then there exists an element $j_{k_x} \in \mathbb{N}_0$ with the property from equation (II). Since the natural numbers are well ordered, we take $j_{k_x} \in \mathbb{N}_0$ as the smallest natural number, which satisfies equation (II). By this we can define the following linear map

$$\psi : \begin{cases} \mathscr{H} \longrightarrow \mathscr{A} \\ x \longmapsto \psi_{j_{k_x}}(x). \end{cases} \tag{III}$$

We claim that ψ is well defined and does not depend on the choice of $k_x \in \mathbb{N}_0$. Assume there exists another k'_x with the property $x \in \mathscr{H}_{\leq (k'_x - 1)}$. Then, either $\mathscr{H}_{\leq (k_x - 1)} \subseteq \mathscr{H}_{\leq (k'_x - 1)}$ or

$\mathcal{H}_{\leq(k'_x-1)} \subseteq \mathcal{H}_{\leq(k_x-1)}$. Assume the latter holds, then because the existence of j_{k_x} and $j_{k'_x}$ is associated to $(\psi_{n+1}-\psi_n)_{n\in\mathbb{N}_0} \to_{\tau_d} 0$, we obtain $j_{k_x} \leq j_{k'_x}$. Because of equation (II) we have

$$\psi(x) = \psi_{j_{k_x}} = \psi_{j_{k'_x}}.$$

The remaining case is analogously shown. Hence, the map ψ is well-defined. The map ψ is linear, since for any $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{K}$ choose $k := \max\{k_x, k_y, k_{\lambda y}, k_{x+\lambda y}\}$ and observe by linearity of ψ_{j_k} that

$$\psi(x+\lambda y) = \psi_{j_k}(x+\lambda y) = \psi_{j_k}(x) + \psi_{j_k}(\lambda y) = \psi_{j_k}(x) + \lambda\psi_{j_k}(y) = \psi(x) + \lambda\psi(y).$$

Let $k \in \mathbb{N}_0$ and $x, \tilde{x} \in \mathcal{H}_{\leq(k-1)}$, then the definition of j_{k_x} and $j_{k_{\tilde{x}}}$ in equation (III) shows that $j_{k_x} = j_{k_{\tilde{x}}}$. Therefore for $k \in \mathbb{N}_0$ we just write $j_k := j_{k_x}$ for all $x \in \mathcal{H}_{\leq(k-1)}$. By this convention the map ψ has the property

$$\forall k \in \mathbb{N}_0 \colon \psi\restriction_{\mathcal{H}_{\leq(k-1)}} = \psi_{j_k}. \tag{IV}$$

Now, we show $(\psi_n)_{n\in\mathbb{N}_0} \to_{\tau_d} \psi$ or equivalently $(\psi-\psi_j)_{j\in\mathbb{N}_0} \to_{\tau_d} 0$. Let $k \in \mathbb{N}_0$, then by equation (IV) there exists $j_k \in \mathbb{N}_0$ such that

$$(\psi-\psi_{j_k})\restriction_{\mathcal{H}_{\leq(k-1)}} = \psi\restriction_{\mathcal{H}_{\leq(k-1)}} - \psi_{j_k}\restriction_{\mathcal{H}_{\leq(k-1)}} = \psi_{j_k} - \psi_{j_k}\restriction_{\mathcal{H}_{\leq(k-1)}} = 0.$$

By equation (II) we can conclude

$$\forall j \geq j_k \colon (\psi-\psi_j)\restriction_{\mathcal{H}_{\leq(k-1)}} = 0$$

which is equivalent to

$$\forall j \geq j_k \colon \psi-\psi_j \in \Omega_k.$$

Since $k \in \mathbb{N}_0$ was arbitrarily chosen we have shown $(\psi-\psi_j)_{j\in\mathbb{N}_0} \to_{\tau_d} 0$, i. e., every Cauchy sequence converges with respect to τ_d which shows that $\mathrm{Lin}(\mathcal{H}, \mathcal{A})$ is complete.

Ad **(c)**: We know that $\mathrm{Lin}(\mathcal{H}, \mathcal{A})$ is complete w. r. t. to the metric induced by topologically admissible family $\mathrm{Lin}^n(\mathcal{H}, \mathcal{A})_{n\in\mathbb{N}_0}$ defined in equation (1.3.9a). Furthermore, we have $\mathrm{Lin}^1(\mathcal{H}, \mathcal{A}) = \mathfrak{g}(\mathcal{H}, \mathcal{A})$ and the unit $e_{\mathcal{H},\mathcal{A}}$ satisfies $e_{\mathcal{H},\mathcal{A}} \in \mathrm{Lin}^0(\mathcal{H}, \mathcal{A}) \setminus \mathrm{Lin}^1(\mathcal{H}, \mathcal{A})$. The well-definedness of the maps $\exp_\star$ and $\log_\star$ now follows from Lemma 1.2.7. The continuity of $\exp_\star$ and $(\log_\star)_1$ is ensured by Lemma 1.2.9. □

1.3.7 Definition ([Grin21, Def. 3.8]). Let $\mathcal{A}$ be a unital algebra and $\mathcal{H}$ be a connected filtered bialgebra. For every $F \in G(\mathcal{H}, \mathcal{A})$, let us define an element $\log_\star F \in \mathfrak{g}(\mathcal{H}, \mathcal{A})$ by

$$\log_\star F := (\log_\star)_1(F - e_{\mathcal{H},\mathcal{A}}). \tag{1.3.10}$$

Then, we put

$$\log_\star \colon \begin{cases} G(\mathcal{H}, \mathcal{A}) \longrightarrow \mathfrak{g}(\mathcal{H}, \mathcal{A}) \\ F \longmapsto \log_\star F. \end{cases} \tag{1.3.11}$$

1.3.8 Remark. By Lemma 1.3.3 **(a)**, we have $G(\mathcal{H}, A) = e_{\mathcal{H},\mathcal{A}} + \mathfrak{g}(\mathcal{H}, \mathcal{A})$. Therefore, equation (1.3.10) is well-defined because

1.3.9 Proposition ([Grin21, Prop. 5.13]). Let $\mathcal{A}$ be a unital algebra and let $\mathcal{H}$ be a connected filtered bialgebra.

(a) Every map $f \in \mathfrak{g}(\mathcal{H}, \mathcal{A})$ satisfies $\log_\star(\exp_\star f) = f$.

(b) Every map $F \in G(\mathscr{H}, \mathscr{A})$ satisfies $\exp_\star(\log_\star F) = F$.

PROOF: By the manner we have defined $\exp_\star \colon \mathfrak{g}(\mathscr{H}, \mathscr{A}) \longrightarrow G(\mathscr{H}, \mathscr{A})$ in equation (1.3.9a) and $\log_\star \colon G(\mathscr{H}, \mathscr{A}) \longrightarrow \mathfrak{g}(\mathscr{H}, \mathscr{A})$ in equation (1.3.11) we can use Proposition 1.2.11 and the assertion follows. A more direct approach which avoids most of the topological arguments, but is more suited to the specific setting of a certain convolution algebra, is provided by the proof of [Grin21, Prop. 5.13]. □

1.3.10 Theorem (Campbell-Baker-Hausdorff-Dynkin). Let $\mathscr{A}$ be a unital algebra and $\mathscr{H}$ a connected filtered bialgebra. For all $f, g \in \mathfrak{g}(\mathscr{H}, \mathscr{A})$ we have

$$(\exp_\star f) \star (\exp_\star g) = \exp_\star(\mathsf{BCH}_\star(f, g)), \tag{1.3.12}$$

where the definition of $\mathsf{BCH}_\star(\,\cdot\,,\,\cdot\,)$ is to be understood in the sense of equations (1.2.20) and (1.2.21), i. e., as a series of right nested brackets $[f^{h_1} \dots g^{k_n}]_\star$ in the Lie algebra associated to the algebra $(\mathrm{Lin}(\mathscr{H}, \mathscr{A}), \star)$.

PROOF: In order to show the assertion we want to apply Theorem 1.2.15. Since the algebra $\mathscr{A}$ is assumed to be unital, the vector space $(\mathrm{Lin}(\mathscr{H}, \mathscr{A}), \star)$ becomes a unital algebra too with unit $e_{\mathscr{H},\mathscr{A}} \in \mathrm{Lin}^0(\mathscr{H}, \mathscr{A}) \setminus \mathrm{Lin}^1(\mathscr{H}, \mathscr{A})$. According to Proposition 1.3.6 **(a)** the convolution algebra $\mathrm{Lin}(\mathscr{H}, \mathscr{A})$ has a topologically admissible family, given by $(\mathrm{Lin}^n(\mathscr{H}, \mathscr{A}))_{n \in \mathbb{N}_0}$. This induces a metrizable topology on $\mathrm{Lin}(\mathscr{H}, \mathscr{A})$. Then, by Proposition 1.3.6 **(b)** $\mathrm{Lin}(\mathscr{H}, \mathscr{A})$ is complete with respect to this metric. Therefore, the assertion now follows from Theorem 1.2.15. □

1.3.11 Remark (Eq. (1.3.12) **in the commutative case).** There is a special case to Theorem 1.3.10 whose proof does not need the full framework of the BCH-formula and in particular no "strong" topological arguments as presented in Section 1.2. For the following assertion we can find such a proof in [Grin21, Prop. 11.1]. Assume we have the same prerequisite as in Theorem 1.3.10. Let $f,\ g \in \mathfrak{g}(\mathscr{H}, \mathscr{A})$ such that $f \star g = g \star f$. Then,

$$(\exp_\star f) \star (\exp_\star g) = \exp_\star(f + g). \tag{1.3.13}$$

1.4 Some results on derivations

We continue our investigations of Section 1.3 by taking derivations into consideration. First of all, recall the definition of a $(\varepsilon_{\mathscr{H}}, \varepsilon_{\mathscr{H}})$-derivation.

1.4.1 Definition ($(\varepsilon_{\mathscr{H}}, \varepsilon_{\mathscr{H}})$-derivation [Grin21, Def. 15.7]). Let $\mathscr{H}$ be a unital algebra, $\mathscr{A}$ be a unital algebra and let $\varepsilon_{\mathscr{H}} \colon \mathscr{H} \longrightarrow \mathbb{C}$ be a unital algebra homomorphism. Let $f \colon \mathscr{H} \longrightarrow \mathscr{A}$ be a linear map. Then,

$$\begin{aligned} &f \text{ is a } (\varepsilon_{\mathscr{H}}, \varepsilon_{\mathscr{H}})\text{-derivation if and only if} \\ &\qquad \forall (a, b) \in \mathscr{H} \times \mathscr{H} \colon f(a \cdot b) = f(a)\varepsilon_{\mathscr{H}}(b) + \varepsilon_{\mathscr{H}}(a) f(b). \end{aligned} \tag{1.4.1}$$

1.4.2 Lemma ($(\varepsilon_{\mathscr{H}}, \varepsilon_{\mathscr{H}})$-derivations form Lie-subalgebra [Grin18, p. 276]). Let $\mathscr{H}$ be a bialgebra, where $\varepsilon_{\mathscr{H}}$ is the counit and define

$$\mathsf{Der}_{(\varepsilon_{\mathscr{H}}, \varepsilon_{\mathscr{H}})}(\mathscr{H}, \mathbb{C}) := \{\, f \colon \mathscr{H} \longrightarrow \mathbb{C} \mid f \text{ is a } (\varepsilon_{\mathscr{H}}, \varepsilon_{\mathscr{H}})\text{-derivation} \,\}, \tag{1.4.2}$$

then

(a) $\mathsf{Der}_{(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})}(\mathcal{H},\mathbb{C}) \subseteq \mathfrak{g}(\mathcal{H},\mathbb{C})$,

(b) $\mathsf{Der}_{(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})}(\mathcal{H},\mathbb{C})$ is a Lie-subalgebra w. r. t. to the Lie-bracket induced by the convolution on $\mathrm{Lin}(\mathcal{H},\mathbb{C})$.

PROOF: AD **(a)**: For any $f \in \mathsf{Der}_{(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})}(\mathcal{H},\mathbb{C})$ we obtain $f(\mathbb{1}_{\mathcal{H}}) = 2f(\mathbb{1}_{\mathcal{H}})$ and by definition of $\mathfrak{g}(\mathcal{H},\mathcal{A})$ in equation (1.3.2) applied to $\mathcal{A} = \mathbb{C}$ the result follows.

AD **(b)**: The calculation is straightforward and can be checked by comparison to [Grin18, Kap. 2, Bem. 1.3, (4) on p. 276]. □

1.4.3 Theorem (Characterization of $(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})$-derivations [Grin21, Thm. 15.9]). Let $(\mathcal{H},*)$ be a unital algebra, $\mathcal{A}$ be a unital algebra and let $\varepsilon_{\mathcal{H}}\colon \mathcal{H} \longrightarrow \mathbb{C}$ be a unital algebra homomorphism. Let $f\colon \mathcal{H} \longrightarrow \mathcal{A}$ be a linear map.

TFAE: **(a)** f is a $(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})$-derivation

(b) $f\Big(\big(\ker\varepsilon_{\mathcal{H}}\big)^2 + \mathbb{K}\cdot\mathbb{1}_{\mathcal{H}}\Big) = 0$, where $\big(\ker\varepsilon_{\mathcal{H}}\big)^2 := \ker\varepsilon_{\mathcal{H}} * \ker\varepsilon_{\mathcal{H}}$.

1.4.4 Theorem ([Grin21, Thm. 15.10]). Let $\mathcal{H}$ be a connected filtered bialgebra and let $\mathcal{A}$ be a commutative unital algebra. Let $f \in \mathfrak{g}(\mathcal{H},\mathcal{A})$ be a linear map.

TFAE: **(a)** f is an $(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})$-derivation

(b) $\exp_\star(f)$ is a unital algebra homomorphism

1.4.5 Lemma. Let $\mathcal{H}$ be a bialgebra, where $\varepsilon_{\mathcal{H}}$ is the counit. Then,

(a) $\exp_\star$ is a map from $\mathsf{Der}_{(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})}(\mathcal{H},\mathbb{C})$ to

$$\mathrm{Alg}(\mathcal{H},\mathbb{C}) := \{\, f \in \mathrm{Lin}(\mathcal{H},\mathbb{C}) \mid f \text{ is unital algebra homomorphism} \,\}. \tag{1.4.3}$$

(b) $\log_\star \restriction_{\mathrm{Alg}(\mathcal{H},\mathbb{C})}$ is a map from $\mathrm{Alg}(\mathcal{H},\mathbb{C})$ to $\mathsf{Der}_{(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})}(\mathcal{H},\mathbb{C})$.

PROOF: AD **(a)**: This is essentially the statement of Theorem 1.4.4.

AD **(b)**: By equation (1.3.3) we have $\mathrm{Alg}(\mathcal{H},\mathbb{C}) \subseteq G(\mathcal{H},\mathbb{C})$. For any $F \in \mathrm{Alg}(\mathcal{H},\mathbb{C})$ we have that $F = \exp_\star\big((\log_\star F)\big)$ from Proposition 1.3.9 **(b)**. From Theorem 1.4.4 now follows that $\log_\star F$ is a $(\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{H}})$-derivation since $\mathbb{C}$ can be seen as a unital commutative algebra. □

For the proof of the next proposition we mention that there is an alternative proof provided by the proof found in [Grin21, p. 248, Sec. 23].

1.4.6 Proposition ([Grin21, Prop. 15.15]). Let $\mathcal{H}$ be a unital algebra and let $\mathcal{A}$ be a commutative unital algebra. For arbitrary unital algebra homomorphisms $f, g\colon \mathcal{H} \longrightarrow \mathcal{A}$ the convolution $f \star g\colon \mathcal{H} \longrightarrow \mathcal{A}$ is a unital algebra homomorphism.

1.4.7 Proposition ([MS17, Thm. 6.1 (a)]). Let $\mathcal{H}$ be a connected graded bialgebra, then $(\mathrm{Alg}(\mathcal{H},\mathbb{C}), \star\restriction_{\mathrm{Alg}(\mathcal{H},\mathbb{C})\times\mathrm{Alg}(\mathcal{H},\mathbb{C})})$ is a group.

PROOF: From Proposition 1.4.6 we obtain $\star\restriction_{\mathrm{Alg}(\mathcal{H},\mathbb{C})\times\mathrm{Alg}(\mathcal{H},\mathbb{C})}\colon \mathrm{Alg}(\mathcal{H},\mathbb{C})\times\mathrm{Alg}(\mathcal{H},\mathbb{C}) \longrightarrow \mathrm{Alg}(\mathcal{H},\mathbb{C})$. Furthermore, it is a well-known result that any graded connected bialgebra $(\mathcal{H},(\mathcal{H}_n)_{n\in\mathbb{N}_0})$ is also a graded Hopfalgebra. A detailed proof can be found in [Grin18, Satz 2.45]. □

1.4.8 Remark ([MS17, Thm. 6.1]). Summarizing the above results we obtain for an $\mathbb{N}_0$-graded connected bialgebra $\mathscr{H}$ that

$$\exp_\star \restriction_{\mathsf{Der}_{(\varepsilon_\mathscr{H},\varepsilon_\mathscr{H})}} : \mathsf{Der}_{(\varepsilon_\mathscr{H},\varepsilon_\mathscr{H})}(\mathscr{H}, \mathbb{C}) \longrightarrow \mathrm{Alg}(\mathscr{H}, \mathbb{C}), \tag{1.4.4a}$$

$$\log_\star \restriction_{\mathrm{Alg}(\mathscr{H},\mathbb{C})} : \mathrm{Alg}(\mathscr{H}, \mathbb{C}) \longrightarrow \mathsf{Der}_{(\varepsilon_\mathscr{H},\varepsilon_\mathscr{H})}(\mathscr{H}, \mathbb{C}) \tag{1.4.4b}$$

are inverse to each other and $\mathsf{Der}_{(\varepsilon_\mathscr{H},\varepsilon_\mathscr{H})}(\mathscr{H}, \mathbb{C})$ forms a Lie-algebra w. r. t. to the Lie-bracket induced by the convolution $\star$ and $\mathrm{Alg}(\mathscr{H}, \mathbb{C})$ forms a group w. r. t. to the convolution $\star$.

Now, we want to investigate certain derivations on the symmetric tensor algebra.

1.4.9 Lemma. Let V be a vector space and let $\varepsilon := \mathcal{S}(0) : \mathrm{Sym}(V) \longrightarrow \mathbb{C}$. Define for any linear functional $\varphi \in \mathrm{Lin}(V, \mathbb{C})$

$$\widehat{\mathrm{D}}(\varphi) : \begin{cases} \mathrm{Sym}(V) \longrightarrow \mathbb{C} \\ b \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}(t\varphi)(b)\big)\Big|_{t=0}. \end{cases} \tag{1.4.5}$$

Then, $\widehat{\mathrm{D}}(\varphi)$ is a $(\varepsilon, \varepsilon)$-derivation and

$$\widehat{\mathrm{D}}(\varphi) \circ \mathrm{i_s} = \varphi. \tag{1.4.6}$$

Proof: First, we shall discuss why the definition for $\widehat{\mathrm{D}}(\varphi)$ is meaningful, i. e., why the total derivative exists. The map $\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{S}(t\varphi)(b)\Big)\Big|_{t=0}$ is linear in the argument $b \in \mathrm{Sym}(V)$. Therefore, it suffices to show that the derivative exists for each basis vector in $\mathrm{Sym}(V)$. Let $b \in \mathrm{Sym}(V)$. Let $(v_i)_{i\in I} \in V^I$ denote a basis sequence for V with totally ordered index set I. Then, the system

$$\{\mathbb{1}_{\mathrm{Sym}(V)}\} \cup \{b_{i_1} \otimes_{\mathrm{Sym}} \cdots \otimes_{\mathrm{Sym}} b_{i_n} \mid (i_j)_{j\in[n]} \in I^{\times n},\ i_1 \leq \ldots \leq i_n\} \tag{I}$$

determines a basis sequence for $\mathrm{Sym}(V)$ ⟦ [BF12, Thm. 10.20] ⟧. Now, for each basis vector of $\mathrm{Sym}(V)$ we can see that $\mathcal{S}(t\varphi)(b)$ must be a polynomial of degree $n \in \mathbb{N}_0$ in t and therefore the derivative exists.

Next, we show that $\widehat{\mathrm{D}}(\varphi) \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$. For this, we show that $\widehat{\mathrm{D}}(\varphi)$ satisfies the property of Theorem 1.4.3 **(b)**. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}(t\varphi)(\mathbb{1}_{\mathrm{Sym}(V)})\big)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(1)\Big|_{t=0} = 0,$$

$$\forall v \in V : \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}(t\varphi)(\mathrm{i_s}(v))\big)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\big((t\varphi)(v)\big)\Big|_{t=0} = \varphi(v),$$

$$\forall n \in \mathbb{N} \setminus \{1\},\ \forall (v_i)_{i\in[n]} \in V^{\times n} : \widehat{\mathrm{D}}(\varphi)\big(\mathrm{i_s}(v_1) \cdot \cdots \cdot \mathrm{i_s}(v_n)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(t^n \prod_{i=1}^{n} \varphi(v_i)\Big)\Big|_{t=0} = 0,$$

where $\mathrm{i_s} : V \hookrightarrow \mathrm{Sym}(V)$ is the canonical insertion map. This shows that $\widehat{\mathrm{D}}(\varphi)$ is a $(\varepsilon, \varepsilon)$-derivation. Since expression (I) determines a basis for $\mathrm{Sym}(V)$, we can deduce that $\widehat{\mathrm{D}}(\varphi)$ fulfills equation (1.4.6). □

1.4.10 Proposition. Let V be a vector space, $\varphi \in \mathrm{Lin}(V, \mathbb{C})$ and let $\varepsilon := \mathcal{S}(0) : \mathrm{Sym}(V) \longrightarrow \mathbb{C}$.

Then, there exists a unique derivation $\mathrm{D}(\varphi) \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ such that

$$\varphi = \mathrm{D}(\varphi) \circ \mathrm{i_s}, \tag{1.4.7}$$

wherein $\mathrm{i_s}\colon V \hookrightarrow \mathrm{Sym}(V)$ is the canonical embedding

Proof: The existence of such a map is provided by Lemma 1.4.9. Next, we show the uniqueness of the map $\mathrm{D}(\varphi)$. Assume there exists another derivation $\tilde{\mathrm{D}}(\varphi) \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ such that $\varphi = \tilde{\mathrm{D}}(\varphi) \circ \mathrm{i_s}$. The map ε is a $\mathbb{C}$-algebra homomorphism which satisfies

$$\varepsilon(v) := \begin{cases} c & \text{for } v = c \cdot \mathbb{1}_{\mathrm{Sym}(V)},\ c \in \mathbb{C} \\ 0 & \text{else.} \end{cases}$$

According to equation (1.1.7) we have $\mathrm{Sym}(V) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{Sym}^n(V)$ and thus $\ker \varepsilon = \mathrm{Sym}(V) \setminus \mathrm{Sym}^0(V) = \mathrm{Sym}(V) \setminus (\mathbb{C} \cdot \mathbb{1}_{\mathrm{Sym}(V)})$. By the equivalent characterization of a derivation from Theorem 1.4.3, we have $\tilde{\mathrm{D}}(\varphi)\big((\ker \varepsilon)^2 + \mathbb{1}_{\mathrm{Sym}(V)}\big) = 0$. Thus,

$$\tilde{\mathrm{D}}(\varphi)\restriction_{\mathrm{Sym}^0(V)+\bigoplus_{k=2}^{\infty}\mathrm{Sym}^k(V)} = 0 = \mathrm{D}(\varphi)\restriction_{\mathrm{Sym}^0(V)+\bigoplus_{k=2}^{\infty}\mathrm{Sym}^k(V)}. \tag{I}$$

By equation (1.4.7), we obtain

$$\tilde{\mathrm{D}}(\varphi) \circ \mathrm{i_s} = \varphi = \mathrm{D}(\varphi) \circ \mathrm{i_s}\,. \tag{II}$$

Equations (I) and (II) now show $\tilde{\mathrm{D}}(\varphi) = \mathrm{D}(\varphi)$. □

1.4.11 Lemma. Let V be a vector space and let $\varepsilon := \mathcal{S}(0)\colon \mathrm{Sym}(V) \longrightarrow \mathbb{C}$. Then,

(a) The map

$$\mathrm{D}\colon \begin{cases} \mathrm{Lin}(V,\mathbb{C}) \longrightarrow \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C}) \\ \varphi \longmapsto \mathrm{D}(\varphi) \end{cases} \tag{1.4.8}$$

is linear.

(b) $\forall \psi \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})\colon\ \mathrm{D}(\psi \circ \mathrm{i_s}) = \psi$.

Proof: Ad **(a)**: It can be directly shown that $\mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ forms a vector subspace of $\mathrm{Lin}\big(\mathrm{Sym}(V), \mathbb{C}\big)$. Hence, for any $\varphi_1, \varphi_2 \in \mathrm{Lin}(V,\mathbb{C})$ and $\alpha \in \mathbb{C}$ we have $\mathrm{D}(\varphi_1) + \mathrm{D}(\varphi_2) \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ and $\alpha\,\mathrm{D}(\varphi_1) \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$. By the defining property for $\mathrm{D}(\,\cdot\,)$ provided in equation (1.4.7) we obtain $\mathrm{D}(\varphi_1 + \varphi_2) = \mathrm{D}(\varphi_1) + \mathrm{D}(\varphi_2)$ and $\mathrm{D}(\alpha\varphi_1) = \alpha\,\mathrm{D}(\varphi_1)$ which shows the linearity of D.

Ad **(b)**: Obviously $\psi \circ \mathrm{i_s} \in \mathrm{Lin}(V,\mathbb{C})$. According to Proposition 1.4.10 $\mathrm{D}(\psi \circ \mathrm{i_s}) \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ is uniquely defined by the equation $\mathrm{D}(\psi \circ \mathrm{i_s}) \circ \mathrm{i_s} = \psi \circ \mathrm{i_s}$. But we have assumed $\psi \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ and trivially $\psi \circ \mathrm{i_s} = \psi \circ \mathrm{i_s}$ and therefore $\mathrm{D}(\psi \circ \mathrm{i_s}) = \psi$. □

Chapter 2

BCH-formula for u.a.u.-products

This chapter introduces all the concepts we need for our setting of noncommutative stochastic independence (Section 2.1). Moreover, we present all the necessary tools which allow us to prove a moment-cumulant formula for a given universal product. For this, we need to define dual semigroups (Section 2.2), Moreover, we need to look for a proper definition of cumulants with respect to a given universal product. We pay attention to this problem in Section 2.3 by using several powerful tools like Schürmann's universal coefficient theorem and the Lachs functor. Once we have shown a moment-cumulant formula (Section 2.4), we focus on implications from this formula for positive u.a.u.-products (Section 2.5).

2.1 Universal products in the category of m-faced algebraic quantum probability spaces

This section provides the axiomatic framework for universal products. First, we recall the free product of (unital) algebras.

2.1.1 Definition (Free product of (unital) algebras [BMM96, Sec. 1.4, Rem. 1.4.1]). Let $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{C}$ be (unital) algebras. Then, a (unital) algebra $\mathscr{A}_1 \sqcup \mathscr{A}_2$ (resp. $\mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2$) satisfies the *universal mapping property (UMP) of the free product* of (unital) algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ if and only if

(a) For each $i \in [2]$ there exist homomorphisms of (unital) algebras $\iota_i : \mathscr{A}_i \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ (resp. $\iota_i : \mathscr{A}_1 \longrightarrow \mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2$), called *canonical homomorphisms*, such that $\iota_1(\mathscr{A}_1) \cup \iota_2(\mathscr{A}_2)$ generates $\mathscr{A}_1 \sqcup \mathscr{A}_2$ (resp. $\mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2$) as a (unital) algebra.

(b) For any (unital) algebra $\mathscr{C}$ and homomorphisms $j_1 : \mathscr{A}_1 \longrightarrow \mathscr{C}$ and $j_2 : \mathscr{A}_2 \longrightarrow \mathscr{C}$ of (unital) algebras there exists a unique homomorphism $j_1 \sqcup j_2 : \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathscr{C}$ (resp. $j_1 \sqcup_{\mathbb{1}} j_2 : \mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2 \longrightarrow \mathscr{C}$) of (unital) algebras such that the following diagram is commutative

$$\begin{array}{ccccc} \mathscr{A}_1 & \xrightarrow{\iota_1} & \mathscr{A}_1 \sqcup \mathscr{A}_2 & \xleftarrow{\iota_2} & \mathscr{A}_2 \\ & \searrow^{j_1} & \downarrow{\scriptstyle j_1 \sqcup j_2} & \swarrow^{j_2} & \\ & & \mathscr{C} & & \end{array} \tag{2.1.1}$$

(respectively in the unital case where $\sqcup$ is replaced by $\sqcup_{\mathbb{1}}$). If such a (unital) algebra $\mathscr{A}_1 \sqcup \mathscr{A}_2$ (resp. $\mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2$) exists we will call it the *free product* of (unital) algebras $\mathscr{A}_1$ and $\mathscr{A}_2$.

2.1.2 Theorem (Existence of the free product of algebras [BMM96, Section 1.4], [Bou98, Ch. III, Ex. 6 for § 5]). Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two algebras. Let I be the ideal in the tensor algebra $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ which is generated by all elements of the form

$$a_1 \otimes a_2 - a_1 a_2 \quad \text{and} \quad a_1' \otimes a_2' - a_1' a_2', \tag{2.1.2}$$

where $a_i \in \mathscr{A}_1$ and $a_i' \in \mathscr{A}_2$ for $i \in \{1, 2\}$. Then, the quotient algebra $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I$ satisfies the universal property of the free product of algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ and the canonical homomorphisms $\iota_1 \colon \mathscr{A}_1 \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ and $\iota_2 \colon \mathscr{A}_2 \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ in equation (2.1.1) are injective.

Proof: We are only going to sketch the proof of the above theorem, since all the details are provided by [BMM96, Section 1.4], although unital algebras have been assumed there. We can slightly modify the proof of [BMM96, Section 1.4] for not necessarily unital algebras. The basic idea is to observe that the canonical inclusion maps $\mathscr{A}_i \hookrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)$, $i \in [2]$ are just linear maps and no homomorphisms of algebras. So, the idea is to make them multiplicative. If I denotes the ideal of $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ generated by elements of equation (2.1.2), then the composition of maps $\mathbf{i}_j \colon \mathscr{A}_j \hookrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2) \twoheadrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I$ defines an homomorphism of algebras for each $j \in [2]$. Since the ideal I is generated by elements of equation (2.1.2), we can see that the map $\mathbf{i}_j$ is injective. Now let $(\mathscr{C}, f_i \colon \mathscr{A}_i \longrightarrow \mathscr{C})_{i \in [2]}$ be a pair consisting of an algebra $\mathscr{C}$ and homomorphisms of algebras. The map $\mathcal{T}(f_1 \oplus f_2) \colon \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2) \longrightarrow \mathscr{C}$ is a homomorphism of algebras and maps the ideal I to zero. As a result $\mathcal{T}(f_1 \oplus f_2)$ can be lifted to a homomorphism of algebras from $\phi \colon \mathcal{T}(f_1 \oplus f_2)/I \longrightarrow \mathscr{C}$. It is clear that ϕ satisfies the property of equation (2.1.1). Moreover, the homomorphisms $\iota_i \colon \mathscr{A}_i \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ are injective, because of the form of the elements in the ideal I from equation (2.1.2). □

2.1.3 Convention. Let I be an arbitrary index set. We define the set

$$\mathbb{A}(I) := \{\, \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in I^{\times n} \mid (n \in \mathbb{N}) \wedge (\forall k \in [n-1] \colon \varepsilon_k \neq \varepsilon_{k+1}) \,\}. \tag{2.1.3}$$

2.1.4 Remark.

(a) In the case of unital algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ we can find a similar construction for the existence of $\mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2$ as we did in the proof of Theorem 2.1.2, but then the ideal I in the tensor algebra $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ is generated by all elements of the form

$$a_1 \otimes a_2 - a_1 a_2, \quad a_1' \otimes a_2' - a_1' a_2', \quad \mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2}, \tag{2.1.4}$$

where $\mathbb{1}_{\mathscr{A}_i}$ denotes the unital element of the unital algebra $\mathscr{A}_i$ for each $i \in [2]$. In this case, the canonical homomorphisms $\iota_i \colon \mathscr{A}_i \longrightarrow \mathscr{A}_1 \sqcup_{\mathbb{1}} \mathscr{A}_2$ are also injective and $\iota_1(\mathscr{A}_1) \cap \iota_2(\mathscr{A}_2) = \mathbb{C}$ (look at [BMM96, Rem. 1.4.1] for a proof of these statements). In other words, we can say that the quotient algebra

$$\mathscr{A}_1 \sqcup \mathscr{A}_2 \Big/ \langle \mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2} \rangle \tag{2.1.5}$$

satisfies the UMP of the free product of unital algebras (by "identification of units").

(b) For two algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ their free product $\mathscr{A}_1 \sqcup \mathscr{A}_2$ exists and is unique up to isomorphism of algebras. Showing uniqueness up to isomorphism is a standard task when dealing with universal mapping properties. We want to discuss a another possible realization of the free product of algebras which is frequently used in the literature. This approach is a bit different to the one presented in the proof of Theorem 2.1.2 and uses the so-called vector space of "alternating tensor chains of finite length" ([NS06, Rem. 6.3]). Here, we want to briefly discuss this approach to show existence of the free product of a family of algebras $(\mathscr{A}_i)_{i\in I}$. If I is an arbitrary index set, $\varepsilon \in \mathbb{A}(I)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{A}(I)$ for some $m \in \mathbb{N}$ and $(V_i)_{i\in I}$ a family of vector spaces, then we put

$$V_\varepsilon := V_{\varepsilon_1} \otimes \cdots \otimes V_{\varepsilon_m}. \tag{2.1.6}$$

Then, the *free product of vector spaces* $(V_i)_{i\in I}$ is defined as the vector space

$$\bigsqcup_{i\in I} V_i := \bigoplus_{\varepsilon\in\mathbb{A}(I)} V_\varepsilon. \tag{2.1.7}$$

We can equip this vector space with a multiplication whenever we are given a family of algebras $(\mathscr{A}_i)_{i\in I}$ by

$$(a_1 \otimes \cdots \otimes a_m) \cdot (b_1 \otimes \cdots b_n) := \begin{cases} a_1 \otimes \cdots a_m \otimes b_1 \otimes \cdots \otimes b_n & \text{for } \varepsilon_m \neq \delta_1 \\ a_1 \otimes \cdots (a_m b_1) \otimes \cdots \otimes b_n & \text{for } \varepsilon_m = \delta_1 \end{cases} \tag{2.1.8}$$

for all $a_1 \otimes \cdots a_m \in \mathscr{A}_{(\varepsilon_1,\dots,\varepsilon_m)}$ and $b_1 \otimes \cdots \otimes b_n \in \mathscr{A}_{(\delta_1,\dots,\delta_n)}$. It can be shown that this algebra satisfies the UMP for the free product of algebras. Again, we can see that the canonical homomorphisms $\iota_i \colon \mathscr{A}_i \longrightarrow \bigsqcup_{j\in I} \mathscr{A}_j$ are injective and therefore we also call them *canonical insertions* to emphasize this fact.

(c) According to [Fra06, Exa. 3.28] the coproduct in the category Alg, in the sense of Rem. 1.1.7 is given by the free product of algebras. Similarly the coproduct in the category of unital algebras uAlg is given by $\sqcup_{\mathbb{1}}$. For each of these coproducts equation (1.1.17) holds. Furthermore, it can be shown that (Alg, $\sqcup$) is a tensor category with $\{0\}$ as unit object and (uAlg, $\sqcup_{\mathbb{1}}$) is a tensor category with $\mathbb{C}$ as unit object (follows in particular from [Fra06, Rem. 3.23]).

2.1.5 Lemma ([Voß07, Satz 1.10.2]). Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two algebras. Then, there exists an isomorphism of unital algebras such that

$$\forall i \in [2] \colon (\mathscr{A}_1)^{\mathbb{1}} \sqcup_{\mathbb{1}} (\mathscr{A}_2)^{\mathbb{1}} \cong \left(\mathscr{A}_1 \sqcup \mathscr{A}_2\right)^{\mathbb{1}}. \tag{2.1.9}$$

Proof: We only give a sketch of the proof which shows that a "canonical" map actually induces an isomorphism. Let $a \in \mathscr{A}_i^{\mathbb{1}}$, then there exist unique elements $\tilde{a} \in \mathbb{C}$ and $\hat{a} \in \mathscr{A}_i$ such that $a = \tilde{a}\mathbb{C} \oplus \hat{a}$. By this we can define a map

$$\alpha_i \colon \begin{cases} (\mathscr{A}_i)^{\mathbb{1}} \longrightarrow \left(\mathscr{A}_1 \sqcup \mathscr{A}_2\right)^{\mathbb{1}} \\ a \longmapsto \tilde{a} \oplus \hat{a}. \end{cases}$$

The maps α_1 and α_2 are homomorphisms of algebras. Therefore, the map $\alpha_1 \sqcup \alpha_2 \colon (\mathscr{A}_1)^{\mathbb{1}} \sqcup (\mathscr{A}_2)^{\mathbb{1}} \longrightarrow \left(\mathscr{A}_1 \sqcup \mathscr{A}_2\right)^{\mathbb{1}}$ is well-defined. The homomorphism $\alpha_1 \sqcup \alpha_2$ is surjective. By the first isomorphism theorem and Remark 2.1.4 **(a)** we are finished if we can show $\ker(\alpha_1 \sqcup \alpha_2) = \langle \mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2} \rangle$, where $\mathbb{1}_{\mathscr{A}_i}$ denotes the unital element of $\mathscr{A}_i$. Let us briefly discuss the proof of this assertion. The statement $\langle \mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2} \rangle \subseteq \ker(\alpha_1 \sqcup \alpha_2)$ is clear. For the other statement one can use the realization of $(\mathscr{A}_1)^{\mathbb{1}} \sqcup (\mathscr{A}_2)^{\mathbb{1}}$ discussed in Remark 2.1.4 **(b)** and perform a proof of induction on the "degree" for an element $x \in \ker(\alpha_1 \sqcup \alpha_2) \subseteq (\mathscr{A}_1)^{\mathbb{1}} \sqcup (\mathscr{A}_2)^{\mathbb{1}}$ denoted by $|x|$ and defined by

$$|x| = \begin{cases} 1 & \text{for } x \in (\mathscr{A}_1)^{\mathbb{1}} \oplus (\mathscr{A}_2)^{\mathbb{1}} \wedge x \neq 0 \\ n & \text{for } \exists n \in \mathbb{N} \colon x \in \bigoplus_{\substack{\mathsf{length}(\varepsilon) \\ \leq n}} \mathscr{A}_\varepsilon^{\mathbb{1}} \wedge x \notin \bigoplus_{\substack{\mathsf{length}(\varepsilon) \\ \leq n-1}} \mathscr{A}_\varepsilon^{\mathbb{1}}. \end{cases} \tag{I}$$

If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ for some $n \in \mathbb{N}$, then put $\mathsf{length}(\varepsilon) := n$. The degree of an element of x is unique and well-defined. Consider the basis of $\bigoplus_{\varepsilon \in \mathbb{A}([2])} \mathscr{A}_\varepsilon^{\mathbb{1}}$ and take the unique representation of x w. r. t. to this basis. Now, define the degree of x as the maximal "length" of each basis vector within this representation of x. Nonetheless, the proof for the induction base is straightforward. For the induction step $n \longrightarrow n+1$ we notice that an element $x \in (\mathscr{A}_1)^{\mathbb{1}} \sqcup (\mathscr{A}_2)^{\mathbb{1}}$ with $|x| = n+1$ has components

$$z \in \bigoplus_{\substack{\mathsf{length}(\varepsilon) \\ \leq n+1}} \mathscr{A}_\varepsilon \quad \text{and} \quad y \in \mathbb{C} \oplus \bigoplus_{\substack{\mathsf{length}(\varepsilon) \\ \leq n}} \mathscr{A}_\varepsilon^{\mathbb{1}},$$

such that $(\alpha_1 \sqcup \alpha_2)(z) = 0 = (\alpha_1 \sqcup \alpha_2)(y)$. Then, the element y has components

$$r \in \bigoplus_{\substack{\mathsf{length}(\varepsilon) \\ \leq n}} \mathscr{A}_\varepsilon^{\mathbb{1}} \quad \text{and} \quad s \in \sum_{i=1}^{d} s_t \in \bigoplus_{\substack{\mathsf{length}(\varepsilon) \\ \leq n+1}} \mathscr{A}_\varepsilon^{\mathbb{1}}$$

Since $(\alpha_1 \sqcup \alpha_2)(z) = 0$ each $s_t \in \mathscr{A}_\varepsilon^{\mathbb{1}}$ has the form for each $t \in [d]$

$$s_t = \sum_j f_{j,t} \otimes (\mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2}) \otimes g_{j,t},$$

where each $f_{j,t}$ and $g_{j,t}$ is a linear combination of pure tensors with maximal length $n-2$. Now, define the element $w := \sum_j f_{j,t} \otimes g_{j,t}$. The element w has the property $|w| \leq n$ and $(\alpha_1 \sqcup \alpha_2)(w) = 0$. It follows that $r + s - w \in \langle \mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2} \rangle$ and finally $x \in \langle \mathbb{1}_{\mathscr{A}_1} - \mathbb{1}_{\mathscr{A}_2} \rangle$. We refer to [Voß07, Satz 1.10.2] for further calculations of the proof. □

2.1.6 Definition (Category of m-faced algebras [MS17, p. 11]). Let $m \in \mathbb{N}$. We want to define the category of so-called *m-faced algebras* denoted by $\mathsf{Alg_m}$. Objects of this category are tuples $(\mathscr{A}, (\mathscr{A}^{(i)})_{i \in [m]})$, where $\mathscr{A}$ is an associative algebra, $(\mathscr{A}^{(i)})_{i \in [m]}$ an m-tuple of associative algebras and furthermore the following properties need to be satisfied;

- **(a)** For all $i \in [m]$ each algebra $\mathscr{A}^{(i)}$ is a subalgebra of $\mathscr{A}$.
- **(b)** The finite family $(\mathscr{A}^{(i)})_{i=1}^m$ *freely generates* $\mathscr{A}$, i. e., the algebra homomorphism $\bigsqcup_{i \in [m]} \mathscr{A}^{(i)} \longrightarrow \mathscr{A}$ defined by

$$\mathscr{A}_\varepsilon \ni a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdot \cdots \cdot a_n \in \mathscr{A} \tag{2.1.10}$$

  is a bijection. In sloppy notation this is denoted by $\mathscr{A} = \bigsqcup_{i=1}^m \mathscr{A}^{(i)}$.

The morphisms $j \in \mathsf{Morph}_{\mathsf{Alg_m}}\Big((\mathscr{B}, (\mathscr{B}^{(i)})_{i \in [m]}), (\mathscr{A}, (\mathscr{A}^{(i)})_{i \in [m]})\Big)$ in this category are defined by the properties

$$j \in \mathsf{Morph}_{\mathsf{Alg}}(\mathscr{B}, \mathscr{A}), \tag{2.1.11}$$

$$\forall\, i \in [m] \colon j(\mathscr{B}^{(i)}) \subseteq \mathscr{A}^{(i)}. \tag{2.1.12}$$

2.1.7 Convention. We will identify the category $\mathsf{Alg_m}$ for $m = 1$ with the category Alg, the category of associative $\mathbb{C}$-algebras.

2.1.8 Definition (Category of (d, m)-algebraic quantum probability spaces [MS17, p. 11]). Let $d, m \in \mathbb{N}$. We want to define the category of *(d, m)-algebraic quantum probability spaces* denoted by $\mathsf{AlgP_{d,m}}$. Objects of this category are triples $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, (\varphi^{(i)})_{i\in[d]})$, where $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}) \in \mathsf{Obj}(\mathsf{Alg_m})$ and $(\varphi^{(i)})_{i\in[d]} \in \big(\mathrm{Lin}(\mathscr{A}, \mathbb{C})\big)^{\times d}$. The morphisms $j \in \mathsf{Morph}_{\mathsf{AlgP_{d,m}}}\Big((\mathscr{B}, (\mathscr{B}^{(i)})_{i\in[m]}, (\psi^{(i)})_{i\in[d]}), (\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, (\varphi^{(i)})_{i\in[d]})\Big)$ in this category are defined by the properties

$$j \in \mathsf{Morph}_{\mathsf{Alg_m}}(\mathscr{B}, \mathscr{A}), \tag{2.1.13}$$

$$\forall\, i \in [d]\colon \varphi^{(i)} \circ j = \psi^{(i)}. \tag{2.1.14}$$

2.1.9 Definition (u.a.u-product in $\mathsf{AlgP_{d,m}}$ [MS17, Sec. 2]). A *universal product* in the category $\mathsf{AlgP_{d,m}}$ is a bifunctor $\odot$ of the form

$$\odot\colon \begin{cases} \mathsf{Obj}(\mathsf{AlgP_{d,m}} \times \mathsf{AlgP_{d,m}}) \ni \Big((\mathscr{A}_1, (\mathscr{A}_1^{(i)})_{i\in[m]}, \underbrace{(\varphi_1^{(i)})_{i\in[d]}}_{=:\varphi_1}), (\mathscr{A}_2, (\mathscr{A}_2^{(i)})_{i\in[m]}, \underbrace{(\varphi_2^{(i)})_{i\in[d]}}_{=:\varphi_2})\Big) \\ \qquad \mapsto \Big(\mathscr{A}_1 \sqcup \mathscr{A}_2, (\mathscr{A}_1^{(i)} \sqcup \mathscr{A}_2^{(i)})_{i\in[m]}, (\varphi_1 \odot \varphi_2)\Big) \in \mathsf{Obj}(\mathsf{AlgP_{d,m}}) \\ \mathsf{Morph}_{\mathsf{AlgP_{d,m}} \times \mathsf{AlgP_{d,m}}}\Big(\big((\mathscr{B}_1, \psi_1), (\mathscr{B}_2, \psi_2)\big), \big((\mathscr{A}_1, \varphi_1), (\mathscr{A}_2, \varphi_2)\big)\Big) \ni (j_1, j_2) \\ \qquad \mapsto j_1 \amalg j_2 \in \mathsf{Morph}_{\mathsf{AlgP_{d,m}}}\big((\mathscr{B}_1 \sqcup \mathscr{B}_2, \psi_1 \odot \psi_2), (\mathscr{A}_1 \sqcup \mathscr{A}_2, \varphi_1 \odot \varphi_2)\big). \end{cases} \tag{2.1.15}$$

(a) A universal product in $\mathsf{AlgP_{d,m}}$ is called *unital* if and only if for all $i \in [2]$, $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_{d,m}})$ the equation

$$\forall\, i \in [d], \forall\, j \in [2] : (\varphi_1 \odot \varphi_2)^{(i)} \circ \iota_j = \varphi_j^{(i)} \tag{2.1.16}$$

is satisfied, where $\iota_j\colon \mathscr{A}_j \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$, $j \in [2]$ are the canonical embeddings.

(b) A universal product in $\mathsf{AlgP_{d,m}}$ is called *associative* if and only if for all $i \in [3]$, $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_{d,m}})$

$$\Big((\varphi_1 \odot \varphi_2) \odot \varphi_3\Big) = \Big(\varphi_1 \odot (\varphi_2 \odot \varphi_3)\Big) \circ \mathsf{can} \tag{2.1.17}$$

holds, wherein $\mathsf{can}\colon (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3 \longrightarrow \mathscr{A}_1 \sqcup (\mathscr{A}_2 \sqcup \mathscr{A}_3)$ is the canonical isomorphism of algebras.

(c) A unital associative universal product in the category $\mathsf{AlgP_{d,m}}$ is abbreviated by *u.a.u.-product*.

(d) A universal product in the category $\mathsf{AlgP_{d,m}}$ is called *symmetric* if and only if for all $i \in [2]$, $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_{d,m}})$

$$\varphi_1 \odot \varphi_2 = (\varphi_2 \odot \varphi_1) \circ \mathsf{can}, \tag{2.1.18}$$

wherein $\mathsf{can}\colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathscr{A}_2 \sqcup \mathscr{A}_1$ is the canonical isomorphism of algebras.

2.1.10 Remark. We strengthen that for a universal product $\odot$ in $\mathsf{AlgP_{d,m}}$ being a bifunctor implies the so-called *universality* condition which says that for each $i \in [2]$, for any $\mathscr{B}_i \in \mathsf{Alg_m}$, $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_{d,m}})$ and $j_i \in \mathsf{Morph}_{\mathsf{Alg_m}}(\mathscr{B}_i, \mathscr{A}_i)$ it holds that

$$\forall j \in [d]\colon \Big((\varphi_1^{(i)} \circ j_1)_{i\in[d]} \odot (\varphi_2^{(i)} \circ j_2)_{i\in[d]}\Big)^{(j)} = (\varphi_1 \odot \varphi_2)^{(j)} \circ (j_1 \amalg j_2). \tag{2.1.19}$$

2.1.11 Convention. From now on we will only consider universal products in the category $\mathsf{AlgP_{1,m}}$ for any $m \in \mathbb{N}$ and we denote this category by $\mathsf{AlgP_m}$. The category denoted by AlgP stands for the category $\mathsf{AlgP_{1,1}}$ (also check Convention 2.1.7).

The next lemma gives us a justification why we actually call the property of Definition 2.1.9 **(a)** unitality.

2.1.12 Lemma. Let $\odot$ be universal product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Then,

TFAE: **(a)** $\odot$ is unital, i. e., $\forall i \in [2]$, $\forall (\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_m})\colon (\varphi_1 \odot \varphi_2)\restriction_{\mathscr{A}_i} = \varphi_i$

(b) $\forall (\mathscr{A}, \varphi) \in \mathsf{Obj}(\mathsf{AlgP_m})\colon \varphi \odot 0 = \varphi = 0 \odot \varphi$

PROOF: The trivial algebra $\{0\}$ has the property that for any algebra $\mathscr{A}$ we have

$$\{0\} \sqcup \mathscr{A} \cong \mathscr{A} \cong \mathscr{A} \sqcup \{0\}. \tag{I}$$

Furthermore, the canonical inclusion maps $\iota_i\colon \mathscr{A}_i \hookrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ can be written as $(\mathsf{id}_{\mathscr{A}_1} \sqcup 0) \circ \mathsf{can}$ resp. $(0 \sqcup \mathsf{id}_{\mathscr{A}_2}) \circ \mathsf{can}$, where can denotes the canonical isomorphism of equation (I). By the universality condition of $\odot$ in equation (2.1.19) and the above observations, both directions follow directly. □

2.1.13 Remark. A u.a.u.-product $\odot$ in $\mathsf{AlgP_m}$ as a bifunctor turns the triple $(\mathsf{AlgP_m}, \odot, (\{0\}, 0 \mapsto 0))$ into a tensor category where the unit object $(\{0\}, 0 \mapsto 0)$ is initial ⟦ [Ger21, Def. 3.3] ⟧. In [Fra06, Sec. 3.3] the notion of independent morphisms of a tensor category has been defined. One might ask why we may restrict ourselves to tensor products of the form $(\mathscr{A}_1 \sqcup \mathscr{A}_2, \varphi_1 \odot \varphi_2)$, where $\odot$ is a u.a.u.-product. The justification for this is given by the so-called "reduction of independence" of Franz which in detail is provided in [Fra06, Prop. 3.45] and [Fra06, Prop. 3.46]. This approach uses the language of tensor categories and represents a much more general framework for the notion of independence. The usual notion of independence for classical probability theory and independences coming from a u.a.u.-product are then to be understood as specific instances of this general notion.

2.1.14 Convention. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two $*$-algebras. Denote the involution of $\mathscr{A}_1$ by $*_1$ and the involution of $\mathscr{A}_2$ by $*_2$. Then, the free product of algebras will always be considered as a $*$-algebra in the canonical way. This means that we define a canonical involution $*$ on the algebra $\mathscr{A}_1 \sqcup \mathscr{A}_2$ on generating elements of $\mathscr{A}_1 \sqcup \mathscr{A}_2$ by

$$*\colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2 \\ \iota_{\varepsilon_1}(a_1) \cdot \dots \cdot \iota_{\varepsilon_n}(a_n) \longmapsto \iota_{\varepsilon_n}\big((a_n)^{*_{\varepsilon_n}}\big) \cdot \dots \cdot \iota_{\varepsilon_1}\big((a_1)^{*_{\varepsilon_1}}\big). \end{cases} \tag{2.1.20}$$

In the above equation $\iota_i\colon \mathscr{A}_i \hookrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ denotes the canonical homomorphic insertion map, $(\varepsilon_i)_{i\in[n]} \in \mathbb{A}([2])$ and $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_i}$. We can homomorphically extend the prescription and obtain a canonical involution on $\mathscr{A}_1 \sqcup \mathscr{A}_2$.

2.1.15 Definition (Positive u.a.u.-product [Lac15, Def. 4.5.3]). Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$. Let $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_m})$ for $i \in [2]$. Assume that for each algebra $\mathscr{A}_i$ there exists an involution $*_i\colon \mathscr{A}_i \longrightarrow \mathscr{A}_i$ which turns $\mathscr{A}_i$ into a $*$-algebra and each subalgebra $\mathscr{A}_i^{(j)}$ is a $*$-subalgebra, i. e., $*_i(\mathscr{A}_i^{(j)}) \subseteq \mathscr{A}_i^{(j)}$. We regard the algebra $\mathscr{A}_1 \sqcup \mathscr{A}_2$ as a $*$-algebra in the sense of Convention 2.1.14. Let φ_1 resp. φ_2 be strongly positive (Def. 1.1.27) on $(\mathscr{A}_1, *_1)$ resp. on $(\mathscr{A}_2, *_2)$. We say that the u.a.u.-product $\odot$ is *positive* if and only if $\varphi_1 \odot \varphi_2$ is a strongly positive linear functional on $(\mathscr{A}_1 \sqcup \mathscr{A}_2, *)$.

2.2 Comonoids in the tensor category of graded m-faced algebras

The definition of a tensor category is well-known in the literature. We have used the definition provided by [Lac15, Def. 2.2.1] but we do not write it here once more. Sometimes we say that a triple $(\mathfrak{C}, \boxtimes, E)$ is a tensor category, where E is the unit object and sometimes we say that a pair $(\mathfrak{C}, \boxtimes)$ is a tensor category without mentioning the unit object. We give the definition of a comonoid in a tensor category to have a point of reference when we speak about so-called "dual semigroups".

2.2.1 Definition (Comonoid in tensor category [Lac15, Def. 2.3.2]). Let $(\mathfrak{C}, \boxtimes)$ be a tensor category. Then, we call an object $\mathscr{C} \in \mathsf{Obj}(\mathfrak{C})$ a *comonoid* if it is equipped with morphisms

- $\Delta\colon \mathscr{C} \longrightarrow \mathscr{C} \boxtimes \mathscr{C}$, called *comultiplication*,
- $\varepsilon\colon \mathscr{C} \longrightarrow E$, called *counit*,

such that the following diagrams commute

$$\begin{array}{ccccc} \mathscr{C}\boxtimes\mathscr{C} & \xleftarrow{\quad\Delta\quad} & \mathscr{C} & \xrightarrow{\quad\Delta\quad} & \mathscr{C}\boxtimes\mathscr{C} \\ {\scriptstyle \mathrm{id}_{\mathscr{C}}\boxtimes\Delta}\downarrow & & & & \downarrow{\scriptstyle \Delta\boxtimes\mathrm{id}_{\mathscr{C}}} \\ \mathscr{C}\boxtimes(\mathscr{C}\boxtimes\mathscr{C}) & & \xrightarrow{\quad\alpha_{\mathscr{C},\mathscr{C},\mathscr{C}}\quad} & & (\mathscr{C}\boxtimes\mathscr{C})\boxtimes\mathscr{C} \end{array} \tag{2.2.1}$$

$$\begin{array}{ccccc} E\boxtimes\mathscr{C} & \xleftarrow{\varepsilon\boxtimes\mathrm{id}_{\mathscr{C}}} & \mathscr{C}\boxtimes\mathscr{C} & \xrightarrow{\mathrm{id}_{\mathscr{C}}\boxtimes\varepsilon} & \mathscr{C}\boxtimes E \\ & {\scriptstyle \ell_{\mathscr{C}}}\searrow & \uparrow{\scriptstyle\Delta} & \swarrow{\scriptstyle r_{\mathscr{C}}} & \\ & & \mathscr{C} & & \end{array} \tag{2.2.2}$$

In the above equations E denotes the unit object in the tensor category, $\alpha_{\cdot,\cdot,\cdot}$ stands for the associativity constraint, $\ell_\cdot, r_\cdot$ are the left resp. right unit constraint.

Dual semigroups have been introduced by Voiculescu in [Voi85]. We give the source from where we have taken its definition.

2.2.2 Definition (Dual semigroup & unital dual semigroup [Lac15, p. 34], m-faced dual semigroup [Ger21, Def. 2.2]). A comonoid in the tensor category $(\mathsf{Alg}, \sqcup, \{0\})$ resp. in the tensor category $(\mathsf{uAlg}, \sqcup_{\mathbb{1}}, \{0\})$ is called *dual semigroup* resp. *unital dual semigroup*. A comonoid in the tensor category $(\mathsf{Alg_m}, \sqcup, \{0\})$ for some $m \in \mathbb{N}$, i. e., the tensor category of m-faced algebras with monoidal structure given by $\sqcup$, is called *m-faced dual semigroup*.

2.2.3 Remark.

(a) Any m-faced dual semigroup is a dual semigroup because any morphism in the category $\mathsf{Alg_m}$ is also a morphism in the category Alg.

(b) In [Lac15, Rem. 2.3.3] we can find the following statement. Since E as the unit object is the initial object $\{0\}$ in Alg, the counit ε of a dual semigroup is always the zero map 0. Therefore, we sometimes omit the counit and refer to the tuple $(\mathcal{C}, \Delta)$ as a dual semigroup.

2.2.4 Lemma. For $m \in \mathbb{N}$ let $(\mathcal{D}_i, \Delta_i, 0)_{i\in[m]}$ be a finite family of comonoids in the tensor category $(\mathsf{Alg}, \sqcup, \{0\})$, then the triple

$$\Big(\bigsqcup_{i=1}^{m} \mathcal{D}_i, \mathsf{can} \circ \big(\coprod_{i=1}^{m} \Delta_i\big), 0\Big) \tag{2.2.3}$$

is a comonoid in $(\mathsf{Alg_m}, \sqcup, \{0\})$, i. e., an m-faced dual semigroup. Herein, the canonical isomorphism can is to be understood in the sense of equation (1.1.17), i. e.,

$$\mathsf{can}\colon \bigsqcup_{i=1}^{m} (\mathcal{D}_i \sqcup \mathcal{D}_i) \longrightarrow \big(\bigsqcup_{i=1}^{m} \mathcal{D}_i\big) \sqcup \big(\bigsqcup_{i=1}^{m} \mathcal{D}_i\big). \tag{2.2.4}$$

PROOF: The proof is straightforward and uses the UMP of the free product of algebras at several occasions. □

2.2.5 Definition (categories Vect^Θ, Alg^Θ, uAlg^Θ, cuAlg^Θ [Lac15, Def. 2.4.1], $\mathsf{Alg}_\mathsf{m}^\Theta$). Let Θ be a commutative monoid with binary operation + and neutral element 0.

(a) By Vect^Θ we denote the category of Θ-graded vector spaces (Definition 1.1.14). Morphisms of this category are homogeneous linear maps. By Alg^Θ we denote the category of Θ-graded algebras (Definition 1.1.19) and morphisms of this category are algebra homomorphisms which are homogeneous as linear maps. By uAlg^Θ we denote the category of Θ-graded algebras which are in addition unital. By cuAlg^Θ we denote the category Θ-graded algebras which are in addition commutative and unital. Morphisms of the category uAlg^Θ and cuAlg^Θ are homomorphisms of unital algebras which are homogeneous as linear maps.

(b) By $\mathsf{Alg}_\mathsf{m}^\Theta$ we denote the category of Θ-graded algebras which are in addition m-faced algebras. Morphisms of this category need to be homogeneous linear maps and m-faced morphisms. We call objects of this category Θ-*graded m-faced algebras*.

2.2.6 Definition (Θ-graded dual semigroup, Θ-graded bialgebra [Lac15, Def. 2.4.1], Θ-graded m-faced dual semigroup). We call a comonoid in the tensor category $(\mathsf{Alg}^\Theta, \sqcup, \{0\})$ a Θ-*graded dual semigroup*. We call a comonoid in the tensor category $\big(\mathsf{uAlg}^\Theta, \otimes, (E_\alpha)_{\alpha\in\Theta}\big)$ with $E_e := \mathbb{C}$ (e neutral element in Θ) and $\forall\, \alpha \in \Theta \setminus \{e\}\colon E_\alpha := \{0\}$ a Θ-*graded bialgebra*. We call a comonoid in the tensor category $(\mathsf{Alg}_\mathsf{m}^\Theta, \sqcup, \{0\})$ a Θ-*graded m-faced dual semigroup*.

2.2.7 Remark.

(a) Let $(V, (V_\alpha)_{\alpha\in\Theta})$ be a Θ-graded vector space. The tensor algebra $\mathrm{T}(V)$ with the Θ-grading

$$\mathrm{T}(V) = \bigoplus_{\alpha\in\Theta} \big(\mathrm{T}(V)\big)_\alpha = \bigoplus_{\alpha\in\Theta}\Big(\bigoplus_{\alpha_1+\cdots+\alpha_r=\alpha} V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_r}\Big) \tag{2.2.5}$$

forms a Θ-graded algebra ⟦ [Gre78, Sec. 3.7] or [Lac15, Exa. 2.4.3] ⟧.

(b) It can be shown that $(\mathsf{Alg}^{\Theta}, \sqcup, \{0\})$ and $(\mathsf{uAlg}^{\Theta}, \otimes, (E_\alpha)_{\alpha\in\Theta})$ are tensor categories ⟦ [Lac15, p. 116] ⟧. Let us discuss the Θ-grading of the free product of algebras $\mathscr{A}_1 \sqcup \mathscr{A}_2$ for $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{Alg}^{\Theta})$. To do this, we shall use the realization of the free product of algebras, discussed in the proof of Theorem 2.1.2. From there we can see that the ideal $I \subseteq \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ is generated by homogeneous elements. Thus, we can transfer the Θ-grading of the tensor algebra $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ to the quotient algebra $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I$. This Θ-grading is also characterized by the property that the canonical embeddings $\mathscr{A}_i \hookrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ are homogeneous.

2.2.8 Example ([MS17, Exa. 4], [Lac15, Exa. 2.4.2], [Lac15, Exa. 2.4.3]). Let V be a vector space. We want to discuss how the tensor algebra $\mathrm{T}(V)$ can be given the structure of an $\mathbb{N}_0$-graded dual semigroup. For this, we need the following canonical isomorphism of algebras

$$\mathsf{can}\colon \mathrm{T}(V_1 \oplus V_2) \longrightarrow \mathcal{T}(V_1) \sqcup \mathcal{T}(V_2) \tag{2.2.6}$$

for two vector spaces V_1 and V_2. From the UMP of the tensor algebra we obtain the following commutative diagram

$$\begin{array}{ccc} V_1 \oplus V_2 & \xhookrightarrow{\iota_1 \oplus \iota_2} & \mathrm{T}(V_1) \sqcup \mathrm{T}(V_2) \\ {\scriptstyle \mathsf{inc}_1 \oplus \mathsf{inc}_2}\big\downarrow & \nearrow{\scriptstyle \mathcal{T}(\iota_1 \oplus \iota_2)} & \\ \mathrm{T}(V_1 \oplus V_2). & & \end{array} \tag{2.2.7}$$

And from the UMP of the tensor algebra and the UMP of the free product of algebras we have the following commutative diagram for each $i \in [2]$

$$\begin{array}{ccc} \mathrm{T}(V_i) & \xhookrightarrow{\iota_i''} & \mathrm{T}(V_1) \sqcup \mathrm{T}(V_2) \\ {\scriptstyle \iota_i'}\big\uparrow & \searrow{\scriptstyle \mathcal{T}(\iota_i')} & \big\downarrow{\scriptstyle \mathcal{T}(\iota_1') \sqcup \mathcal{T}(\iota_2')} \\ V_i & \xhookrightarrow{\mathsf{inc}_i} & \mathrm{T}(V_1) \oplus \mathrm{T}(V_2) \end{array} . \tag{2.2.8}$$

Since the algebras $\mathrm{T}(V_1 \oplus V_2)$ and $\mathrm{T}(V_1) \sqcup \mathrm{T}(V)_2$ are generated by elements from $V_1 \cup V_2$, it can be easily shown that the maps $\mathcal{T}(\iota_1 \oplus \iota_2)$ and $\mathcal{T}(\iota_1') \sqcup \mathcal{T}(\iota_2')$ are inverse to each other. Note that $\iota_i = \iota_i'' \circ \iota_i'$.

Now, we come back to equip $\mathrm{T}(V)$ with a structure of a dual semigroup. Consider the natural inclusion maps

$$\mathsf{inc}_1\colon \begin{cases} V \longrightarrow V \oplus V \\ v \longmapsto (v, 0) \end{cases} \qquad \mathsf{inc}_2\colon \begin{cases} V \longrightarrow V \oplus V \\ v \longmapsto (0, v). \end{cases} \tag{2.2.9}$$

For the linear map $\mathsf{inc}_1 + \mathsf{inc}_2\colon V \longrightarrow V \oplus V$ due to equation (1.1.5) there exists an algebra homomorphism $\mathrm{T}(\mathsf{inc}_1 + \mathsf{inc}_2)\colon \mathrm{T}(V) \longrightarrow \mathrm{T}(V \oplus V)$. Using the canonical isomorphism from equation (2.2.6) it is easily shown for generating elements of $\mathrm{T}(V)$ that

$$\big(\mathrm{T}(V), \mathsf{can} \circ \mathrm{T}(\mathsf{inc}_1 + \mathsf{inc}_2), 0\big) \tag{2.2.10}$$

is a dual semigroup. We refer to its comultiplication as *primitive comultiplication* of $\mathrm{T}(V)$. By Remark 2.2.7 **(a)**, the tensor algebra $\mathrm{T}(V)$ carries a natural $\mathbb{N}_0$-grading if V is trivially graded, i. e., $V_1 = V$ and $\forall n \in \mathbb{N}_0 \setminus \{1\}: V_n = \{0\}$. The 0-th part of the $\mathbb{N}_0$-grading of $\mathrm{T}(V)$ is $\{0\}$. Then, the primitive comultiplication from equation (2.2.10) is easily seen to be homogeneous. In other words, the tensor algebra w. r. t. to the primitive comultiplication becomes an $\mathbb{N}_0$-graded dual semigroup.

2.3 Reduction of convolution

We seek for a "good" definition of cumulants w. r. t. a given u.a.u.-product. Cumulants are classical objects dating back to the danish mathematician Thiele in the late 19th century. They can be seen as families of multilinear maps tightly related to the notion of independence, as they vanish when evaluated with independent entries. Indeed, for each type of independence a particular family of cumulants exist. They are characterized through so-called moment-cumulant relations, which express moments in terms of cumulants by summing over specific set partitions.

In order to find a possible realization of cumulants w. r. t. a given u.a.u.-product we use dual semigroups and define a convolution on their dual space. Once we have exhibited Schürmann's universal coefficient theorem and the Lachs functor, we are in the position to define an "exponential" w. r. t. a given u.a.u.-product $\odot$ (this corresponds to the notion of "moment"). We will show that this exponential has a two-sided inverse and we will call this inverse a "logarithm" w. r. t. $\odot$ (this corresponds to the notion of "cumulant").

2.3.1 Definition (Convolution on the dual of an m-faced dual semigroup [MS17, eq. (5.7)]). Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ and $(\mathcal{D}, \Delta, 0)$ be an m-faced dual semigroup. Then, we define the *convolution* $\circledast$ on $\mathcal{D}$ by

$$\circledast\colon \begin{cases} \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \times \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \\ (\varphi_1, \varphi_2) \longmapsto (\varphi_1 \odot \varphi_2) \circ \Delta. \end{cases} \tag{2.3.1}$$

2.3.2 Remark. Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$.

(a) The associativity of $\odot$ ensures the associativity of $\circledast$. The unitality of $\odot$ and its equivalent characterization provided in Lemma 2.1.12 ensures that $(\mathrm{Lin}(\mathcal{D}, \mathbb{C}), \circledast)$ is unital with unit 0. Thus, $(\mathrm{Lin}(\mathcal{D}, \mathbb{C}), \circledast)$ is a monoid.

(b) Let $(\mathcal{D}, \Delta, \varepsilon)$ be an m-faced dual semigroup. For the convolution power regarding $\circledast$ of an element $\varphi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ we define

$$\varphi^{\circledast n} := \begin{cases} \varepsilon = 0 & \text{for } n = 0 \\ \varphi & \text{for } n = 1 \\ \varphi \circledast \varphi^{\circledast(n-1)} & \text{for } n \geq 2. \end{cases} \tag{2.3.2}$$

In the following we try to formulate an equivalent characterization of the n-fold convolution power $\varphi^{\circledast n}$ which involves the usage of the following map. We recursively define for

$$\forall n \in \mathbb{N}_0\colon \Delta^{(n)}\colon \begin{cases} \mathcal{D} \longrightarrow \mathcal{D}^{\sqcup(n+1)} \\ d \longmapsto \big(\mathsf{can} \circ (\mathsf{id} \amalg \Delta^{(n-1)}) \circ \Delta\big)(d) \end{cases} \tag{2.3.3}$$

as composition of the following maps

$$\Delta\colon \mathscr{D} \longrightarrow \mathscr{D} \sqcup \mathscr{D}, \tag{2.3.4a}$$

$$\Delta^{(n-1)} \amalg \mathsf{id}\colon \mathscr{D} \sqcup \mathscr{D} \longrightarrow \mathscr{D} \sqcup \mathscr{D}^{\sqcup n}, \tag{2.3.4b}$$

$$\mathsf{can}\colon \mathscr{D}^{\sqcup n} \sqcup \mathscr{D} \longrightarrow \mathscr{D}^{\sqcup (n+1)}. \tag{2.3.4c}$$

For $n = 0$ we set

$$\Delta^{(-1)} := \varepsilon = 0\colon \mathscr{D} \longrightarrow \mathscr{D}^{\sqcup 0}, \tag{2.3.5}$$

since $\mathscr{D}^{\sqcup 0} = \{0\}$ ⟦ $\{0\}$ is the unit object in the tensor category $(\mathsf{Alg}, \sqcup)$ ⟧. Obviously, the map $\Delta^{(n)}$ is a morphism in the category $\mathsf{Alg_m}$ for each $n \in \mathbb{N}_0$. The maps $\Delta^{(n)}$ have the following properties:

- Because of the counit property, we obtain $\Delta^{(0)} = \mathsf{id}\colon \mathscr{D} \longrightarrow \mathscr{D}$. On the other hand, we have $\mathscr{D}^{\sqcup 1} = \mathscr{D}$ and $\Delta^{(1)} = \Delta$.
- By coassociativity in equation (2.2.1), we have

$$(\Delta \amalg \mathsf{id}) \circ \Delta = \Delta^{(2)} = (\mathsf{id} \amalg \Delta) \circ \Delta \tag{2.3.6}$$

  up to canonical isomorphism.
- We claim that the following equation is valid up to canonical isomorphism

$$(\mathsf{id} \amalg \Delta^{(n-1)}) \circ \Delta = \Delta^{(n)} = (\Delta^{(n-1)} \amalg \mathsf{id}) \circ \Delta. \tag{2.3.7}$$

  A similar proof can be found in [Grin18, Bem. 2.1.$\frac{1}{2}$ 3) on p. 73]. But there, the monoidal structure is given by the ordinary tensor product of vector spaces. Then, an analogous map $\Delta^{(n)}\colon \mathscr{C} \longrightarrow \mathscr{C}^{\otimes n+1}$ can be defined. Thus, we need to replace $\otimes$ by $\sqcup$ and the unit object $\mathbb{K}$, where $\mathbb{K}$ is a field, by $\{0\}$ in the proof of [Grin18, Bem. 2.1.$\frac{1}{2}$ 3) on p. 73].

For an m-faced dual semigroup $(\mathscr{D}, \Delta, 0)$ and $\varphi \in \mathrm{Lin}(\mathscr{D}, \mathbb{C})$

$$\forall n \in \mathbb{N}_0\colon \varphi^{\circledast n} = \varphi^{\odot n} \circ \Delta^{(n-1)} \tag{2.3.8}$$

holds. We prove this statement by induction over $n \in \mathbb{N}_0$. The statement is true for $n = 0$. Performing the induction step $n \longrightarrow n+1$ we calculate for $n > 0$

$$\begin{aligned}
\varphi^{\circledast(n+1)} &= (\varphi \odot \varphi^{(\circledast n)}) \circ \Delta \\
&= \Big(\varphi \odot (\varphi^{\odot n} \circ \Delta^{(n-1)})\Big) \circ \Delta \\
&= \Big((\varphi \circ \mathsf{id}) \odot (\varphi^{\odot n} \circ \Delta^{(n-1)})\Big) \circ \Delta \\
&= \Big((\varphi \odot \varphi^{\odot n}) \circ (\mathsf{id} \amalg \Delta^{(n-1)})\Big) \circ \Delta \quad [\![\text{ eq. (2.1.19) }]\!] \\
&= (\varphi \odot \varphi^{\odot n}) \circ \Delta^{(n)} \quad [\![\text{ eq. (2.3.3) }]\!] \\
&= \varphi^{\odot(n+1)} \circ \Delta^{(n)}.
\end{aligned} \tag{2.3.9}$$

(c) Since we are mainly interested in a "good" notion of a cumulant with respect to $\odot$, we need to find a "good" definition of a moment formula with respect to $\odot$ as its mutual inverse. We can try to define something like an "exponential map" $\exp_{\circledast}$ on $(\mathrm{Lin}(\mathcal{D},\mathbb{C}),\circledast)$ by

$$\forall\varphi\in\mathrm{Lin}(\mathcal{D},\mathbb{C})\colon\ \exp_{\circledast}\varphi:=\sum_{k=0}^{\infty}\frac{1}{k!}\varphi^{\circledast k}. \tag{2.3.10}$$

Leaving issues of convergence aside, we may ask if this definition deserves being called an convolution exponential for φ. The answer is in general no, because the convolution $\circledast$ from equation (2.3.1) is not bilinear in general. This has an effect for instance on commuting elements $\varphi_1,\ \varphi_2\in\mathrm{Lin}(\mathcal{D},\mathbb{C})$ w. r. t. $\circledast$. Then, the formula

$$\exp_{\circledast}(\varphi_1+\varphi_2)=(\exp_{\circledast}\varphi_1)\circledast(\exp_{\circledast}\varphi_2) \tag{2.3.11}$$

will not hold in general. But this would give us somehow a moral claim to call it exponential for $\circledast$. A better definition for an exponential for $\circledast$ is not obvious and heavily relies on the following machinery

1. Schürmann's so-called "universal coefficient theorem"
2. "reduction of convolution" to the dual space of a commutative bialgebra
3. characterization of continuous convolution semigroups on comonoids in Alg_m

Because the following theorem is of such great importance for us, we stick to its original notation used in [MS17, Thm. 4.2]. First, we present the statement and discuss its notation afterwards in Remark 2.3.4.

2.3.3 Theorem (Universal coefficient theorem [MS17, Thm. 4.2]). Let $\odot$ be a universal product in the category AlgP_m for $m\in\mathbb{N}$. Moreover, let $k,\,n\in\mathbb{N}$, $(\mathcal{A}_i,(\mathcal{A}_i^{(j)})_{j\in[m]},\varphi_i)_{i\in[k]}\in\big(\mathrm{Obj}(\mathsf{AlgP}_\mathsf{m})\big)^{\times k}$, $\varepsilon:=(\varepsilon_{i,1},\varepsilon_{i,2})_{i\in[n]}\in\mathbb{A}([k]\times[m])$ and $(a_i)_{i\in[n]}\in\prod_{i=1}^{n}\mathcal{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$. Then, there exist complex constants $\alpha^{(\varepsilon)}_{\pi_1,\ldots,\pi_k}\in\mathbb{C}$, uniquely determined by $\odot$, indexed by $\varepsilon\in([k]\times[m])^{\times n}$ and $\pi_i\in\mathcal{M}(X^{(i)})$ such that

$$\begin{aligned}(\varphi_1\odot\cdots\odot\varphi_k)(a_1\cdot\cdots\cdot a_n)=\sum_{\pi_1\in\mathcal{M}(X_\varepsilon^{(1)})}\cdots\sum_{\pi_k\in\mathcal{M}(X_\varepsilon^{(k)})}\Bigg(&\alpha^{(\varepsilon)}_{\pi_1,\ldots,\pi_k}\\ &\prod_{M_1\in\pi_1}\varphi_1\Big(\big(j(a_1,\ldots,a_n)(M_1)\big)\Big)\\ &\cdots\prod_{M_k\in\pi_k}\varphi_k\Big(\big(j(a_1,\ldots,a_n)\big)(M_k)\Big)\Bigg).\end{aligned} \tag{2.3.12}$$

2.3.4 Remark.

(a) Let $X_n:=\{x_1,\ldots,x_n\}$ be the set formed by $n\in\mathbb{N}$ "indeterminates" $x_1,\ldots,x_n$. The map

$$j(a_1,\ldots,a_n)\colon\mathbb{C}\langle X_n\rangle\longrightarrow\coprod_{j=1}^{m}\coprod_{i=1}^{k}\mathcal{A}_i^{(j)} \tag{2.3.13}$$

is the unique algebra homomorphism defined by

$$\forall i \in [n]\colon\ j(a_1, \dots, a_n)(x_i) = a_i. \tag{2.3.14}$$

Herein $\mathbb{C}\langle X_n\rangle$ denotes the free algebra over X_n, i.e. $\mathbb{C}\langle X_n\rangle \cong \mathrm{T}(\mathbb{C}X_n)$.

(b) We put for any $\varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in \mathbb{A}([k] \times [m])$

$$\forall j \in [k]\colon\ X_\varepsilon^{(j)} := \{\, x_\ell \in X_n \mid \varepsilon_{\ell,1} = j \,\}. \tag{2.3.15}$$

For each $j \in [k]$ the set $\mathcal{M}(X_\varepsilon^{(j)})$ is formed by all monomials of the (commutative, unital) polynomial algebra

$$\mathbb{C}\big[\{\, M \in \mathbb{C}\langle X_\varepsilon^{(j)}\rangle \mid M \text{ is a monomial} \,\}\big] \tag{2.3.16}$$

with the property that each indeterminate of $X_\varepsilon^{(j)}$ appears exactly once ([MS17, p. 9]). By $\mathcal{OM}(X_\varepsilon^{(j)})$ we denote the subset of $\mathcal{M}(X_\varepsilon^{(j)})$ of *right-ordered* elements of $\mathcal{M}(X_\varepsilon^{(j)})$, i. e., of elements

$$M_1 \bullet \cdots \bullet M_\ell \quad \text{such that} \quad M_r = x_{i(r,1)} \cdot \cdots \cdot x_{i(r,s_r)} \tag{2.3.17}$$

with $i(r,1) < \cdots < i(r,s_r),\ r \in [\ell]$ for any $r \in [\ell]$. Here we use the symbol $\cdot$ to indicate the multiplication in the free algebra $\mathbb{C}\langle X_\varepsilon^{(j)}\rangle$ and the symbol $\bullet$ for the multiplication in the polynomial algebra of equation (2.3.16). Elements of $\mathcal{M}(X_\varepsilon^{(j)}) \setminus \mathcal{OM}(X_\varepsilon^{(j)})$ are called *wrong-ordered* ([MS17, p. 24]). By $\mathbb{1}_j$ we denote the unique monomial in $\mathcal{OM}(X_\varepsilon^{(j)})$ such that there exists a monomial M in $\mathbb{C}\langle X_\varepsilon^{(j)}\rangle$ with the property $\mathbb{1}_j = M$ and for the degree of the monomial holds $\deg(M) = |X_\varepsilon^{(j)}|$.

(c) Let $\pi = M_1 \bullet M_2 \bullet \cdots \bullet M_r \in \mathcal{M}(X_\varepsilon^{(j)})$ with definition of $X_\varepsilon^{(j)}$ from equation (2.3.15) and $r \in \mathbb{N}$, then by abuse of notation we write $M_i \in \pi$ for all $i \in [r]$.

2.3.5 Convention. Let $k,\ m,\ n \in \mathbb{N}$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in ([k] \times [m])^{\times n}$. This is a typical index expression which appears whenever we want to evaluate a u.a.u.-product in the sense of equation (2.3.12). Occasionally, we use the notation

$$\mathsf{type}(\varepsilon) := (\varepsilon_{i,1})_{i\in[n]} \in \mathbb{N}^{\times n}, \qquad \mathsf{type}(\varepsilon_i) := \varepsilon_{i,1} \in \mathbb{N} \tag{2.3.18}$$

to gain access for the “type index” of ε and

$$\mathsf{col}(\varepsilon) := (\varepsilon_{i,2})_{i\in[n]} \in \mathbb{N}^{\times n}, \qquad \mathsf{col}(\varepsilon_i) := \varepsilon_{i,2} \in \mathbb{N} \tag{2.3.19}$$

to gain access for the “color index” or sometimes we also call it the “face index” of ε. Why we call the face index a color index will become more clearer later when we deal with “colored” partitions.

2.3.6 Proposition (Characterization of a universal product [MS17, Prop. 5.1]). Let $m \in \mathbb{N}$ and $\odot$ be a universal product in $\mathsf{AlgP_m}$. Then, for all $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP_m}),\ i \in [2]$ there exists a unique mapping

$$\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}\colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Sym}(\mathscr{A}_1) \otimes \mathrm{Sym}(\mathscr{A}_2) \tag{2.3.20}$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathscr{A}_1 \sqcup \mathscr{A}_2 & \xrightarrow{\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}} & \mathrm{Sym}(\mathscr{A}_1) \otimes \mathrm{Sym}(\mathscr{A}_2) \\ & \searrow^{\varphi_1 \odot \varphi_2} & \Big\downarrow {\scriptstyle \mathcal{S}(\varphi_1)\otimes\mathcal{S}(\varphi_2)} \\ & & \mathbb{C} \end{array} . \tag{2.3.21}$$

Proof: In [MS17, eq. (5.2)] the map $\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Sym}(\mathscr{A}_1) \otimes \mathrm{Sym}(\mathscr{A}_2)$ is defined for $m, n \in \mathbb{N}$, $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}) \in \mathsf{Obj}(\mathsf{Alg_m})$, $\varepsilon = (\varepsilon_{i,1}, \varepsilon_{i,2})_{i\in[n]} \in \mathbb{A}([2] \times [m]), (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$ by

$$\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}(a_1 \cdot \dots \cdot a_n) = \sum_{\pi_1 \in \mathcal{M}(X_\varepsilon^{(1)})} \sum_{\pi_2 \in \mathcal{M}(X_\varepsilon^{(2)})} \Bigg(\alpha^{(\varepsilon)}_{\pi_1,\pi_2} \Big(\prod_{M_1\in\pi_1} \mathrm{i_s}\big(j(a_1, \dots, a_n)(M_1)\big)\Big) \otimes \Big(\prod_{M_2\in\pi_2} \mathrm{i_s}\big(j(a_1, \dots, a_n)(M_2)\big)\Big)\Bigg)$$

with the constants $\alpha^{(\varepsilon)}_{\pi_1,\pi_2}$ of Theorem 2.3.3 and notation introduced in Remark 2.3.4. The canonical injection $\mathrm{i_s}$ denotes the injection into the symmetric tensor algebra introduced in equation (1.1.12). By the UMP of the symmetric tensor algebra, equation (2.3.21) follows. □

The next proposition shows how important the mappings $\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}$ are for a given universal product $\odot$. Also compare this statement to [BS02, p. 539] for the case $d = m = 1$.

2.3.7 Proposition ([MS17, Prop. 5.2]). Let $\odot$ be a universal product in $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Then, for the family of mappings $(\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2})_{\mathscr{A}_1,\mathscr{A}_2\in\mathsf{Obj}(\mathsf{Alg_m})}$ (each family member satisfies equation (2.3.21)) we have the following properties.

(a) For all $i \in [2]$ and for all $\mathscr{A}_i, \mathscr{B}_i \in \mathsf{Obj}(\mathsf{Alg_m})$ and morphisms $j_i \in \mathsf{Morph}_{\mathsf{Alg_m}}(\mathscr{B}_i, \mathscr{A}_i)$

$$\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ j_1 \amalg j_2 = \big(\mathrm{Sym}((j_1) \otimes \mathrm{Sym}(j_2)\big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2} \tag{2.3.22}$$

holds.

(b) The universal product is associative if and only if for all $i \in [3]$, $\mathscr{A}_i, \in \mathsf{Obj}(\mathsf{Alg_m})$

$$\Big(\mathsf{id}_{\mathrm{Sym}(\mathscr{A}_1)} \otimes \mathcal{S}(\sigma^{\odot}_{\mathscr{A}_2,\mathscr{A}_3})\Big) \circ \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2\sqcup\mathscr{A}_3} = \Big(\mathcal{S}(\sigma^{\odot}_{\mathscr{A}_2,\mathscr{A}_3}) \otimes \mathsf{id}_{\mathrm{Sym}(\mathscr{A}_3)}\Big) \circ \sigma^{\odot}_{\mathscr{A}_1\sqcup\mathscr{A}_2,\mathscr{A}_3} \circ \mathsf{can} \tag{2.3.23}$$

holds. Herein, $\mathsf{can} \colon \mathscr{A}_1 \sqcup (\mathscr{A}_2 \sqcup \mathscr{A}_3) \longrightarrow (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3$ denotes the canonical isomorphism of algebras.

(c) The universal product is unital if and only if for all $i \in [2]$, $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{Alg_m})$

$$\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ \iota_1 = \mathsf{inc}_1, \tag{2.3.24a}$$

$$\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ \iota_2 = \mathsf{inc}_2, \tag{2.3.24b}$$

where $\iota_i \colon \mathscr{A}_i \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ and $\mathsf{inc}_i \colon \mathscr{A}_i \longrightarrow \mathrm{Sym}(\mathscr{A}_1) \otimes \mathrm{Sym}(\mathscr{A}_2)$ for each $i \in [2]$ as the canonical inclusion maps.

(d) The universal product is symmetric if and only if for all $i \in [2]$, $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{Alg_m})$

$$\widetilde{\mathsf{can}} \circ \sigma^{\odot}_{\mathscr{A}_2,\mathscr{A}_1} = \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ \mathsf{can}, \tag{2.3.25}$$

where $\widetilde{\mathsf{can}} \colon \mathrm{Sym}(\mathscr{A}_2) \otimes \mathrm{Sym}(\mathscr{A}_1) \longrightarrow \mathrm{Sym}(\mathscr{A}_1) \otimes \mathrm{Sym}(\mathscr{A}_2)$ denotes the canonical isomorphism of vector spaces and $\mathsf{can} \colon \mathscr{A}_2 \sqcup \mathscr{A}_1 \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ the canonical isomorphism of algebras.

Proof: Ad **(a)**: Let us assume that $\odot$ is a universal product in the category $\mathsf{AlgP_m}$. Let $(\mathscr{A}_1, \varphi_1) \in \mathsf{Obj}(\mathsf{Alg_m})$ and $(\mathscr{A}_2, \varphi_2) \in \mathsf{Obj}(\mathsf{Alg_m})$. Now, we can calculate for the left hand side of equation (2.3.22)

$$\begin{aligned}\Big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\Big) \circ \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ (j_1 \amalg j_2) &= (\varphi_1 \odot \varphi_2) \circ (j_1 \amalg j_2)^d \\ &= (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2).\end{aligned}$$

For the right hand side of equation (2.3.22) we obtain

$$\begin{aligned}&\big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big) \circ \big(\mathrm{Sym}(j_1) \otimes \mathrm{Sym}(j_2)\big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2} \\ &\qquad = \Big((\mathcal{S}(\varphi_1) \circ \mathrm{Sym}(j_1)) \otimes (\mathcal{S}(\varphi_2) \circ \mathrm{Sym}(j_2))\Big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2} \\ &\qquad = \Big(\mathcal{S}(\varphi_1 \circ j_1) \otimes \mathcal{S}(\varphi_2 \circ j_2)\Big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2} \\ &\qquad = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2).\end{aligned}$$

Thus, this yields $\forall b \in \mathscr{B}_1 \sqcup \mathscr{B}_2$

$$\begin{aligned}&\big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big)\Big(\Big(\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ (j_1 \amalg j_2) - \big(\mathrm{Sym}(j_1) \otimes \mathrm{Sym}(j_2)\big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2}\Big)(b)\Big) = 0 \\ \Longleftrightarrow\ &\Big((\mathsf{can}^{-1} \circ \mathsf{can})\big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big) \circ (\mathsf{can}^{-1} \circ \mathsf{can})\Big) \\ &\quad\Big(\Big(\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ (j_1 \amalg j_2) - \big(\mathrm{Sym}(j_1) \otimes \mathrm{Sym}(j_2)\big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2}\Big)(b)\Big) = 0 \\ &\quad [\![\ \mathsf{can} \colon \mathrm{Sym}(\mathscr{A}_1) \otimes \mathrm{Sym}(\mathscr{A}_2) \longrightarrow \mathrm{Sym}(\mathscr{A}_1 \oplus \mathscr{A}_2) \text{ from Lem. 1.1.6 }]\!] \\ \Longleftrightarrow\ &\big(\mathsf{can}^{-1} \circ \mathcal{S}(\varphi_1 \oplus \varphi_2) \circ \mathsf{can}\big) \qquad\qquad \text{(I)} \\ &\quad\Big(\Big(\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2} \circ (j_1 \amalg j_2) - \big(\mathrm{Sym}(j_1) \otimes \mathrm{Sym}(j_2)\big) \circ \sigma^{\odot}_{\mathscr{B}_1,\mathscr{B}_2}\Big)(b)\Big) = 0.\end{aligned}$$

At this point we provide an intermediate result for elements in the symmetric tensor algebra. Consider the following linear map for a vector space V

$$\Phi \colon \begin{cases} \mathrm{Sym}(V) \longrightarrow \{\, f \colon \mathbb{C}^I \longrightarrow \mathbb{C} \,\} \\ P \longmapsto \big(\mathcal{S}(\varphi)(P)\big)_{\varphi \in \mathrm{Lin}(V,\mathbb{C})}, \end{cases} \tag{II}$$

where I is the index set of a basis $(x_i)_{i \in I}$ of V. The above prescription is well-defined since any linear functional $\varphi \in \mathrm{Lin}(V, \mathbb{C})$ is uniquely determined by its values on basis vectors in V.

The basis of V induces a basis of $\mathrm{Sym}(V)$ and thus the value $\mathcal{S}(\varphi)(P)$ corresponds to "inserting" $\mathcal{S}(\varphi)(x_i)$ into P, if P is expanded into basis vectors x_i of V. It can be directly shown that above map Φ is injective, since the characteristic of $\mathbb{C}$ as a field is 0. Now, equation (I) holds for any linear functionals $\varphi_1 \oplus \varphi_2 \in \mathrm{Lin}(\mathscr{A}_1 \oplus \mathscr{A}_2, \mathbb{C})$ and $b \in \mathscr{B}_1 \sqcup \mathscr{B}_2$ was arbitrarily chosen and therefore equation (2.3.22) follows from the fact that the map Φ from equation (II) is injective.
AD **(b)**: Assume $\odot$ is an associative universal product in the category $\mathsf{AlgP_m}$. Let $(\mathscr{A}_i)_{i\in[3]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times 3}$, then we can calculate for any linear functionals $(\varphi_i)_{i\in[3]} \in (\mathrm{Lin}(\mathscr{A}_i, \mathbb{C}))^{\times 3}$

$$
\begin{aligned}
(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2) \otimes \mathcal{S}(\varphi_3)) &\circ \left(\mathsf{id}_{\mathrm{Sym}(\mathscr{A}_1)^{\otimes d}} \otimes \mathcal{S}(\sigma^{\odot}_{\mathscr{A}_2,\mathscr{A}_3})\right) \circ \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2\sqcup\mathscr{A}_3} \\
&= \left(\mathcal{S}(\varphi_1) \otimes \left((\mathcal{S}(\varphi_2) \otimes \mathcal{S}(\varphi_3)) \circ \mathcal{S}(\sigma^{\odot}_{\mathscr{A}_2,\mathscr{A}_3})\right)\right) \circ \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2\sqcup\mathscr{A}_3} \\
&= \left(\mathcal{S}(\varphi_1) \otimes \mathcal{S}\left((\mathcal{S}(\varphi_2) \otimes \mathcal{S}(\varphi_3)) \circ \sigma^{\odot}_{\mathscr{A}_2,\mathscr{A}_3}\right)\right) \circ \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2\sqcup\mathscr{A}_3} \\
&= \left(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2 \odot \varphi_3)\right) \circ \sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2\sqcup\mathscr{A}_3} \\
&= \varphi_1 \odot (\varphi_2 \odot \varphi_3).
\end{aligned}
$$

We could analogously perform the calculation for the right hand side of equation (2.3.23). Once again, the assertion now follows from Lemma 1.1.6 and the fact that the map Φ from equation (II) is injective. The other direction is clear.

The remaining proofs for **(c)** and **(d)** follow in a similar fashion from above and therefore are straightforward to do. □

Although we have to deal with universal products in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$, we can still apply the "machinery" of reduction of convolution, presented in [Lac15, Sec. 5.2]. Let us argue why we are allowed to do this. In [Lac15, Sec. 5.2] universal products in the category AlgP are considered. In the following we will establish a cotensor functor from the tensor category $(\mathsf{Alg}^{\Theta}_{\mathsf{m}}, \sqcup, \{0\})$ to the tensor category $(\mathsf{cuAlg}^{\Theta}, \otimes, \mathbb{C})$. In [Lac15, Sec. 5.2] a similar functor from $(\mathsf{Alg}^{\Theta}, \sqcup, \{0\})$ to the tensor category $(\mathsf{cuAlg}^{\Theta}, \otimes, \mathbb{C})$ has been constructed which essentially depends on the definition of the mappings $\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}$ for $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{Alg})$, $i \in [2]$. The proofs of Lemma 2.3.9 and Theorem 2.3.10 can be taken from [Lac15, Sec. 5.2], where we need to adjust indices of the mappings $\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}$ from $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{Alg})$, $i \in [2]$ to $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{Alg_m})$, $i \in [2]$. We recall that any m-faced algebra is in particular an algebra. This fact allows us for the definition to be made in equation (2.3.26), to use a forgetful functor from $\mathsf{Alg}^{\Theta}_{\mathsf{m}}$ to Alg^{Θ} and then take the definition of the functor $\mathfrak{L}$ from [Lac15, eq. 5.7]. Also compare this discussion to the statement of [Ger21, Sec. 4].

2.3.8 Definition (Lachs functor [Lac15, eq. 5.7]). We define the *Lachs functor* $\mathfrak{L}\colon \mathsf{Alg}^{\Theta}_{\mathsf{m}} \longrightarrow \mathsf{cuAlg}^{\Theta}$ as a functor by

$$
\mathfrak{L}\colon \begin{cases}
\mathsf{Obj}(\mathsf{Alg}^{\Theta}_{\mathsf{m}}) \ni \left(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, (\mathscr{A}_\alpha)_{\alpha\in\Theta}\right) \\
\qquad \mapsto \left(\mathrm{Sym}(\mathscr{A}), \left((\mathrm{Sym}(\mathscr{A}))_\alpha\right)_{\alpha\in\Theta}\right) \in \mathsf{Obj}(\mathsf{cuAlg}^{\Theta}) \\
\mathsf{Morph}_{\mathsf{Alg}^{\Theta}_{\mathsf{m}}}\left(\left(\mathscr{B}, (\mathscr{B}^{(i)})_{i\in[m]}, (\mathscr{B}_\alpha)_{\alpha\in\Theta}\right), \left(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, (\mathscr{A}_\alpha)_{\alpha\in\Theta}\right)\right) \ni j \\
\qquad \mapsto \mathrm{Sym}(j) \in \mathsf{Morph}_{\mathsf{cuAlg}^{\Theta}}\left(\mathrm{Sym}(\mathscr{B}), \mathrm{Sym}(\mathscr{A})\right).
\end{cases} \tag{2.3.26}
$$

That the above prescription is a functor, follows from the UMP for the symmetric tensor algebra $\mathrm{Sym}(V)$, where V is a Θ-graded vector space. We want to consider the following composition of functors

$$\mathfrak{L} \circ \sqcup = \mathfrak{L}(\,\cdot\, \sqcup \,\cdot\,)\colon\ \mathsf{Alg}_{\mathsf{m}}^{\Theta} \times \mathsf{Alg}_{\mathsf{m}}^{\Theta} \longrightarrow \mathsf{cuAlg}^{\Theta}, \tag{2.3.27}$$

$$\otimes \circ (\mathfrak{L}, \mathfrak{L}) = \mathfrak{L}(\,\cdot\,) \otimes \mathfrak{L}(\,\cdot\,)\colon\ \mathsf{Alg}_{\mathsf{m}}^{\Theta} \times \mathsf{Alg}_{\mathsf{m}}^{\Theta} \longrightarrow \mathsf{cuAlg}^{\Theta}. \tag{2.3.28}$$

2.3.9 Lemma ([Lac15, Lem. 5.2.3]). Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP}_{\mathsf{m}}$ for some $m \in \mathbb{N}$. Then the family $\mathcal{S}(\sigma^{\odot}) := (\mathcal{S}(\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}))_{\mathscr{A}_1,\mathscr{A}_2 \in \mathsf{Obj}(\mathsf{Alg}_{\mathsf{m}}^{\Theta})}$ is a natural transformation

$$\mathcal{S}(\sigma^{\odot})\colon\ \mathfrak{L}(\,\cdot\, \sqcup \,\cdot\,) \Rightarrow \mathfrak{L}(\,\cdot\,) \otimes \mathfrak{L}(\,\cdot\,). \tag{2.3.29}$$

2.3.10 Theorem ([Lac15, Thm. 5.2.4]). Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP}_{\mathsf{m}}$ for some $m \in \mathbb{N}$. Consider the tensor categories $(\mathsf{Alg}_{\mathsf{m}}^{\Theta}, \sqcup, \{0\})$ and $(\mathsf{cuAlg}^{\Theta}, \otimes, \mathbb{C})$, then the triple $(\mathfrak{L}, \mathcal{S}(\sigma^{\odot}), g_0)$ consisting of the functor $\mathfrak{L}\colon \mathsf{Alg}_{\mathsf{m}}^{\Theta} \longrightarrow \mathsf{cuAlg}^{\Theta}$, the natural transformation $\mathcal{S}(\sigma^{\odot})\colon \mathfrak{L}(\,\cdot\, \sqcup \,\cdot\,) \Rightarrow \mathfrak{L}(\,\cdot\,) \otimes \mathfrak{L}(\,\cdot\,)$ and the map $g_0\colon \mathrm{Sym}(\{0\}) \longrightarrow \mathbb{C}$ with $g_0(\mathbb{1}_{\mathrm{Sym}(\{0\})}) = 1$ is a cotensor functor.

2.3.11 Corollary ([MS17, Thm. 5.2], [Lac15, p. 118]). Let $(\mathfrak{L}, \mathcal{S}(\sigma^{\odot}), g_0)\colon (\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\}) \longrightarrow (\mathsf{cuAlg}^{\mathbb{N}_0}, \otimes, \mathbb{C})$ be the cotensor functor of Theorem 2.3.10.

(a) If $\big((\mathscr{D}, (\mathscr{D}_n)_{n\in\mathbb{N}_0}, (\mathscr{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathscr{D}_0 = \{0\}$, then

$$\big(\mathfrak{L}(\mathscr{D}), \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}}) \circ \mathfrak{L}(\Delta), g_0 \circ \mathfrak{L}(0)\big) \tag{2.3.30}$$

is a comonoid in the tensor category $(\mathsf{cuAlg}^{\mathbb{N}_0}, \otimes, \mathbb{C})$, i. e., a graded, commutative bialgebra (Definition 1.1.20 **(c)**) which is also connected as a graded bialgebra (Definition 1.1.23) and which is also connected as a filtered bialgebra (Definition 1.1.22).

(b) If $\big((\mathscr{D}, (\mathscr{D}_n)_{n\in\mathbb{N}_0}, (\mathscr{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathscr{D}_0 = \{0\}$, then the prescription

$$\mathrm{Lin}(\mathscr{D}, \mathbb{C}) \ni \varphi \longmapsto \mathcal{S}(\varphi) \in \mathrm{Lin}\big(\mathrm{Sym}(\mathscr{D}), \mathbb{C}\big) \tag{2.3.31}$$

defines a homomorphism between the monoids $(\mathrm{Lin}(\mathscr{D}, \mathbb{C}), \circledast)$ and $\big(\mathrm{Lin}(\mathrm{Sym}(\mathscr{D}), \mathbb{C}), \star_{\odot}\big)$, whereby $\star_{\odot}$ denotes the convolution induced by the comultiplication given in **(a)**. This means for all $i \in [2]$, $\varphi_i \in \mathrm{Lin}(\mathscr{D}, \mathbb{C})$

$$\mathcal{S}(\varphi_1 \circledast \varphi_2) = \mathcal{S}(\varphi_1) \star_{\odot} \mathcal{S}(\varphi_2). \tag{2.3.32}$$

Proof: Ad **(a)**: In [Lac15, Cor. 2.3.5] it is stated that cotensor functors map comonoids to comonoids. From Theorem 2.3.10 we know that $(\mathfrak{L}, \mathcal{S}(\sigma^{\odot}), g_0)$ is a cotensor functor. By this, we conclude that $(\mathfrak{L}(\mathscr{D}), \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}}) \circ \mathfrak{L}(\Delta), g_0 \circ \mathfrak{L}(0))$ is a graded and commutative bialgebra. A similar proof which avoids the terminology of cotensor functors can be found in [BS05, Thm. 3.4]. The algebra structure of the bialgebra $\mathfrak{L}(\mathscr{D})$ is the natural one induced by $\otimes_{\mathrm{Sym}}$ for pure vectors. Furthermore, it is a connected graded bialgebra because by Lemma 1.1.16 we can see that $\big(\mathrm{Sym}(\mathscr{D})\big)_0 = \mathbb{C} \cdot \mathbb{1}_{\mathrm{Sym}(\mathscr{D})}$ (we have assumed $\mathscr{D}_0 = \{0\}$). This shows that the graded bialgebra is connected. Moreover, due to Lemma 1.1.21 the graded bialgebra becomes a filtered bialgebra with $\big(\mathrm{Sym}(\mathscr{D})\big)_{\leq 0} = \mathbb{C} \cdot \mathrm{Sym}(\mathscr{D})$ and by Definition 1.1.22 it is also a connected filtered bialgebra.
Ad **(b)**: In [BS05, Thm. 3.4] this has been proven for the more general case $m = 1$ and without assumption about a grading of the dual semigroup. But the same reasoning also applies in our

case $m > 1$, since the formal steps of reasoning do not interfere with properties of morphisms of m-faced algebras or the $\mathbb{N}_0$-grading of the dual semigroup $\mathcal{D}$. For an alternative proof, we refer to the proof of [Lac15, eq. (5.8)]. □

2.3.12 Convention. Corollary 2.3.11 **(a)** tells us that a u.a.u.-product $\odot$ in the category $\mathsf{AlgP_m}$ takes a certain $\mathbb{N}_0$-graded m-faced dual semigroup and associates with it a connected graded bialgebra with comultiplication $\Delta_\odot$ realized on a certain symmetric tensor algebra. With respect to this comultiplication $\Delta_\odot$ we can define a convolution $\star_\odot$. Without burdening ourselves too much with heavy notation, we set

$$\star := \star_\odot \tag{2.3.33}$$

for a u.a.u.-product $\odot$. This means that it should be clear from the context if a convolution is induced by $\odot$ in the sense of Corollary 2.3.11 **(a)** and we do not further highlight its dependence on the specific u.a.u.-product $\odot$.

We may convince ourselves that all the nice properties of a convolution exponential are encoded in an object called "convolution semigroup". Hence, seeking for the right definition of the convolution exponential is the same as characterizing convolution semigroups. The result of this connection, after gathering the necessary tools and describing their properties, is given in Theorem 2.3.14. We notice that in [BS05, Def. 4.2] and [BS05, Thm. 4.6] u.a.u.-products $\odot$ in the category AlgP have been considered. But, we can easily extend the definition and statement to the multi-faced case $m > 1$ since the same formal steps of reasoning can be applied. The crucial step for the machinery of the Lachs functor is the equivalent characterization of a u.a.u.-product in Proposition 2.3.7 in terms of the family of mappings $\sigma_{\mathcal{A}_1,\mathcal{A}_2}$. From there we can see that we need to run through all linear functionals $\varphi \in \mathrm{Lin}(\mathcal{A}, \mathbb{C})$ to achieve the existence of such family of mappings. Here it is not important if the algebra $\mathcal{A}$ has m faces or only one face. Furthermore, we have the technical obstacle to demand an $\mathbb{N}_0$-graded dual semigroup $(\mathcal{D}, (\mathcal{D}_i)_{i\in\mathbb{N}_0})$ with $\mathcal{D}_0 = \{0\}$ because then convergence of the convolution exponential on the dual space of a connected graded bialgebra is ensured in our framework (Corollary 2.3.11 **(a)**). The work of [BS05] does not need this assumption since their justification for the existence of the convolution exponential is different (see Remark 1.3.4). By these comments in mind, we may state the following definition and Theorem 2.3.14, where we refer for its proof to the proof done for [BS05, Thm. 4.6].

2.3.13 Definition (Convolution semigroup on a m-faced dual semigroup [BS05, Def. 4.2]). Let $(\mathcal{D}, \Delta, \varepsilon)$ be an m-faced dual semigroup and assume $\odot$ is a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. A family $(\varphi_t)_{t\in\mathbb{R}_+} \in \big(\mathrm{Lin}(\mathcal{D}, \mathbb{C})\big)^{\mathbb{R}_+}$ is called a *convolution semigroup* on the dual semigroup $(\mathcal{D}, \Delta, \varepsilon)$ if and only if for all $s, t \in \mathbb{R}_+$ we have

$$\varphi_s \circledast \varphi_t = \varphi_{s+t}, \tag{2.3.34a}$$

$$\varphi_0 = \varepsilon = 0. \tag{2.3.34b}$$

A convolution semigroup is said to be *weakly continuous* if and only if in addition

$$\forall b \in \mathcal{D}\colon \lim_{t\to 0^+} \varphi_t(b) = \varepsilon(b) = 0. \tag{2.3.35}$$

2.3.14 Theorem (Characterization of convolution semigroup on an m-faced dual semigroup [BS05, Thm. 4.6]). Let $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ be a comonoid in the tensor category

$(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$ and assume $\odot$ is a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Then,

Tfae: **(a)** The family $(\varphi_t)_{t\in\mathbb{R}_+} \in \big(\mathrm{Lin}(\mathcal{D}, \mathbb{C})\big)^{\mathbb{R}_+}$ forms a weakly continuous convolution semigroup on the dual semigroup $\mathcal{D}$.

(b) There exists a linear functional $\Psi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ such that

$$\forall t \in \mathbb{R}_+ \colon \varphi_t = \exp_\star(t\,\mathrm{D}(\Psi)) \circ \mathrm{i_s}, \tag{2.3.36}$$

wherein $\mathrm{i_s} \colon \mathcal{D} \hookrightarrow \mathrm{Sym}(\mathcal{D})$ is the canonical embedding, $\exp_\star$ denotes the well-defined exponential on the convolution algebra $\mathrm{Lin}\big(\mathrm{Sym}(\mathcal{D}), \star\big)$ in the sense of equation (1.3.9a) using Convention 2.3.12 and $\mathrm{D}(\,\cdot\,)$ has been defined in Proposition 1.4.10.

2.3.15 Remark. Let $m \in \mathbb{N}$, a comonoid $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$, $\odot$ a u.a.u.-product in the category $\mathsf{AlgP_m}$, a linear functional $\varphi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ and $b \in \mathcal{D}$. We put

$$(\exp_\odot \varphi)(b) := \left(\left(\sum_{n=0}^{\infty} \mathrm{D}(\varphi)^{\star n}\right) \circ \mathrm{i_s}\right)(b). \tag{2.3.37}$$

By the above equivalent characterization of weakly continuous convolution semigroups, we can conclude from equation (2.3.34a) that

$$\big(\exp_\odot s\varphi\big) \circledast \big(\exp_\odot t\varphi\big) = \big(\exp_\odot (s+t)\varphi\big). \tag{2.3.38}$$

Finally, this result gives us a moral claim what to call an exponential on the monoid $(\mathrm{Lin}(\mathcal{D}, \mathbb{C}), \circledast)$ and $\exp_\odot \varphi$ seems to be promisingly.

2.4 Moment-cumulant formula

In the previous section we have collected convincing reasons, how an exponential and logarithm with respect to a u.a.u.-product might be defined. This in turn allows us to establish our desired moment-cumulant formula which is of great importance for our further investigations. Here, we bring into effect our considerations from Sections 1.2–1.4.

2.4.1 Definition (Exponential and logarithm on the dual space of a dual semigroup). Assume that a u.a.u-product $\odot$ in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ is given. Let $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ be a comonoid in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$. Then, we define

$$\exp_\odot \colon \begin{cases} \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \\ \varphi \longmapsto \Big(\exp_\star\big(\mathrm{D}(\varphi)\big)\Big) \circ \mathrm{i_s}, \end{cases} \tag{2.4.1}$$

$$\log_\odot \colon \begin{cases} \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \\ \varphi \longmapsto \Big(\log_\star\big(\mathcal{S}(\varphi)\big)\Big) \circ \mathrm{i_s}\,. \end{cases} \tag{2.4.2}$$

wherein $\star$ denotes the convolution using Convention 2.3.12.

2.4.2 Remark. We want to discuss why the definitions of $\exp_\odot$ and $\log_\odot$ make sense. According to Corollary 2.3.11 **(a)** the Lachs functor associates to the comonoid $\mathcal{D} \in \mathsf{Obj}(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0})$ a connected filtered bialgebra $\mathrm{Sym}(\mathcal{D})$. In particular, by Proposition 1.3.6 **(c)** the mappings $\exp_\star$ and $(\log_\star)_1$ are well-defined and therefore $\log_\star$ is well-defined because of Definition 1.3.7. In detail this means: for $\varphi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ the symbol $\mathrm{D}(\varphi)$ denotes the unique derivation in $\mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C})$ which satisfies equation (1.4.7). Here, ε is the counit of the bialgebra $\mathrm{Sym}(\mathcal{D})$ coming from the Lachs functor (equation (2.3.30)). If we consult Remark 1.4.8, then we can see for the exponential

$$\exp_\star \restriction_{\mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(\mathcal{D}),\mathbb{C})} \colon \mathsf{Der}_{(\varepsilon,\varepsilon)}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big) \longrightarrow \mathrm{Alg}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big). \tag{2.4.3}$$

Therefore, $\exp_\odot$ is well-defined. Furthermore, since $\mathcal{S}(\varphi) \in \mathrm{Alg}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big)$ $\log_\odot$ is well-defined because we have for the logarithm

$$\log_\star \restriction_{\mathrm{Alg}(\mathrm{Sym}(\mathcal{D}),\mathbb{C})} \colon \mathrm{Alg}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big) \longrightarrow \mathsf{Der}_{(\varepsilon,\varepsilon)}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big). \tag{2.4.4}$$

We recall that by Lemma 1.4.2 **(b)** the subspace $\mathsf{Der}_{(\varepsilon,\varepsilon)}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big) \subseteq \mathrm{Lin}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big)$ is a Lie algebra w. r. t. the Lie bracket induced by $\star$. By Proposition 1.4.7 $\mathrm{Alg}\big(\mathrm{Sym}(\mathcal{D}), \mathbb{C}\big)$ is a group w. r. t. $\star$.

2.4.3 Proposition. Assume that a u.a.u-product $\odot$ in $\mathsf{AlgP}_{\mathsf{m}}$ is given and a comonoid $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ is fixed. Then, $\exp_\odot \colon \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ and $\log_\odot \colon \mathrm{Lin}(\mathcal{D}, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ are inverse to each other.

Proof: We calculate for an arbitrary $\varphi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$

$$\begin{aligned}
\exp_\odot(\log_\odot \varphi) &= \Big(\exp_\star\big(\mathrm{D}(\log_\odot \varphi)\big)\Big) \circ \mathrm{i}_{\mathrm{s}} && [\![\text{ eq. (2.4.1) }]\!] \\
&= \Big(\exp_\star\Big(\underbrace{\mathrm{D}\big((\log_\star \mathcal{S}(\varphi)) \circ \mathrm{i}_{\mathrm{s}}\big)}_{\substack{=\log_\star \mathcal{S}(\varphi) \\ [\![\text{ Lem. 1.4.11 (b), eq. (2.4.4) }]\!]}} \Big)\Big) \circ \mathrm{i}_{\mathrm{s}} && [\![\text{ eq. (2.4.2) }]\!] \\
&= \underbrace{\Big(\exp_\star\big(\log_\star \mathcal{S}(\varphi)\big)\Big)}_{\substack{=\mathcal{S}(\varphi) \\ [\![\text{ Prop. 1.3.9 (b) }]\!]}} \circ \mathrm{i}_{\mathrm{s}} = \mathcal{S}(\varphi) \circ \mathrm{i}_{\mathrm{s}} = \varphi
\end{aligned}$$

and

$$\begin{aligned}
\log_\odot(\exp_\odot \varphi) &= \Big(\log_\star\big(\mathcal{S}(\exp_\odot \varphi)\big)\Big) \circ \mathrm{i}_{\mathrm{s}} && [\![\text{ eq. (2.4.2) }]\!] \\
&= \Bigg(\log_\star\underbrace{\Big(\mathcal{S}\Big(\big(\exp_\star \mathrm{D}(\varphi)\big) \circ \mathrm{i}_{\mathrm{s}}\Big)\Big)}_{\substack{=\exp_\star \mathrm{D}(\varphi) \\ [\![\text{ eq. (1.1.12) }]\!]}}\Bigg) \circ \mathrm{i}_{\mathrm{s}} && [\![\text{ eq. (2.4.1) }]\!] \\
&= \underbrace{\Big(\log_\star\big(\exp_\star \mathrm{D}(\varphi)\big)\Big)}_{\substack{=\mathrm{D}(\varphi) \\ [\![\text{ Prop. 1.3.9 (a) }]\!]}} \circ \mathrm{i}_{\mathrm{s}} = \mathrm{D}(\varphi) \circ \mathrm{i}_{\mathrm{s}} = \varphi && [\![\text{ eq. (1.4.7) }]\!] \quad \square
\end{aligned}$$

2.4.4 Definition. Assume that a u.a.u-product $\odot$ in $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ is given and $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_\mathsf{m}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$.

(a) For any $k \in \mathbb{N}_0$ we define the following map

$$T_{\boxplus k}\colon \begin{cases} \big(\operatorname{Lin}(\mathcal{D}, \mathbb{C})\big)^{\times k} \longrightarrow \operatorname{Lin}(\mathcal{D}, \mathbb{C}) \\ (\psi_1, \dots, \psi_k) \longmapsto \big(\mathrm{D}(\psi_1) \star \cdots \star \mathrm{D}(\psi_k)\big) \circ \mathrm{i_s}\,. \end{cases} \tag{2.4.5}$$

By $\psi_1 \boxplus \cdots \boxplus \psi_k$ we mean $T_{\boxplus k}(\psi_1, \dots, \psi_k)$ and $\psi^{\boxplus k}$ is the k-th power of the element ψ. We put $\psi^{\boxplus 0} := 0$.

(b) We set

$$[\,\cdot\,,\,\cdot\,]_\boxplus\colon \begin{cases} \operatorname{Lin}(\mathcal{D}, \mathbb{C}) \times \operatorname{Lin}(\mathcal{D}, \mathbb{C}) \longrightarrow \operatorname{Lin}(\mathcal{D}, \mathbb{C}) \\ (\varphi, \psi) \longmapsto \varphi \boxplus \psi - \psi \boxplus \varphi = [\mathrm{D}(\varphi), \mathrm{D}(\psi)]_\star \circ \mathrm{i_s}, \end{cases} \tag{2.4.6}$$

where $[\varphi_1, \varphi_2]_\star := \varphi_1 \star \varphi_2 - \varphi_2 \star \varphi_1$ denotes the Lie bracket induced by $\star$ for $\varphi_1,\ \varphi_2 \in \operatorname{Lin}\big(\operatorname{Sym}(\mathcal{D}), \mathbb{C}\big)$.

The next proposition shows why the k-fold operation $T_{\boxplus k}$ is so important for our framework.

2.4.5 Proposition. Assume that a u.a.u-product $\odot$ in $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ is given and $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_\mathsf{m}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$. Then,

(a) $T_{\boxplus k}\colon \big(\operatorname{Lin}(\mathcal{D}, \mathbb{C})\big)^{\times k} \longrightarrow \operatorname{Lin}(\mathcal{D}, \mathbb{C})$ is a $\mathbb{C}$-multilinear map.

(b) For all $\psi \in \operatorname{Lin}(\mathcal{D}, \mathbb{C})$ holds

$$\exp_\odot \psi = \sum_{k=0}^{\infty} \frac{1}{k!} \psi^{\boxplus k}, \tag{2.4.7}$$

$$\log_\odot \psi = \psi - \sum_{k=2}^{\infty} \frac{1}{k!} \big(\log_\odot \psi\big)^{\boxplus k}. \tag{2.4.8}$$

(c) $\big(\operatorname{Lin}(\mathcal{D}, \mathbb{C}), [\,\cdot\,,\,\cdot\,]_\boxplus\big)$ is a Lie algebra.

(d) $\forall\, \varphi, \psi \in \operatorname{Lin}(\mathcal{D}, \mathbb{C}),\ \forall\, n \in \mathbb{N},\ \forall\, (h_i)_{i\in[n]} \in \mathbb{N}_0^{\times n},\ \forall\, (k_i)_{i\in[n]} \in \mathbb{N}_0^{\times n}$:

$$\Big[\varphi^{h_1}\psi^{k_1} \dots \varphi^{h_n}\psi^{k_n}\Big]_\boxplus = \Big[(\mathrm{D}(\varphi))^{h_1}(\mathrm{D}(\psi))^{k_1} \dots (\mathrm{D}(\varphi))^{h_n}(\mathrm{D}(\psi))^{k_n}\Big]_\star \circ \mathrm{i_s}, \tag{2.4.9}$$

where we use Convention 1.2.14.

(e) $\forall\, k \in \mathbb{N} \setminus \{1, 2\},\ \forall\, (\psi_i)_{i\in[k]} \in \big(\operatorname{Lin}(\mathcal{D}, \mathbb{C})\big)^{\times k},\ \forall\, \ell \in \{\ell, \dots, k-1\} \subseteq \mathbb{N}$:

$$\begin{aligned} &\psi_1 \boxplus \cdots \boxplus \psi_{\ell-1} \boxplus \psi_\ell \boxplus \psi_{\ell+1} \boxplus \psi_{\ell+2} \boxplus \cdots \boxplus \psi_k \\ &\qquad - \psi_1 \boxplus \cdots \boxplus \psi_{\ell-1} \boxplus \psi_{\ell+1} \boxplus \psi_\ell \boxplus \psi_{\ell+2} \boxplus \cdots \boxplus \psi_k \\ &\qquad\qquad = \psi_1 \boxplus \cdots \boxplus \psi_{\ell-1} \boxplus [\psi_\ell, \psi_{\ell+1}]_\boxplus \boxplus \psi_{\ell+2} \boxplus \cdots \boxplus \psi_k. \end{aligned} \tag{2.4.10}$$

Proof: Ad **(a)**: We have to show that $T_{\boxplus k}$ is multilinear map, i. e., it is linear in each argument.

We claim that $T_{\boxplus k}$ is composition of linear maps. The canonical insertion map $\mathrm{i}_s\colon V \hookrightarrow \mathrm{Sym}(V)$ is by definition linear, the map $\mathrm{D}\colon \mathrm{Lin}(\mathscr{D},\mathbb{C}) \longrightarrow \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(\mathscr{D},\mathbb{C})$ is linear by Lemma 1.4.11 **(a)** and it is well known the convolution $\star$ determines the structure of an algebra on the dual space of $\mathrm{Sym}(\mathscr{D})$. This proves the assertion.

Ad **(b)**: For equation (2.4.7) we calculate

$$\begin{aligned}
\exp_{\odot}\psi &= \Big(\exp_{\star}\big(\mathrm{D}(\varphi)\big)\Big)\circ \mathrm{i}_s && [\![\text{ definition of } \exp_{\odot}(\,\cdot\,) \text{ in eq. (2.4.1) }]\!]\\
&= \left(\sum_{k=0}^{\infty}\frac{1}{k!}\big(\mathrm{D}(\varphi)\big)^{\star k}\right)\circ \mathrm{i}_s && [\![\text{ definition of } \exp_{\star}(\,\cdot\,) \text{ in Prop. 1.3.6 \textbf{(c)} }]\!]\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\Big(\big(\mathrm{D}(\varphi)\big)^{\star k}\circ \mathrm{i}_s\Big)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\psi^{\boxplus k} && [\![\text{ definition of } (\,\cdot\,)^{\boxplus k} \text{ in Def. 2.4.4 }]\!].
\end{aligned}$$

For equation (2.4.8) we calculate

$$\begin{aligned}
\psi &= \mathcal{S}(\psi)\circ \mathrm{i}_s \quad [\![\text{ UMP of symmetric algebra }]\!]\\
&= \exp_{\star}\big(\log_{\star}\mathcal{S}(\psi)\big)\circ \mathrm{i}_s \quad [\![\ \mathcal{S}(\psi)\in G\big(\mathrm{Sym}(\mathscr{D}),\mathbb{C}\big),\ \text{Prop. 1.3.9 \textbf{(b)} }]\!]\\
&= \left(\sum_{k=0}^{\infty}\frac{1}{k!}\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star k}\right)\circ \mathrm{i}_s \quad [\![\text{ def. of } \exp_{\star}\,\cdot\, \text{ in Prop. 1.3.6 \textbf{(c)} }]\!]\\
&= \Bigg(\underbrace{\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star 0}}_{=e_{\mathrm{Sym}(\mathscr{D}),\mathbb{C}}} + \underbrace{\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star 1}}_{=\log_{\star}\mathcal{S}(\psi)} + \sum_{k=2}^{\infty}\frac{1}{k!}\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star k}\Bigg)\circ \mathrm{i}_s\\
&\qquad [\![\ e_{\mathrm{Sym}(\mathscr{D}),\mathbb{C}} \text{ is the unital element in the convolution algebra } \mathrm{Lin}\big(\mathrm{Sym}(\mathscr{D}),\mathbb{C}\big)\ ]\!]\\
&= \left(\mathcal{S}(0) + \log_{\star}\mathcal{S}(\psi) + \sum_{k=2}^{\infty}\frac{1}{k!}\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star k}\right)\circ \mathrm{i}_s \quad [\![\text{ Conv. 1.1.9 \& Cor. 2.3.11 \textbf{(a)} }]\!]\\
&= \mathcal{S}(0)\circ \mathrm{i}_s + \log_{\star}\mathcal{S}(\psi)\circ \mathrm{i}_s + \sum_{k=2}^{\infty}\left(\frac{1}{k!}\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star k}\circ \mathrm{i}_s\right)\\
&= \log_{\odot}\psi + \sum_{k=2}^{\infty}\left(\frac{1}{k!}\big(\log_{\star}\mathcal{S}(\psi)\big)^{\star k}\circ \mathrm{i}_s\right) \quad [\![\text{ UMP of symmetric algebra \& eq. (2.4.2) }]\!]\\
&= \log_{\odot}\psi + \sum_{k=2}^{\infty}\left(\frac{1}{k!}\Big(\mathrm{D}\big(\log_{\star}\mathcal{S}(\psi)\circ \mathrm{i}_s\big)\Big)^{\star k}\circ \mathrm{i}_s\right) \quad [\![\text{ eq. (1.4.4b), then apply Lem. 1.4.11 \textbf{(b)} }]\!]\\
&= \log_{\odot}\psi + \sum_{k=2}^{\infty}\left(\frac{1}{k!}\Big(\mathrm{D}\big(\log_{\odot}\psi\big)\Big)^{\star k}\circ \mathrm{i}_s\right) \quad [\![\text{ def. of } \log_{\odot}\,\cdot\, \text{ in eq. (2.4.2) }]\!]
\end{aligned}$$

$$= \log_{\odot} \psi + \sum_{k=2}^{\infty} \frac{1}{k!} (\log_{\odot} \psi)^{\boxplus k} \quad [\![\text{ def. of } \cdot^{\boxplus k} \text{ in eq. (2.4.5) }]\!].$$

AD **(c)**: From Lemma 1.4.2 **(b)** we know that $(\mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(V), \mathbb{C}), [\,\cdot\,,\,\cdot\,]_{\star})$ is a Lie subalgebra of $\mathrm{Lin}(\mathscr{D}, \mathbb{C})$. Thus $[\,\cdot\,,\,\cdot\,]_{\boxplus} \colon \mathrm{Lin}(\mathscr{D}, \mathbb{C}) \times \mathrm{Lin}(\mathscr{D}, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathscr{D}, \mathbb{C})$ is $\mathbb{C}$-bilinear map. Furthermore, $[\varphi, \varphi]_{\boxplus} = 0$, since $[\,\cdot\,,\,\cdot\,]_{\star}$ is a Lie bracket. For the Jacobi identity we observe the following for any $(\varphi_i)_{i\in[3]} \in \big(\mathrm{Lin}(\mathscr{D}, \mathbb{C})\big)^{\times 3}$

$$\begin{aligned}
\big[[\varphi_1, \varphi_2]_{\boxplus}, \varphi_3\big]_{\boxplus} &= \Big[\mathrm{D}\big([\mathrm{D}(\varphi_1), \mathrm{D}(\varphi_2)]_{\star} \circ \mathrm{i_s}\big), \mathrm{D}(\varphi_3)\Big]_{\star} \circ \mathrm{i_s} \quad [\![\text{ eq. (2.4.6) }]\!] \\
&= \big[[\mathrm{D}(\varphi_1), \mathrm{D}(\varphi_2)]_{\star}, \mathrm{D}(\varphi_3)\big]_{\star} \circ \mathrm{i_s} \\
&\qquad [\![\ [\mathrm{D}(\varphi_1), \mathrm{D}(\varphi_2)]_{\star} \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(\mathscr{D}), \mathbb{C}) \ \& \text{ Lemma 1.4.11 } \textbf{(b)} \ ]\!].
\end{aligned}$$

By this result the Jacobi identity for $[\,\cdot\,,\,\cdot\,]_{\boxplus}$ follows from the Jacobi identity of $[\,\cdot\,,\,\cdot\,]_{\star}$.
AD **(d)**: Since $\big(\mathrm{Lin}(\mathscr{D}, \mathbb{C}), [\,\cdot\,,\,\cdot\,]_{\boxplus}\big)$ is a Lie algebra, the right hand side of equation (2.4.9) is well defined according to equation (1.2.18). We claim

$$\begin{aligned}
\forall \varphi, \psi \in \mathrm{Lin}(\mathscr{D}, \mathbb{C}),\ \forall m \in \mathbb{N} \colon & \underbrace{[\varphi, [\varphi, \dots, [\varphi,}_{m \text{ times}} \psi]_{\boxplus}]_{\boxplus}]_{\boxplus} \\
&= \underbrace{[\mathrm{D}(\varphi), [\mathrm{D}(\varphi), \dots, [\mathrm{D}(\varphi),}_{m \text{ times}} \mathrm{D}(\psi)]_{\star}]_{\star}]_{\star} \circ \mathrm{i_s}\,.
\end{aligned} \tag{I}$$

We show this by induction over m. For $m = 1$ this is the definition of $[\,\cdot\,,\,\cdot\,]_{\boxplus}$ provided in equation (2.4.6). Assume equation (I) holds for some $m \in \mathbb{N}$, then we perform the induction step $m \to m+1$ and calculate

$$\begin{aligned}
&\underbrace{[\varphi, [\varphi, \dots, [\varphi,}_{(m+1) \text{ times}} \psi]_{\boxplus}]_{\boxplus}]_{\boxplus} \\
&= \Big[\mathrm{D}(\varphi), \mathrm{D}\Big(\underbrace{[\varphi, [\varphi, \dots, [\varphi,}_{m \text{ times}} \psi]_{\boxplus}]_{\boxplus}]_{\boxplus}\Big)\Big]_{\star} \circ \mathrm{i_s} \quad [\![\text{ def. of } [\,\cdot\,,\,\cdot\,]_{\boxplus} \text{ in eq. (2.4.6) }]\!] \\
&= \Big[\mathrm{D}(\varphi), \mathrm{D}\Big(\underbrace{[\mathrm{D}(\varphi), [\mathrm{D}(\varphi), \dots, [\mathrm{D}(\varphi),}_{m \text{ times}} \mathrm{D}(\psi)]_{\star}]_{\star}]_{\star} \circ \mathrm{i_s}\Big)\Big]_{\star} \circ \mathrm{i_s} \quad [\![\text{ induction hypothesis }]\!] \\
&= \Big[\mathrm{D}(\varphi), \underbrace{[\mathrm{D}(\varphi), [\mathrm{D}(\varphi), \dots, [\mathrm{D}(\varphi),}_{m \text{ times}} \mathrm{D}(\psi)]_{\star}]_{\star}]_{\star}\Big]_{\star} \circ \mathrm{i_s} \quad [\![\text{ Lemma 1.4.11 }]\!] \\
&= \underbrace{[\mathrm{D}(\varphi), [\mathrm{D}(\varphi), \dots, [\mathrm{D}(\varphi),}_{(m+1) \text{ times}} \mathrm{D}(\psi)]_{\star}]_{\star}]_{\star} \circ \mathrm{i_s}\,.
\end{aligned}$$

In particular, choosing $\psi = \varphi$ in equation (I) and using the assertion of Lemma 1.4.11 we have $\forall \psi \in \mathrm{Lin}(\mathscr{D}, \mathbb{C}),\ \forall m \in \mathbb{N}$:

$$\mathrm{D}\Big(\underbrace{[\psi, [\psi, \dots [\psi,}_{m \text{ times}} \psi]_{\boxplus}]_{\boxplus}]_{\boxplus}\Big) = \Big(\underbrace{[\mathrm{D}(\psi), [\mathrm{D}(\psi), \dots [\mathrm{D}(\psi),}_{m \text{ times}} \mathrm{D}(\psi)]_{\star}]_{\star}]_{\star}. \tag{II}$$

Combining the results of equations (I) and (II), we obtain

$$\forall \varphi, \psi \in \operatorname{Lin}(\mathscr{D}, \mathbb{C}),\ \forall h_1 \in \mathbb{N}\ \forall k_1 \in \mathbb{N}:$$

$$[\underbrace{\varphi, [\varphi, \dots, [\varphi,}_{h_1 \text{ times}} [\underbrace{\psi, [\psi, \dots [\psi, \psi}_{k_1 \text{ times}}]]_{\boxplus}]_{\boxplus}]_{\boxplus}]_{\boxplus}$$

$$= [\underbrace{\mathrm{D}(\varphi), [\mathrm{D}(\varphi), \dots, [\mathrm{D}(\varphi),}_{h_1 \text{ times}} [\underbrace{\mathrm{D}(\psi), [\mathrm{D}(\psi), \dots [\mathrm{D}(\psi), \mathrm{D}(\psi)}_{k_1 \text{ times}}]]_{\star}]_{\star}]_{\star}]_{\star} \circ \mathrm{i_s}$$

which shows the induction base for $n = 1$ in equation (2.4.9). By several applications of equation (I) and Lemma 1.4.11 the induction step $n \to n+1$ in equation (2.4.9) is easily done.
Ad **(e)**: We calculate

$$\begin{aligned}
&\psi_1 \boxplus \cdots \boxplus \psi_\ell \boxplus \psi_{\ell+1} \boxplus \cdots \boxplus \psi_k - \psi_1 \boxplus \cdots \boxplus \psi_{\ell+1} \boxplus \psi_\ell \boxplus \cdots \boxplus \psi_k \\
&\quad = \big(\mathrm{D}(\psi_1) \star \cdots \star \mathrm{D}(\psi_\ell) \star \mathrm{D}(\psi_{\ell+1}) \star \cdots \star \mathrm{D}(\psi_k) \\
&\qquad - \mathrm{D}(\psi_1) \star \cdots \star \mathrm{D}(\psi_{\ell+1}) \star \mathrm{D}(\psi_\ell) \star \cdots \star \mathrm{D}(\psi_k)\big) \circ \mathrm{i_s} \quad [\![\text{ Def. 2.4.4 }]\!] \\
&\quad = \big(\mathrm{D}(\psi_1) \star \cdots \star [\mathrm{D}(\psi_\ell), \mathrm{D}(\psi_{\ell+1})]_\star \star \cdots \star \mathrm{D}(\psi_k)\big) \circ \mathrm{i_s} \\
&\quad = \Big(\mathrm{D}(\psi_1) \star \cdots \star \mathrm{D}\big([\mathrm{D}(\psi_\ell), \mathrm{D}(\psi_{\ell+1})]_\star \circ \mathrm{i_s}\big) \star \cdots \star \mathrm{D}(\psi_k)\Big) \circ \mathrm{i_s} \\
&\qquad\qquad [\![\ [\mathrm{D}(\psi_\ell), \mathrm{D}(\psi_{\ell+1})]_\star \in \mathsf{Der}_{(\varepsilon,\varepsilon)}(\mathrm{Sym}(\mathscr{D}^d), \mathbb{C}) \ \&\ \text{Lemma 1.4.11 } \mathbf{(b)} \ ]\!] \\
&\quad = \Big(\mathrm{D}(\psi_1) \star \cdots \star \mathrm{D}\big([\psi_\ell, \psi_{\ell+1}]_{\boxplus}\big) \star \cdots \star \mathrm{D}(\psi_k)\Big) \circ \mathrm{i_s} \quad [\![\text{ Def. 2.4.4 }]\!] \\
&\quad = \psi_1 \boxplus \cdots \boxplus [\psi_\ell, \psi_{\ell+1}]_{\boxplus} \boxplus \cdots \boxplus \psi_k \quad [\![\text{ Def. 2.4.4 }]\!].
\end{aligned}$$
□

The next lemma might help us to get a rough impression why the operation $\boxplus$ does not need to be associative. A concrete example is postponed and will be delivered in Example 5.1.7. On the other hand, the next lemma will be important in Section 2.5 in particular for Lemma 2.5.11 and Lemma 2.5.12 whenever a given u.a.u.-product possesses so-called highest coefficients.

2.4.6 Lemma. Assume that a u.a.u-product $\odot$ in $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ is given and $\big((\mathscr{D}, (\mathscr{D}_n)_{n\in\mathbb{N}_0}, (\mathscr{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_\mathsf{m}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathscr{D}_0 = \{0\}$. For any $k \in \mathbb{N}$ and for any k-tuple $(\varphi_i)_{i\in[k]} \in \big(\operatorname{Lin}(\mathscr{D}, \mathbb{C})\big)^{\times k}$

$$\forall b \in \mathrm{Sym}(\mathscr{D}) \colon (\varphi_1 \boxplus \cdots \boxplus \varphi_k)(b) = \frac{\partial}{\partial t_1 \dots \partial t_k} \Big(\big((t_1\varphi_1) \circledast \cdots \circledast (t_k\varphi_k)\big)(b) \Big) \Big|_{t_1 = \ldots = t_k = 0} \tag{2.4.11}$$

holds.

Proof: We can calculate for any $b \in \mathscr{D}$

$$\begin{aligned}
&(\varphi_1 \boxplus \cdots \boxplus \varphi_n)(b) \\
&\quad = \big(\mathrm{D}(\varphi_1) \star \cdots \star \mathrm{D}(\varphi_k)\big)\big(\mathrm{i_s}(b)\big) \quad [\![\text{ eq. (2.4.5) }]\!] \\
&\quad = \bigg(\Big(\frac{\mathrm{d}}{\mathrm{d}t_1} \big(\mathcal{S}(t_1\varphi_1)(\,\cdot\,)\big)\Big|_{t_1=0} \Big) \star \cdots \star \Big(\frac{\mathrm{d}}{\mathrm{d}t_k} \big(\mathcal{S}(t_k\varphi_k)(\,\cdot\,)\big)\Big|_{t_k=0} \Big) \bigg) \big(\mathrm{i_s}(b)\big)
\end{aligned}$$

⟦ Lem. 1.4.9, Prop. 1.4.10 ⟧

$$= \left(\left(\left(\frac{\mathrm{d}}{\mathrm{d}t_1}\big(\mathcal{S}(t_1\varphi_1)(\,\cdot\,)\big)\Big|_{t_1=0}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}}{\mathrm{d}t_k}\big(\mathcal{S}(t_k\varphi_k)(\,\cdot\,)\big)\Big|_{t_k=0}\right)\right) \circ \tilde{\Delta}^{(n)}\right)\big(\mathrm{i_s}(b)\big)$$

⟦ let be $\tilde{\Delta}$ the comultiplication coming from the Lachs functor, $\exists$ similar eq. as in eq. (2.3.8) but here $\tilde{\Delta}\colon \mathrm{Sym}(\mathcal{D}) \longrightarrow \mathrm{Sym}(\mathcal{D}) \otimes \mathrm{Sym}(\mathcal{D})$ ⟧

$$= \frac{\partial}{\partial t_1 \ldots \partial t_n}\left(\left(\big(\mathcal{S}(t_1\varphi_1) \otimes \cdots \otimes \mathcal{S}(t_n\varphi_n)\big) \circ \tilde{\Delta}^{(n)}\right)\big(\mathrm{i_s}(b)\big)\right)\Big|_{t_1=\ldots=t_n=0}$$

⟦ write $\tilde{\Delta}^{(n)}\big(\mathrm{i_s}(b)\big)$ as a linear combination of basis vectors ⟧

$$= \frac{\partial}{\partial t_1 \ldots \partial t_n}\Big(\big(\mathcal{S}(t_1\varphi_1) \star \cdots \star \mathcal{S}(t_n\varphi_n)\big)\big(\mathrm{i_s}(b)\big)\Big)\Big|_{t_1=\ldots=t_n=0}$$

$$= \frac{\partial}{\partial t_1 \ldots \partial t_n}\Big(\mathcal{S}\big((t_1\varphi_1) \circledast \cdots \circledast (t_n\varphi_n)\big)\big(\mathrm{i_s}(b)\big)\Big)\Big|_{t_1=\ldots=t_n=0} \quad [\![\text{ Cor. 2.3.11 \textbf{(b)} }]\!]$$

$$= \frac{\partial}{\partial t_1 \ldots \partial t_n}\Big(\big((t_1\varphi_1) \circledast \cdots \circledast (t_n\varphi_n)\big)(b)\Big)\Big|_{t_1=\ldots=t_n=0} \quad [\![\text{ UMP of } \mathrm{Sym}(\mathcal{D})]\!].$$ □

2.4.7 Lemma. Assume that a u.a.u-product $\odot$ in $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ is given and $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$. If $\odot$ is symmetric, then $[\,\cdot\,,\,\cdot\,]_{\boxplus} = 0$.

Proof: We calculate for $\varphi_1,\ \varphi_2 \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$ and some $b \in \mathcal{D}$

$$[\varphi_1, \varphi_2]_{\boxplus}(b)$$

$$= (\varphi_1 \boxplus \varphi_2 - \varphi_2 \boxplus \varphi_1)(b) \quad [\![\text{ eq. (2.4.6) }]\!]$$

$$= \frac{\partial^2}{\partial t_1\, \partial t_2}\Big(\big((t_1\varphi_1) \circledast (t_2\varphi_2)\big) - \big((t_1\varphi_2) \circledast (t_2\varphi_1)\big)\Big)(b)\Big|_{t_1=t_2=0} \quad [\![\text{ eq. (2.4.11) }]\!]$$

$$= \frac{\partial^2}{\partial t_1\, \partial t_2}\left(\Big(\big((t_1\varphi_1) \odot (t_2\varphi_2)\big) \circ \Delta\Big) - \Big(\big((t_1\varphi_2) \odot (t_2\varphi_1)\big) \circ \Delta\Big)\right)(b)\Big|_{t_1=t_2=0} \quad [\![\text{ eq. (2.3.1) }]\!]$$

$$= \frac{\partial^2}{\partial t_1\, \partial t_2}\left(\Big(\big((t_1\varphi_1) \odot (t_2\varphi_2)\big) \circ \Delta\Big) - \Big(\big((t_2\varphi_1) \odot (t_1\varphi_2)\big) \circ \Delta\Big)\right)(b)\Big|_{t_1=t_2=0} \quad [\![\ \odot \text{ is symmetric }]\!]$$

$$= \frac{\partial^2}{\partial t_1\, \partial t_2}\left(\Big(\big((t_1\varphi_1) \odot (t_2\varphi_2)\big) \circ \Delta\Big) - \Big(\big((t_1\varphi_1) \odot (t_2\varphi_2)\big) \circ \Delta\Big)\right)(b)\Big|_{t_1=t_2=0}$$

⟦ rename indices, Schwarz's theorem ⟧

$$= 0.$$ □

2.4.8 Theorem ("BCH-formula" for convolution w. r. t. $\odot$ [MS17, eq. 7.3]). Assume that a u.a.u-product $\odot$ in $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ is given and $\big((\mathcal{D}, (\mathcal{D}_n)_{n\in\mathbb{N}_0}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ is a comonoid in the tensor category $(\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ with $\mathcal{D}_0 = \{0\}$. Then, for the convolution exponential defined in equation (2.4.1) holds:

(a) $\forall\, \varphi, \psi \in \mathrm{Lin}(\mathscr{D}, \mathbb{C})$:

$$\exp_{\odot} \varphi \circledast \exp_{\odot} \psi = \exp_{\odot}\Big(\mathsf{BCH}_{\boxplus}(\varphi, \psi)\Big), \tag{2.4.12}$$

(b) $\forall\, \varphi, \psi \in \mathrm{Lin}(\mathscr{D}, \mathbb{C})$:

$$\log_{\odot}(\varphi \circledast \psi) = \mathsf{BCH}_{\boxplus}\big(\log_{\odot} \varphi, \log_{\odot} \psi\big). \tag{2.4.13}$$

Here, $\mathsf{BCH}_{\boxplus}(\,\cdot\,,\,\cdot\,)$ denotes the series in the sense of equation(1.2.21), i. e., a series of right-nested brackets $[(\,\cdot\,)^{h_1} \ldots (\,\cdot\,)^{k_n}]_{\boxplus}$ in the Lie algebra $\big(\mathrm{Lin}(\mathscr{D}, \mathbb{C}), [\,\cdot\,,\,\cdot\,]_{\boxplus}\big)$.

Proof: Ad **(a)**: We calculate

$$\begin{aligned}
&\exp_{\odot} \varphi \circledast \exp_{\odot} \psi \\
&\quad = (\exp_{\odot} \varphi \odot \exp_{\odot} \psi) \circ \Delta && [\![\text{ definition of } \odot \text{ in eq. (2.3.33) }]\!] \\
&\quad = \Big(\big(\exp_{\star} \mathrm{D}(\varphi) \circ \mathrm{i_s}\big) \odot \big(\exp_{\star} \mathrm{D}(\psi) \circ \mathrm{i_s}\big)\Big) \circ \Delta && [\![\text{ definition of } \exp_{\odot}(\,\cdot\,) \text{ in eq. (2.4.1) }]\!] \\
&\quad = \Big(\mathcal{S}(\exp_{\star} \mathrm{D}(\varphi) \circ \mathrm{i_s}) \otimes \mathcal{S}(\exp_{\star} \mathrm{D}(\psi) \circ \mathrm{i_s})\Big) \circ \sigma_{\mathscr{D},\mathscr{D}} \circ \Delta && [\![\text{ Proposition 2.3.6 }]\!] \\
&\quad = \Big(\exp_{\star} \mathrm{D}(\varphi) \otimes \exp_{\star} \mathrm{D}(\psi)\Big) \circ \sigma_{\mathscr{D},\mathscr{D}} \circ \Delta && \left[\!\!\left[\begin{array}{c} \exp_{\star} \mathrm{D}(\varphi) \in \mathrm{Alg}(\mathrm{Sym}(\mathscr{D}^d), \mathbb{C}), \\ \text{UMP of symm. tensor algebra} \end{array}\right]\!\!\right] \\
&\quad = \Big(\exp_{\star} \mathrm{D}(\varphi) \otimes \exp_{\star} \mathrm{D}(\psi)\Big) \circ \mathcal{S}(\sigma_{\mathscr{D},\mathscr{D}} \circ \Delta) \circ \mathrm{i_s} && [\![\text{ UMP of symm. tensor algebra }]\!] \\
&\quad = \Big(\exp_{\star} \mathrm{D}(\varphi) \star \exp_{\star} \mathrm{D}(\psi)\Big) \circ \mathrm{i_s} && \left[\!\!\left[\begin{array}{l} \text{Cor. 2.3.11 } \textbf{(a)} \text{ says} \\ \mathcal{S}(\sigma_{\mathscr{D},\mathscr{D}} \circ \Delta) = \mathcal{S}(\sigma_{\mathscr{D},\mathscr{D}}) \circ \mathrm{Sym}(\Delta) \\ \text{is comultiplication for } \mathrm{Sym}(\mathscr{D}), \\ \text{then use def. of convolution } \star \\ \text{from eq. (1.1.21)} \end{array}\right]\!\!\right] \\
&\quad = \bigg(\exp_{\star}\Big(\mathsf{BCH}_{*}\big(\mathrm{D}(\varphi), \mathrm{D}(\psi)\big)\Big)\bigg) \circ \mathrm{i_s} && [\![\text{ Theorem 1.3.10 }]\!] \\
&\quad = \bigg(\exp_{\star}\Big(\mathrm{D}\Big(\mathsf{BCH}_{\boxplus}(\varphi, \psi)\Big)\Big)\bigg) \circ \mathrm{i_s} && [\![\text{ eq. (2.4.9) \& Prop. 1.4.10 }]\!] \\
&\quad = \exp_{\odot}\big(\mathsf{BCH}_{\boxplus}(\varphi, \psi)\big) && [\![\text{ definition of } \exp_{\odot}(\,\cdot\,) \text{ in eq. (2.4.1) }]\!].
\end{aligned}$$

Ad **(b)**: We calculate

$$\begin{aligned}
&\log_{\odot}(\varphi \circledast \psi) \\
&\quad = \log_{\star}\big(\mathcal{S}(\varphi \circledast \psi)\big) \circ \mathrm{i_s} && [\![\text{ definition of } \log_{\odot} \text{ in eq. (2.4.2) }]\!] \\
&\quad = \log_{\star}\Big(\mathcal{S}(\varphi) \star \mathcal{S}(\psi)\Big) \circ \mathrm{i_s} && [\![\ \mathcal{S} \text{ is monoid homomorphism by Cor. 2.3.11 } \textbf{(b)}\ ]\!]
\end{aligned}$$

$$= \log_\star\Big(\exp_\star\big(\log_\star \mathcal{S}(\varphi)\big) \star \exp_\star\big(\log_\star \mathcal{S}(\psi)\big)\Big) \circ \mathrm{i_s} \quad \left[\!\!\left[\begin{array}{l}\text{because } \mathcal{S}(\varphi) \in G(\mathcal{S}(\mathcal{D}), \mathbb{C})\\ \text{Prop. 1.3.9 } \textbf{(b)} \text{ applies}\end{array}\right]\!\!\right]$$

$$= \log_\star\Bigg(\exp_\star\Big(\mathsf{BCH}_\star\big(\log_\star \mathcal{S}(\varphi), \log_\star \mathcal{S}(\psi)\big)\Big)\Bigg) \circ \mathrm{i_s} \quad \left[\!\!\left[\begin{array}{l}\log_\star \mathcal{S}(\varphi) \in \mathfrak{g}(\mathcal{S}(\mathcal{D}), \mathbb{C}),\\ \text{Thm. 1.3.10}\end{array}\right]\!\!\right]$$

$$= \Big(\mathsf{BCH}_\star\big(\log_\star \mathcal{S}(\varphi), \log_\star \mathcal{S}(\psi)\big)\Big) \circ \mathrm{i_s} \quad [\![\text{ Prop. 1.3.9 } \textbf{(a)}]\!]$$

$$= \Big(\mathsf{BCH}_\star\big(\mathrm{D}(\log_\odot \varphi), \mathrm{D}(\log_\odot \psi)\big)\Big) \circ \mathrm{i_s} \quad [\![\text{ def. of } \log_\odot \text{ in eq. (2.4.2), Prop. 1.4.10 }]\!]$$

$$= \mathsf{BCH}_\boxplus\big(\log_\odot \varphi, \log_\odot \psi\big) \quad [\![\text{ eq. (2.4.9) }]\!]. \qquad \square$$

2.4.9 Corollary. If the u.a.u-product $\odot$ is additionally symmetric, then

(a) $\forall\, \varphi, \psi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$:

$$\exp_\odot \varphi \circledast \exp_\odot \psi = \exp_\odot\big(\varphi + \psi\big), \tag{2.4.14}$$

(b) $\forall\, \varphi, \psi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$:

$$\log_\odot(\varphi \circledast \psi) = \log_\odot \varphi + \log_\odot \psi. \tag{2.4.15}$$

Proof: We have two possibilities to prove these equations. The first possibility is: Lemma 2.4.7 tells us that if $\odot$ is symmetric, then $[\,\cdot\,,\,\cdot\,]_\boxplus = 0$. The equations then follow from the BCH-formula of Theorem 2.4.8.

The second possibility for a proof relies on the statements presented in Remark 1.3.11 and therefore in total avoids once again any "strong" topological arguments as presented in Section 1.2. By Lemma 2.4.7 we have: if $\odot$ is symmetric, then this implies $[\,\cdot\,,\,\cdot\,]_\boxplus = 0$. Equation (2.4.6) then yields

$$\forall \varphi,\, \psi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})\colon [\mathrm{D}(\varphi), \mathrm{D}(\psi)]_\star \circ \mathrm{i_s} = 0.$$

Since $\mathsf{Der}_{(\mathcal{S}(0),\mathcal{S}(0))}(\mathrm{Sym}(\mathcal{D}), \mathbb{C})$ forms a Lie subalgebra in $(\mathrm{Lin}(\mathrm{Sym}(\mathcal{D}), \mathbb{C}), [\,\cdot\,,\,\cdot\,]_\star$ (because of Proposition 2.4.5 **(c)**), we can conclude $[\mathrm{D}(\varphi), \mathrm{D}(\psi)]_\star \in \mathsf{Der}_{(\mathcal{S}(0),\mathcal{S}(0))}(\mathrm{Sym}(\mathcal{D}), \mathbb{C})$. From Theorem 1.4.3 we can now see that

$$\forall\, n \in \mathbb{N}_0 \setminus \{1\},\ \forall\, b \in \mathrm{Sym}^n(\mathcal{D})\colon [\mathrm{D}(\varphi), \mathrm{D}(\psi)]_\star(b) = 0.$$

Hence, we obtain $[\mathrm{D}(\varphi), \mathrm{D}(\psi)]_\star = 0$. Now, the claims follow from similar calculations as we did in the proof of Theorem 2.4.8 but instead of the application of Theorem 1.3.10 we may now apply equation (1.3.13). $\square$

2.4.10 Convention. Let $m \in \mathbb{N}$. Assume we are given an m-tuple of $\mathbb{C}$-vector spaces $(V_i)_{i\in[m]}$. Furthermore let us denote by $\Delta_i\colon \mathrm{T}(V_i) \longrightarrow \mathrm{T}(V_i) \sqcup \mathrm{T}(V_i)$ the map, defined in the sense of equation (2.2.10) which turns $\mathrm{T}(V_i)$ into a $\mathbb{N}_0$-graded dual semigroup and with primitive comultiplication Δ_i (Example 2.2.8). Each V_i is trivially graded , i. e., $V_i^{\langle 1\rangle} = V_i$ and $\forall\, n \in \mathbb{N}_0 \setminus \{1\}\colon V_i^{\langle n\rangle} = \{0\}$. Here, we have used angled brackets for the index of the $\mathbb{N}_0$-grading. By this $\mathbb{N}_0$-grading for each V_i, the direct sum $\bigoplus_{i=1}^m V_i$ is $\mathbb{N}_0$-graded and by Remark 2.2.7 **(a)** the tensor algebra $\mathrm{T}(\bigoplus_{i=1}^m V_i)$ is $\mathbb{N}_0$-graded. From equation (2.2.6) we have $\bigsqcup_{i=1}^m \mathrm{T}(V_i) \cong \mathrm{T}(\bigoplus_{i=1}^m V_i)$ and we can even see that this is an isomorphism of $\mathbb{N}_0$-graded algebras (Remark 2.2.7 **(b)**).

Then, similar to the assertion of Lemma 2.2.4 the triple

$$\Big(\bigsqcup_{i=1}^{m} \mathrm{T}(V_i), \coprod_{i=1}^{m} \Delta_i, 0\Big) \tag{2.4.16}$$

defines a comonoid in the category $\mathsf{Alg}_{\mathsf{m}}^{\mathbb{N}_0}$, where the 0-th part of the $\mathbb{N}_0$-grading is $\{0\}$. In equation (2.4.16) we no longer explicitly mention the canonical isomorphism of equation (2.2.4). By the term $\mathbb{N}_0$*-graded* m*-faced dual semigroup with primitive comultiplication* $\coprod_{i=1}^{m} \Delta_i$ *(w. r. t.* $(V_i)_{i\in[m]}$*)* we mean the triple of equation (2.4.16).

2.4.11 Remark. Let V be a vector space and assume $(e_i)_{i\in I} \in V^I$ is a basis for V. Then,

$$\big(e_{i_1} \otimes \cdots \otimes e_{i_k}\big)_{k\in\mathbb{N},(i_j)_{j\in[k]}\in I^{\times k}} \tag{2.4.17}$$

is a basis for $\mathrm{T}(V)$. For a proof of this fact we refer to [BF12, Prop. 2.35].

2.4.12 Theorem (Moment-cumulant formula of a u.a.u.-product [MS17, eq. (7.4)]). Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Let $\big(\mathcal{A}_i, (\mathcal{A}_i^{(\ell)})_{\ell\in[m]}, \varphi_i\big)_{i\in[2]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times 2}$. Define for any $\ell \in [m]$

$$\mathrm{j}_1^{(\ell)} := \mathcal{T}(\mathsf{id}_{\mathcal{A}_1^{(\ell)}} \oplus 0)\colon\ \mathrm{T}(\mathcal{A}_1^{(\ell)} \oplus \mathcal{A}_2^{(\ell)}) \longrightarrow \mathcal{A}_1^{(\ell)} \tag{2.4.18}$$

and

$$\mathrm{j}_2^{(\ell)} := \mathcal{T}(0 \oplus \mathsf{id}_{\mathcal{A}_2^{(\ell)}})\colon\ \mathrm{T}(\mathcal{A}_1^{(\ell)} \oplus \mathcal{A}_2^{(\ell)}) \longrightarrow \mathcal{A}_2^{(\ell)}. \tag{2.4.19}$$

Put

$$\forall\, \ell \in [m]\colon V_\ell := \mathcal{A}_1^{(\ell)} \oplus \mathcal{A}_2^{(\ell)} \tag{2.4.20}$$

and consider $(\bigsqcup_{\ell=1}^{m} \mathrm{T}(V_\ell), \coprod_{\ell=1}^{m} \Delta_\ell, 0)$ as the $\mathbb{N}_0$-graded m-faced dual semigroup with primitive comultiplication $\coprod_{\ell=1}^{m} \Delta_\ell$ using Convention 2.4.10. Then,

$$\varphi_1 \odot \varphi_2 = \Bigg(\underbrace{\Big(\varphi_1 \circ \Big(\coprod_{\ell\in[m]} \mathrm{j}_1^{(\ell)}\Big)\Big)}_{=:\tilde{\varphi}_1} \circledast \underbrace{\Big(\varphi_2 \circ \Big(\coprod_{\ell\in[m]} \mathrm{j}_2^{(\ell)}\Big)\Big)}_{=:\tilde{\varphi}_2}\Bigg) \circ \mathsf{inc}_{\mathcal{A}_1\sqcup\mathcal{A}_2, \bigsqcup_{\ell=1}^{m} \mathrm{T}(V_\ell)}, \tag{2.4.21}$$

where $\circledast$ denotes the convolution w. r. t. the primitive comultiplication $\coprod_{\ell=1}^{m} \Delta_\ell$ and $\mathsf{inc}_{\mathcal{A}_1\sqcup\mathcal{A}_2, \bigsqcup_{\ell=1}^{m} \mathrm{T}(V_\ell)}$ is a canonical inclusion of vector spaces. Furthermore, it holds that

$$\varphi_1 \odot \varphi_2 = \Big(\exp_\odot\Big(\mathsf{BCH}_{\boxplus}(\log_\odot \tilde{\varphi}_1, \log_\odot \tilde{\varphi}_2)\Big)\Big) \circ \mathsf{inc}_{\mathcal{A}_1\sqcup\mathcal{A}_2, \bigsqcup_{j=1}^{m} \mathrm{T}(V_j)}. \tag{2.4.22}$$

Proof: Let $(a_{1,i}^{(j)})_{i\in I_1^{(j)}} \in (\mathcal{A}_1^{(j)})^{I_1^{(j)}}$ denote a basis for $\mathcal{A}_1^{(j)}$ with index set $I_1^{(j)}$ for each $j \in [m]$. Likewise, let $(a_{2,i}^{(j)})_{i\in I_2^{(j)}} \in (\mathcal{A}_2^{(j)})^{I_2^{(j)}}$ denote a basis sequence for $\mathcal{A}_2^{(j)}$ with index set $I_2^{(j)}$ for each $j \in [m]$. Let $\iota_i^{(j)}\colon \mathcal{A}_i^{(j)} \hookrightarrow \mathcal{A}_1 \sqcup \mathcal{A}_2$ denote the homomorphic insertion map for each $i \in [2]$ and

$j \in [m]$. It can be shown that

$$\left\{ \iota_{\varepsilon_1}^{(\delta_1)}\left(a_{\varepsilon_1,k_1}^{(\delta_1)}\right) \cdot \dots \cdot \iota_{\varepsilon_n}^{(\delta_n)}\left(a_{\varepsilon_n,k_n}^{(\delta_n)}\right) \,\middle|\, \begin{array}{l} n \in \mathbb{N},\ (\varepsilon_i, \delta_i)_{i\in[n]} \in \mathbb{A}([2]\times[m]), \\ (k_i)_{i\in[n]} \in \prod_{j=1}^n I_{\varepsilon_j}^{(\delta_j)} \end{array} \right\}$$

determines a basis for $\mathscr{A}_1 \sqcup \mathscr{A}_2$. This can be seen, if we take a possible realization of the free product of algebras $\mathscr{A}_1 \sqcup \mathscr{A}_2$, discussed in Remark 2.2.7 **(b)**. By the above fact for a basis of $\mathscr{A}_1 \sqcup \mathscr{A}_2$ we can define the following linear map on basis elements of $\mathscr{A}_1 \sqcup \mathscr{A}_2$

$$\tilde{\iota}\colon \begin{cases} \left(\coprod_{i=1}^m \mathscr{A}_1^{(i)}\right) \sqcup \left(\coprod_{i=1}^m \mathscr{A}_2^{(i)}\right) \longrightarrow \coprod_{i=1}^m \mathrm{T}\big(\underbrace{\mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)}}_{\equiv V_i}\big) \\ \iota_{\varepsilon_1}^{(\delta_1)}\left(a_{\varepsilon_1,k_1}^{(\delta_1)}\right) \cdot \dots \cdot \iota_{\varepsilon_n}^{(\delta_n)}\left(a_{\varepsilon_n,k_n}^{(\delta_n)}\right) \longmapsto \mathsf{can}\big(a_{\varepsilon_1,k_1}^{(\delta_1)} \otimes \dots \otimes a_{\varepsilon_n,k_n}^{(\delta_n)}\big), \end{cases} \tag{I}$$

where $\mathsf{can}\colon\ \mathrm{T}\big(\bigoplus_{i=1}^m \mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)}\big) \longrightarrow \coprod_{i=1}^m \mathrm{T}\big(\mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)}\big)$ is the canonical isomorphism of algebras (Example 2.2.8). The map $\tilde{\iota}$ is injective and therefore deserves to be seen as a canonical inclusion map of vector spaces, i. e., $\tilde{\iota} = \mathsf{inc}_{\mathsf{inc}_{\mathscr{A}_1 \sqcup \mathscr{A}_2, \coprod_{\ell=1}^m \mathrm{T}(V_\ell)}}$. Now, we claim that

$$\mathsf{id}_{\mathscr{A}_1 \sqcup \mathscr{A}_2} = \left(\left(\coprod_{\ell=1}^m \mathrm{j}_1^{(\ell)}\right) \amalg \left(\coprod_{\ell=1}^m \mathrm{j}_2^{(\ell)}\right)\right) \circ \left(\coprod_{i=1}^m \Delta_i\right) \circ \tilde{\iota}. \tag{II}$$

Since the right hand side of the above equation is a composition of linear maps, it suffices to show the above equation for basis elements of $\mathscr{A}_1 \sqcup \mathscr{A}_2$. From equation (I) we can see how $\tilde{\iota}$ maps basis elements of $\mathscr{A}_1 \sqcup \mathscr{A}_2$ to certain basis elements of $\mathrm{T}(\bigoplus_{i=1}^m (\mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)}))$. But all the maps appearing on the right hand side of equation (II) composed after $\tilde{\iota}$ are homomorphisms of algebras. By equation (1.1.3) the binary operation $\otimes$ is the multiplication for pure tensors in the tensor algebra $\mathrm{T}(\bigoplus_{i=1}^m (\mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)}))$. The statement of equation (II) now follows from the following calculation

$$\begin{aligned}
&\left(\left(\left(\coprod_{\ell=1}^m \mathrm{j}_1^{(\ell)}\right) \amalg \left(\coprod_{\ell=1}^m \mathrm{j}_2^{(\ell)}\right)\right) \circ \left(\coprod_{i=1}^m \Delta_i\right)\right)\big(\underbrace{a_{\varepsilon_r,k_r}^{(\delta_r)}}_{\in \coprod_{\ell=1}^m \mathrm{T}(V_\ell)}\big) \\
&= \left(\left(\left(\coprod_{\ell=1}^m \mathrm{j}_1^{(\ell)}\right) \amalg \left(\coprod_{\ell=1}^m \mathrm{j}_2^{(\ell)}\right)\right) \circ \left(\left(\coprod_{\ell=1}^m \iota_1^{(\ell)}\right) \amalg \left(\coprod_{\ell=1}^m \iota_2^{(\ell)}\right)\right)\right)\big(\Delta_{\delta_r}(a_{\varepsilon_r,k_r}^{(\delta_r)})\big) \\
&\qquad [\![\ \text{UMP of free product of algebras}\ ]\!] \\
&= \big(\mathrm{j}_1^{(\delta_r)} \amalg \mathrm{j}_2^{(\delta_r)}\big)\Big(\widetilde{\mathsf{can}}\big((0, a_{\varepsilon_r,k_r}^{(\delta_r)}) + (a_{\varepsilon_r,k_r}^{(\delta_r)}, 0)\big)\Big) \\
&\qquad \left[\!\!\left[\ \begin{array}{l} \text{UMP of free product of algebras,} \\ \text{primitive comultiplication def. in eq. (2.2.10)} \\ \widetilde{\mathsf{can}} \text{ is isomorphism similar to eq. (2.2.6)} \end{array}\ \right]\!\!\right] \\
&= \mathsf{id}_{\mathscr{A}_{\varepsilon_r}^{(\delta_r)}}(a_{\varepsilon_r,k_r}^{(\delta_r)}) \quad [\![\ \text{def. of } \mathrm{j}_i^{(\ell)} \text{ in eq. (2.4.18) \& (2.5.4)}\ ]\!]
\end{aligned}$$

Having proved equation (II) we can now proceed with the following calculation

$$\varphi_1 \odot \varphi_2 = (\varphi_1 \odot \varphi_2) \circ \left(\left(\coprod_{\ell=1}^{m} \mathrm{j}_1^{(\ell)} \right) \amalg \left(\coprod_{\ell=1}^{m} \mathrm{j}_2^{(\ell)} \right) \right) \circ \left(\coprod_{i=1}^{m} \Delta_i \right) \circ \tilde{\iota} \quad [\![\text{eq. (II)}]\!]$$

$$= \left(\left(\varphi_1 \circ \left(\coprod_{\ell=1}^{m} \mathrm{j}_1^{(\ell)} \right) \right) \odot \left(\varphi_2 \circ \left(\coprod_{\ell=1}^{m} \mathrm{j}_2^{(\ell)} \right) \right) \right) \circ \left(\coprod_{i=1}^{m} \Delta_i \right) \circ \tilde{\iota}$$

$[\![$ $\forall\, i \in [2]\colon\ \coprod_{\ell=1}^{m} \mathrm{j}_i^{(\ell)} \in \mathsf{Morph}_{\mathsf{Alg_m}}\big(\bigsqcup_{\ell=1}^{m} \mathrm{T}(V_\ell), \mathscr{A}_i\big)$
because $\mathscr{A}_i = \bigsqcup_{r=1}^{m} \mathscr{A}_i^{(r)}$ and we have $(\coprod_{r=1}^{m} \mathrm{j}_i^{(r)})(\mathrm{T}(V_\ell)) = \mathscr{A}_i^{(\ell)}$,
then we can use eq. (2.1.19) for $d = 1$ $]\!]$

$$= \left(\left(\varphi_1 \circ \left(\coprod_{\ell=1}^{m} \mathrm{j}_1^{(\ell)} \right) \right) \circledast \left(\varphi_2 \circ \left(\coprod_{\ell=1}^{m} \mathrm{j}_2^{(\ell)} \right) \right) \right) \circ \tilde{\iota}$$

This shows equation (2.4.21). According to Convention 2.4.10 $(\bigsqcup_{i=1}^{m} \mathrm{T}(V_i), \coprod_{i=1}^{m} \Delta_i, 0)$ is a comonoid in the tensor category $(\mathsf{Alg}_\mathsf{m}^{\mathbb{N}_0}, \sqcup, \{0\})$. Equation (2.4.22) now follows from equation (2.4.13) and Proposition 2.4.3. ◻

2.5 Highest coefficients

Now, we focus on a special class of u.a.u.-products and apply the moment-cumulant formula to these. This derived formula will serve as a role model for our coming investigations in this work.

2.5.1 Definition (Right-ordered monomials property). We say that a u.a.u.-product $\odot$ in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ has the *right-ordered monomials property* if and only if in the expression of the universal product by the universal coefficient theorem (Theorem 2.3.3) only right ordered monomials appear, i. e., sums are to be taken over the set $\mathcal{OM}(\,\cdot\,)$ which means

$\forall k \in \mathbb{N} \setminus \{1\},\ \forall \varepsilon := (\varepsilon_{i,1}, \varepsilon_{i,2})_{i\in[n]} \in \mathbb{A}([k] \times [m]),\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})},\ \forall j \in [k],\ \forall \pi_j \in \mathcal{M}(X^{(j)}) \setminus \mathcal{OM}(X^{(j)})$:

$$\alpha_{\pi_1,\dots,\pi_k}^{(\varepsilon)} = 0 \tag{2.5.1}$$

Here, we have used the notation introduced in Remark 2.3.4.

2.5.2 Remark. According to [MS17, Rem. 4.4] a positive u.a.u.-product has the right-ordered monomials property. In Chapter 3 we are going to define another class of symmetric u.a.u.-product which come from partitions. They have the right-ordered monomials property by design but it is not directly clear if they will lead to a positive u.a.u.-product.

2.5.3 Definition (Highest coefficient [Spe97]). Let $\odot$ be a u.a.u.-product with right-ordered monomials property in the category $\mathsf{AlgP_m}$. Due to the universal coefficient theorem (Thm. 2.3.3) the u.a.u.-product $\odot$ is uniquely determined by constants $\alpha_{\pi_1,\dots,\pi_k}^{(\varepsilon)} \in \mathbb{C}$ for any $\varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in \mathbb{A}([k] \times [m])$ and $\pi_j \in \mathcal{OM}(X^{(j)})$ (because it has the right-ordered monomials property). Among these coefficients we set

$$\forall n,\, k \in \mathbb{N},\ \forall \varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in \mathbb{A}([k] \times [m])\colon\ \alpha_{\max}^{(\varepsilon)} := \alpha_{\mathbb{1}_1,\dots,\mathbb{1}_k}^{(\varepsilon)} \tag{2.5.2}$$

and call $\alpha_{\max}^{(\varepsilon)}$ the *highest coefficient* (for a given ε-tuple) w. r. t. $\odot$.

2.5.4 Remark. For the above definition we notice that if a u.a.u.-product does not have the right-ordered monomials property, then we can not speak of exactly one highest coefficient. Instead, there can now exist several highest coefficients for one ε-tuple.

2.5.5 Convention.

(a) Let $I \subseteq \mathbb{N}$ be a set and $|I| = n$ for some $n \in \mathbb{N}$. We have a natural order on I. Hence, there exist elements $i_j \in \mathbb{N}$ such that $I = \{i_1, \dots, i_n\}$ and $i_j < i_{j+1}$. For each $i \in \{1, 2, \dots, n\}$, let A_i be a set and assume $\forall j \in [n]\colon a_{i_j} \in A_j$. By $(a_i)_{i\in I}$ we will mean the n-tuple $(a_{i_1}, \dots, a_{i_n}) \in \prod_{j=1}^{n} A_j$.

(b) Let us assume we have some multiplication of an algebra denoted by $\cdot$ on a set $\mathscr{A}$ and $(a_i)_{i\in I} \in \mathscr{A}^I$ for some finite index set $I \subseteq \mathbb{N}$. We want to have some convention for the large operators $\prod$ regarding $\cdot$ although the regarding algebraic operation $\cdot$ does not necessarily need to be commutative. Hence, an expression like $\prod_{i\in I} a_i$ would not be well defined since there is some arbitrariness in the way to express the product. Instead, we want to define large operators like $\overrightarrow{\prod}_{i\in I} a_i$ and $\overrightarrow{\bigotimes}_{i\in I} v_i$ for the multiplication on the tensor algebra $\mathrm{T}(V)$ (V a vector space) between pure tensors. For $I = \{i_1, \dots, i_n\}$ with $i_1 < i_2 < \cdots < i_n$ and $n \in \mathbb{N}$ we set

$$\forall (a_i)_{i\in I} \in \mathscr{A}^{\times n}\colon \overrightarrow{\prod_{i\in I}} a_i := a_{i_1} \cdot a_{i_2} \cdot \dots \cdot a_{i_n} \in \mathscr{A}, \tag{2.5.3a}$$

$$\forall (v_i)_{i\in I} \in V^{\times n}\colon \overrightarrow{\bigotimes_{i\in I}} v_i := v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_n} \in \mathrm{T}(V). \tag{2.5.3b}$$

2.5.6 Definition (Block over $\mathbb{N}$, position of an element in a block). Let $n \in \mathbb{N}$.

(a) For an n-tuple $x := (x_i)_{i\in[n]} \in \mathbb{N}^{\times n}$ we define

$$\mathsf{set}\, x := \{x_1, x_2, \dots, x_n\} \subseteq \mathbb{N}. \tag{2.5.4}$$

If all entries of x are pairwise distinct, then we write for x as an n-tuple

$$|x| := |\mathsf{set}\, x|. \tag{2.5.5}$$

(b) We call an n-tuple $(x_i)_{i\in[n]} \in \mathbb{N}^{\times n}$ a *block* (over $\mathbb{N}$), if $\forall i \in [n-1]\colon x_i < x_{i+1}$.

(c) Let X be a set with n distinct natural numbers for some $n \in \mathbb{N}$, i. e., $X = \{x_1, \dots, x_n\} \subseteq \mathbb{N}$ such that $\forall i, j \in [n]\colon i \neq j \implies x_i \neq x_j$. By the well-ordering principle we can sort the elements of X in ascending order. We denote this sorted tuple by $\mathsf{block}\, X \in \mathbb{N}^{\times n}$ which is the associated block originated from the finite set X.

(d) For any block $b := (b_i)_{i\in[n]} \in \mathbb{N}^{\times n}$ and some element $x \in \mathsf{set}(b)$ we denote by $\mathsf{pos}_b(x)$ the position of x in the n-tuple b. Hence, $\mathsf{pos}_b(x)$ is uniquely determined, because if $x \in \mathsf{set}(b)$, then there exists $j \in [n]$ such that $b_j = x$ and for all $i \in [n]$ with $i \neq j$ we have $b_i \neq x$ by Definition 2.5.6 **(b)**.

2.5.7 Convention. Let V be a vector space, $n \in \mathbb{N}$ and $x := (x_i)_{i\in[n]} \in V^{\times n}$ be some n-tuple of elements in V. Then, we define for x and for any block $b = (b_i)_{i\in[\ell]} \in [n]^{\times \ell}$, $\ell \in [n]$

$$x_b := \overrightarrow{\bigotimes_{i\in[\ell]}} x_{b_i} \equiv x_{b_1} \otimes \cdots \otimes x_{b_\ell} \in \mathrm{T}(V), \tag{2.5.6}$$

where the last equation holds by Convention 2.5.5 **(b)**. If $x = x_1 \otimes \cdots \otimes x_n \in V^{\otimes n}$, then we can associate to x the finite tuple $(x_1, \ldots, x_n) \in V^{\times n}$ and x_b still makes sense w. r. t. to this n-tuple under usage of equation (2.5.6).

2.5.8 Convention (Reduced tuple). Let $n,\ k \in \mathbb{N}$. There exists a unique procedure to associate to any given tuple $\varepsilon = (\varepsilon_{i,1}, \varepsilon_{i,2})_{i\in[n]} \in ([k]\times[m])^{\times n}$ an element $\tilde{\varepsilon} = (\tilde{\varepsilon}_{i,1}, \tilde{\varepsilon}_{i,2})_{i\in[\tilde{n}]} \in \mathbb{A}([k]\times[m])$ with $\tilde{n} \leq n$. We denote the uniquely by ε determined element by red ε. As an example for

$$\varepsilon = \big((1,2),(1,2),(1,2),(1,1),(1,1),(2,3),(3,3)\big) \tag{2.5.7}$$

we obtain for red ε

$$\operatorname{red}\varepsilon = \big((1,2),(1,1),(2,3),(3,3)\big). \tag{2.5.8}$$

Now, we can apply this setting to the calculation of a u.a.u.-product. Assume $\odot$ is a u.a.u.-product with right-ordered monomials property in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. We set

$\forall k \in \mathbb{N},\ \forall n \in \mathbb{N}\setminus[k-1],\ \forall \varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big) \in ([k]\times[m])^{\times n},\ \forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$:

$$\big(\varphi_1 \odot \cdots \odot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) := \big(\varphi_1 \odot \cdots \odot \varphi_k\big)(\tilde{a}_1 \cdot \cdots \cdot \tilde{a}_{\tilde{n}}), \tag{2.5.9}$$

where the evaluation of the right hand side of equation (2.5.9) is covered by the universal coefficient theorem (Thm. 2.3.3) and $(\tilde{a}_i)_{i\in[\tilde{n}]} \in \prod_{i=1}^n \mathscr{A}_{\tilde{\varepsilon}_{i,1}}^{(\tilde{\varepsilon}_{i,2})}$ is the unique tuple which emerges from $(a_i)_{i\in[n]}$ with the property

$$\exists!\ \tilde{n} \in [n]\colon \big((\tilde{\varepsilon}_{i,1}, \tilde{\varepsilon}_{i,2})\big)_{i\in[\tilde{n}]} = \operatorname{red}(\varepsilon) \in \mathbb{A}\big([k]\times[m]\big). \tag{2.5.10}$$

We can think of the tuple $(\tilde{a}_i)_{i\in[\tilde{n}]}$ as the tuple $(a_i)_{i\in[n]}$, where we have multiplied "segmentwise repeated" occurrences. Without getting to technical we illustrate this procedure for an example. Let $m = k = 3$, $n = 7$ and $\varepsilon \in ([k]\times[m])^{\times n}$ given by equation (2.5.7). If $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$, then

$$\tilde{a}_1 = a_1 \cdot a_2 \cdot a_3, \quad \tilde{a}_2 = a_4 \cdot a_5, \quad \tilde{a}_3 = a_6, \quad \tilde{a}_4 = a_7. \tag{2.5.11}$$

For matter of comparison the next statements will be formulated in a first instance for the single-faced case, i. e., $m = 1$ and then extend the statement to the multi-faced case, i. e., $m \geq 1$.

2.5.9 Lemma. Assume $\odot$ is a u.a.u.-product with right-ordered monomials property in the category AlgP. Let $k \in \mathbb{N}\setminus\{1\}$. Then, we have

$\forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [k]^{\times n},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_i}$:

$|\mathsf{set}\,\varepsilon| < k$

$$\Longrightarrow \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)\big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0} = 0, \tag{2.5.12}$$

where we denote

$$\forall i \in [k]\colon\ \iota_i\colon \mathscr{A}_i \hookrightarrow \bigsqcup_{j=1}^{k} \mathscr{A}_j \quad \text{as canonical inclusions.} \tag{2.5.13}$$

PROOF: We prove this assertion by induction over $k \in \mathbb{N} \setminus \{1\}$. For the induction base $k = 2$ we assume $\varepsilon = (1)$ and then we calculate

$$\begin{aligned}
&\frac{\partial^2}{\partial t_1\,\partial t_2}\Big(\big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(\iota_{\varepsilon_1}(a_1)\big)\Big)\Big|_{t_1=t_2=0}\\
&= \frac{\partial^2}{\partial t_1\,\partial t_2}\Big((t_1\varphi_1)\big(\iota_{\varepsilon_1}(a_1)\big)\Big)\Big|_{t_1=t_2=0} \quad [\![\text{ unitality of } \odot\ ]\!]\\
&= \frac{\partial}{\partial t_2}\Big(\varphi_1\big(\iota_{\varepsilon_1}(a_1)\big)\Big)\Big|_{t_2=0} = 0.
\end{aligned}$$

The other case $\varepsilon = (2)$ is shown analogously. For induction step $k-1 \to k$ we make a case consideration of 2 cases. First we assume $|\mathsf{set}\,\varepsilon| < k$ and $1 \notin \mathsf{set}\,\varepsilon$. Then, we calculate

$$\begin{aligned}
&\frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big(\big((t_1\varphi_1)\odot\cdots\odot(t_k\varphi_k)\big)\big(\iota_{\varepsilon_1}(a_1)\cdot\cdots\cdot\iota_{\varepsilon_n}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0}\\
&= \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Bigg(\Big((t_1\varphi_1)\odot\big((t_2\varphi_2)\cdots\odot(t_k\varphi_k)\big)\Big)\Big(\mathsf{can}\Big(\big(\iota_{\varepsilon_1}(a_1)\cdot\cdots\cdot\iota_{\varepsilon_n}(a_n)\big)\Big)\Big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}\\
&\quad [\![\ \textstyle\bigsqcup_{i=1}^k \mathscr{A}_i \overset{\mathsf{can}}{\cong} \mathscr{A}_1 \sqcup \bigsqcup_{i=2}^k \mathscr{A}_i,\ \odot \text{ is associative }]\!]\\
&= \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Bigg(\big((t_2\varphi_2)\odot\cdots\odot(t_k\varphi_k)\big)\Big(\mathsf{can}\Big(\big(\iota_{\varepsilon_1}(a_1)\cdot\cdots\cdot\iota_{\varepsilon_n}(a_n)\big)\Big)\Big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}\\
&\quad [\![\ 1 \notin \mathsf{set}\,\varepsilon, \text{ unitality of } \odot\ ]\!]\\
&= \frac{\partial}{\partial t_1}\Bigg(\frac{\partial^{k-1}}{\partial t_2 \dots \partial t_k}\Bigg(\big((t_2\varphi_2)\odot\cdots\odot(t_k\varphi_k)\big)\Big(\mathsf{can}\Big(\big(\iota_{\varepsilon_1}(a_1)\cdot\cdots\cdot\iota_{\varepsilon_n}(a_n)\big)\Big)\Big)\Bigg)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}\\
&\quad [\![\text{ Schwarz's theorem }]\!]\\
&= 0 \quad [\![\text{ no dependence of } t_1\ ]\!].
\end{aligned}$$

For the next case we assume $|\mathsf{set}\,\varepsilon| < k$ and $1 \in \mathsf{set}\,\varepsilon$. Then, we calculate

$$\begin{aligned}
&\frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big(\big((t_1\varphi_1)\odot\cdots\odot(t_k\varphi_k)\big)\big(\iota_{\varepsilon_1}(a_1)\cdot\cdots\cdot\iota_{\varepsilon_n}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0}\\
&= \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Bigg(\Big((t_1\varphi_1)\odot\big((t_2\varphi_2)\cdots\odot(t_k\varphi_k)\big)\Big)\underbrace{\Big(\mathsf{can}\Big(\big(\iota_{\varepsilon_1}(a_1)\cdot\cdots\cdot\iota_{\varepsilon_n}(a_n)\big)\Big)\Big)}_{=:a_1\cdot\cdots\cdot a_n \in\, \mathscr{A}_1 \sqcup \bigsqcup_{i=2}^k \mathscr{A}_i}\Bigg)\Bigg|_{t_1=\ldots=t_k=0}\\
&\quad [\![\ \textstyle\bigsqcup_{i=1}^k \mathscr{A}_i \overset{\mathsf{can}}{\cong} \mathscr{A}_1 \sqcup \bigsqcup_{i=2}^k \mathscr{A}_i,\ \odot \text{ is associative }]\!]
\end{aligned}$$

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\sum_{\pi_1 \in \mathcal{OM}(X^{(1)})} \sum_{\pi_2 \in \mathcal{OM}(X^{(2)})} \alpha^{(\varepsilon)}_{\pi_1,\pi_2}$$

$$\prod_{M_1 \in \pi_1} (t_1\varphi_1)\big(j(a_1, \dots, a_n)(M_1)\big)$$

$$\prod_{M_1 \in \pi_2} \big((t_2\varphi_1) \odot \dots \odot (t_k\varphi_k)\big)\big(j(a_1, \dots, a_n)(M_2)\big)\Bigg)\Bigg|_{t_1=\dots=t_k=0}$$

⟦ Thm. 2.3.3 applied to $\varepsilon \in \mathbb{A}([2])$ ⟧

$$= \frac{\partial^{k-1}}{\partial t_2 \dots \partial t_k} \Bigg(\sum_{\pi_2 \in \mathcal{OM}(X^{(2)})} \alpha^{(\varepsilon)}_{\mathbb{1}_1,\pi_2}$$

$$\varphi_1\big(j(a_1, \dots, a_n)(\mathbb{1}_1)\big)$$

$$\prod_{M_1 \in \pi_2} \big((t_2\varphi_1) \odot \dots \odot (t_k\varphi_k)\big)\big(j(a_1, \dots, a_n)(M_2)\big)\Bigg)\Bigg|_{t_2=\dots=t_k=0}$$

⟦ Schwarz's theorem, product rule ⟧

$$= \sum_{\pi_2 \in \mathcal{OM}(X^{(2)})} \alpha^{(\varepsilon)}_{\mathbb{1}_1,\pi_2} \varphi_1\big(j(a_1, \dots, a_n)(\mathbb{1}_1)\big)$$

$$\prod_{M_2 \in \pi_2} \underbrace{\frac{\partial^{k-1}}{\partial t_2 \dots \partial t_k}\Big(\big((t_2\varphi_2) \odot \dots \odot (t_k\varphi_k)\big)\big(j(a_1, \dots, a_n)(M_2)\big)\Big)\Big|_{t_1=\dots=t_k=0}}_{=0}$$

$$= 0 \quad \left[\!\!\left[\begin{array}{l} \forall \pi_2 \in \mathcal{OM}(X^{(2)}),\ \forall M_2 \in \pi_2,\ \exists \tilde{n} \in \mathbb{N},\ \exists \tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i\in[\tilde{n}]} \in \{2, \dots, k\}^{\times \tilde{n}}, \\ \exists (\delta_i)_{i\in[\tilde{n}]} \in [n]^{\times \tilde{n}} \colon \big(j(a_1, \dots, a_n)(M_2)\big) = \tilde{\iota}_{\tilde{\varepsilon}_1}(a_{\delta_1}) \cdot \dots \cdot \tilde{\iota}_{\tilde{\varepsilon}_n}(a_{\delta_n}) \in \bigsqcup_{i=2}^k \mathscr{A}_i, \\ |\mathsf{set}\, \tilde{\varepsilon}| < k-1 \\ \implies \text{application of induction hypothesis for each monomial} \end{array} \right]\!\!\right].$$

□

We also have a version of Lemma 2.5.9 for the case $m \geq 1$.

2.5.10 Lemma. Assume $\odot$ is a u.a.u.-product with right-ordered monomials property in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Let $k \in \mathbb{N} \setminus \{1\}$. Then, we have

$\forall (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k},\ \forall \varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in ([k] \times [m])^{\times n}, \forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}^{(\varepsilon_{i,2})}_{\varepsilon_{i,1}}$:

$$|\mathsf{set}\big((\varepsilon_{i,1})_{i\in[n]}\big)| < k$$

$$\implies \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big(\big((t_1\varphi_1) \odot \dots \odot (t_k\varphi_k)\big)\big(\iota^{(\varepsilon_{1,2})}_{\varepsilon_{1,1}}(a_1) \cdot \dots \cdot \iota^{(\varepsilon_{n,2})}_{\varepsilon_{n,1}}(a_n)\big)\Big)\Big|_{t_1=\dots=t_k=0} = 0, \quad (2.5.14)$$

where we denote

$$\forall\, i \in [k],\ \forall\, j \in [m]\colon\ \iota_i^{(j)}\colon \mathscr{A}_i^{(j)} \hookrightarrow \bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{m} \mathscr{A}_i^{(j)} \quad \text{as canonical inclusions.} \tag{2.5.15}$$

Proof: The proof is formally the same as the proof of Lemma 2.5.9. We just need to apply the universal coefficient theorem in the case $m \geq 1$. □

2.5.11 Lemma. Let $\odot$ be a u.a.u.-product in the category AlgP, which has the right ordered monomials property. Let V be a $\mathbb{C}$-vector space and consider $(\mathrm{T}(V), \Delta, 0)$ as a $\mathbb{N}_0$-graded single-faced dual semigroup with primitive comultiplication Δ according to Convention 2.4.10. Assume that $(v_i)_{i\in I} \in V^I$ is a basis of V with index set I and $\psi,\ \tilde{\psi} \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$, then

(a) $\forall\, k \in \mathbb{N} \setminus \{1\},\ \forall n \geq k,\ \forall (i_j)_{j\in[n]} \in I^{\times n}$:

$$\psi^{\boxplus k}(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n},\\ |\mathsf{set}\,\varepsilon|=k}} \Big(\alpha_{\max}^{(\mathrm{red}\,\varepsilon)} \prod_{i=1}^{k} \psi\big(\tilde{v}_{b_i}\big)\Big). \tag{2.5.16}$$

Herein, $\tilde{v}_{b_i}$ is defined as follows. For each $j \in [k]$ let $\iota_j\colon T(V) \hookrightarrow \bigsqcup_{j=1}^{k} \mathrm{T}(V)$ denote the canonical homomorphic embeddings. Then, define for each $\varepsilon \in [k]^{\times n}$ with $|\mathsf{set}\,\varepsilon| = k$

$$\tilde{v}_1 \cdot \cdots \cdot \tilde{v}_{\tilde{n}} \coloneqq \iota_{\varepsilon_1}(v_{i_1}) \cdot \cdots \cdot \iota_{\varepsilon_n}(v_{i_n}) \tag{2.5.17}$$

such that $(\tilde{v}_{i_j})_{j\in[\tilde{n}]}$ denotes the transition from $(v_{i_j})_{j\in[n]}$ to a tuple, where segmentwise repeated occurrences have been multiplied (eq. (2.5.9) for $m = 1$). Then, set

$$\forall j \in [k]\colon\ b_j \coloneqq \mathsf{block}\big(\{\, \ell \in [\tilde{n}] \mid (\mathrm{red}\,\varepsilon)_\ell = j \,\}\big). \tag{2.5.18}$$

Due to Convention 2.5.7 the expression $\tilde{v}_{b_i}$ is well-defined .

(b) $\forall\, k \in \mathbb{N} \setminus \{1\},\ \forall n < k,\ \forall (i_j)_{j\in[n]} \in I^{\times n}$:

$$\psi^{\boxplus k}(v_{i_1} \otimes \cdots \otimes v_{i_n}) = 0. \tag{2.5.19}$$

(c) $\forall\, n \in \mathbb{N},\ \forall (i_j)_{j\in[n]} \in I^{\times n}$:

$$(\exp_\odot \psi)(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \sum_{k=1}^{n}\Bigg(\frac{1}{k!} \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n},\\ |\mathsf{set}\,\varepsilon|=k}} \Big(\alpha_{\max}^{(\mathrm{red}\,\varepsilon)} \prod_{i=1}^{k} \psi\big(\tilde{v}_{b_i}\big)\Big)\Bigg). \tag{2.5.20}$$

(d) $\forall\, n \in \mathbb{N}\setminus,\ \forall (i_j)_{j\in[n]} \in I^{\times n}$:

$$\begin{aligned}&(\log_\odot \psi)(v_{i_1} \otimes \cdots \otimes v_{i_n})\\ &\quad= \psi(v_{i_1} \otimes \cdots \otimes v_{i_n}) - \sum_{k=2}^{n}\Bigg(\frac{1}{k!} \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n},\\ |\mathsf{set}\,\varepsilon|=k}} \Big(\alpha_{\max}^{(\mathrm{red}\,\varepsilon)} \prod_{i=1}^{k} (\log_\odot \psi)\big(\tilde{v}_{b_i}\big)\Big)\Bigg).\end{aligned} \tag{2.5.21}$$

(e) $\forall\, i \in I$:

$$[\psi, \tilde{\psi}]_{\boxplus}(v_i) = 0 \tag{2.5.22}$$

and $\forall\, n \in \mathbb{N} \setminus \{1\}$, $\forall (i_j)_{j\in[n]} \in I^{\times n}$:

$$[\psi, \tilde{\psi}]_{\boxplus}(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[2]^{\times n}, \\ |\mathrm{set}\,\varepsilon|=2}} \Big(\alpha_{\max}^{(\mathrm{red}\,\varepsilon)}\big(\psi(\tilde{v}_{b_1})\tilde{\psi}(\tilde{v}_{b_2}) - \tilde{\psi}(\tilde{v}_{b_1})\psi(\tilde{v}_{b_2})\big)\Big). \tag{2.5.23}$$

Proof: Ad **(a)**: We calculate

$$\psi^{\boxplus k}(v_{i_1} \otimes \cdots \otimes v_{i_n})$$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\psi) \circledast \cdots \circledast (t_k\psi)\big)(v_{i_1} \otimes \cdots \otimes v_{i_n})\Big)\Big|_{t_1=\ldots=t_k=0}$$

⟦ apply Lem. 2.4.6 for $m = 1$ ⟧

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\bigg(\Big(\big((t_1\psi) \odot \cdots \odot (t_k\psi)\big) \circ \Delta^{(k-1)}\Big)(v_{i_1} \otimes \cdots \otimes v_{i_n})\bigg)\bigg|_{t_1=\ldots=t_k=0} \quad \text{⟦ eq. (2.3.8) ⟧}$$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\bigg(\Big(\big((t_1\psi) \odot \cdots \odot (t_k\psi)\big) \circ \big((\mathrm{id} \amalg \Delta^{(k-2)}) \circ \Delta\big)\Big)(v_{i_1} \otimes \cdots \otimes v_{i_n})\bigg)\bigg|_{t_1=\ldots=t_n=0}$$

⟦ eq. (2.3.3) ⟧

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\bigg(\Big(\big((t_1\psi) \odot \cdots \odot (t_k\psi)\big) \circ \Big((\mathrm{id} \amalg \Delta^{(k-2)}) \circ \big(\mathrm{can} \circ \mathrm{T}(\mathrm{inc}_1 + \mathrm{inc}_2)\big)\Big)\Big)$$

$$(v_{i_1} \otimes \cdots \otimes v_{i_n})\bigg)\bigg|_{t_1=\ldots=t_k=0}$$

⟦ def. of comultiplication on $\mathrm{T}(V)$ in eq. (2.2.10) ⟧

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\bigg(\Big(\big((t_1\psi) \odot \cdots \odot (t_k\psi)\big) \circ \Big((\mathrm{id} \amalg \Delta^{(k-2)})\Big)\Big)$$

$$\Big(\overrightarrow{\bigotimes_{j\in[n]}} \mathrm{can}\big(\mathrm{inc}_1(v_{i_j}) + \mathrm{inc}_2(v_{i_j})\big)\Big)\bigg)\bigg|_{t_1=\ldots=t_k=0}$$

⟦ $\mathrm{T}(\mathrm{inc}_1 + \mathrm{inc}_2)$ is homomorphism of algebras ⟧

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\bigg(\Big(\big((t_1\psi) \odot \cdots \odot (t_k\psi)\big) \circ \Big((\mathrm{id} \amalg \Delta^{(k-2)})\Big)\Big)$$

$$\Big(\overrightarrow{\prod_{j\in[n]}}\Big(\big(\mathrm{can} \circ \mathrm{inc}_1\big)(v_{i_j}) + \big(\mathrm{can} \circ \mathrm{inc}_2\big)(v_{i_j})\Big)\Big)\bigg)\bigg|_{t_1=\ldots=t_k=0}$$

⟦ $\mathrm{can}\colon \mathrm{T}(V \oplus V) \longrightarrow \mathrm{T}(V) \sqcup \mathrm{T}(V)$ is canonical isomorphism of algebras ⟧

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\Big(((t_1\psi) \odot \cdots \odot (t_k\psi)) \circ \Big((\mathsf{id} \amalg \Delta^{(k-2)}) \Big) \Big)$$

$$\underbrace{\Big(\prod_{j\in[n]}^{\rightarrow} (\iota_{1,2}(v_{i_j}) + \iota_{2,2}(v_{i_j})) \Big)}_{\in \mathrm{T}(V) \sqcup \mathrm{T}(V)} \Bigg) \Bigg|_{t_1=\ldots=t_k=0}$$

⟦ introduce k-copies of $\mathrm{T}(V)$ by $\forall\, i \in [k]\colon\ \mathscr{A}_i \coloneqq \mathrm{T}(V)$
set $\forall\, j \in \{2, \dots, k\},\ \forall\, i \in [j]\colon\ \iota_{i,j}\colon \mathscr{A}_i \hookrightarrow \bigsqcup_{i=1}^{j} \mathscr{A}_i$
then we have $\forall\, i \in [2]\colon\ \iota_{i,2} = \mathsf{can} \circ \mathsf{inc}_i$ ⟧

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\Big(((t_1\psi) \odot \cdots \odot (t_k\psi)) \Big)$$

$$\Big(\prod_{j\in[n]}^{\rightarrow} \Big((\mathsf{id} \amalg \Delta^{(k-2)}) (\iota_{1,2}(v_{i_j}) + \iota_{2,2}(v_{i_j})) \Big) \Big) \Bigg) \Bigg|_{t_1=\ldots=t_k=0}$$

⟦ $\mathsf{id} \amalg \Delta^{(k-2)}$ is homomorphism of algebras ⟧

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\Big(((t_1\psi) \odot \cdots \odot (t_k\psi)) \Big)$$

$$\Big(\prod_{j\in[n]}^{\rightarrow} \Big(\iota_{1,2}(v_{i_j}) + \Delta^{(k-2)} (\iota_{2,2}(v_{i_j})) \Big) \Big) \Bigg) \Bigg|_{t_1=\ldots=t_k=0}$$

⟦ UMP of $\mathscr{A}_1 \sqcup \mathscr{A}_2$ ⟧

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\Big(((t_1\psi) \odot \cdots \odot (t_k\psi)) \Big)$$

$$\Big(\prod_{j\in[n]}^{\rightarrow} \Big(\iota_{1,2}(v_{i_j}) + \Big((\mathsf{id} \amalg \Delta^{(k-3)}) \circ \Delta \Big) (\iota_{2,2}(v_{i_j})) \Big) \Big) \Bigg) \Bigg|_{t_1=\ldots=t_k=0}$$

⟦ $\Delta^{(k-2)} = ((\mathsf{id} \amalg \Delta^{(k-3)}) \circ \Delta)$ ⟧

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\Big(((t_1\psi) \odot \cdots \odot (t_k\psi)) \Big)$$

$$\Big(\prod_{j\in[n]}^{\rightarrow} \Big(\iota_{1,3}(v_{i_j}) + (\mathsf{id} \amalg \Delta^{(k-3)}) (\iota_{2,3}(v_{i_j}) + \iota_{3,3}(v_{i_j})) \Big) \Big) \Bigg) \Bigg|_{t_1=\ldots=t_k=0}$$

⟦ def. of Δ, $\mathscr{A}_1 \sqcup (\mathscr{A}_2 \sqcup \mathscr{A}_3) \cong \bigsqcup_{i\in[3]} \mathscr{A}_i$ ⟧

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Bigg(\Big(((t_1\psi) \odot \cdots \odot (t_k\psi)) \Big)$$

$$\Bigg(\overrightarrow{\prod_{j\in[n]}}\Big(\sum_{\ell=1}^{k-2}\iota_{\ell,k-1}(v_{i_j})+(\mathsf{id}\amalg\Delta)\big(\iota_{k-1,k-1}(v_{i_j})\big)\Big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$$\left[\!\left[\begin{array}{l}\text{recursive definition of } \Delta^{(k-3)} \text{ until we reach } \Delta^{(1)}=\Delta,\\ \text{proof can be made rigorously by induction over } k\in\mathbb{N}\end{array}\right]\!\right]$$

$$=\frac{\partial^k}{\partial t_1\ldots\partial t_k}\Bigg(\Big(\big((t_1\psi)\odot\cdots\odot(t_k\psi)\big)\Big(\overrightarrow{\prod_{j\in[n]}}\Big(\sum_{\ell=1}^{k}\iota_{\ell,k}(v_{i_j})\Big)\Big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$[\![$ def. of Δ $]\!]$

$$=\frac{\partial^k}{\partial t_1\ldots\partial t_k}\Bigg(\big((t_1\psi)\odot\cdots\odot(t_k\psi)\big)\Big(\sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n}}}\iota_{\varepsilon_1}(v_{i_1})\cdot\cdots\cdot\iota_{\varepsilon_n}(v_{i_n})\Big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$[\![$ set $\forall\, i\in[k]\colon\ \iota_i\coloneqq\iota_{i,k}$ $]\!]$

$$=\sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n}}}\frac{\partial^k}{\partial t_1\ldots\partial t_k}\Big(\big((t_1\psi)\odot\cdots\odot(t_k\psi)\big)\big(\iota_{\varepsilon_1}(v_{i_1})\cdot\cdots\cdot\iota_{\varepsilon_n}(v_{i_n})\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

$[\![$ mapping $\big((t_1\psi)\odot\cdots\odot(t_k\psi)\big)$ and k-fold partial derivatives are linear $]\!]$

$$=\sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n},\\ |\mathsf{set}\,\varepsilon|<k}}\underbrace{\frac{\partial^k}{\partial t_1\ldots\partial t_k}\Big(\big((t_1\psi)\odot\cdots\odot(t_k\psi)\big)\big(\iota_{\varepsilon_1}(v_{i_1})\cdot\cdots\cdot\iota_{\varepsilon_n}(v_{i_n})\big)\Big)}_{=0\quad [\![\text{ Lem. 2.5.9 }]\!]}\Big|_{t_1=\ldots=t_k=0}$$

$$+\sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n},\\ |\mathsf{set}\,\varepsilon|=k}}\frac{\partial^k}{\partial t_1\ldots\partial t_k}\Big(\big((t_1\psi)\odot\cdots\odot(t_k\psi)\big)\big(\iota_{\varepsilon_1}(v_{i_1})\cdot\cdots\cdot\iota_{\varepsilon_n}(v_{i_n})\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

$[\![$ assumption $n\geq k$ $]\!]$

$$=\sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[k]^{\times n},\\ |\mathsf{set}\,\varepsilon|=k}}\Big(\alpha_{\max}^{(\mathrm{red}\,\varepsilon)}\prod_{i=1}^{k}\psi\big(\tilde v_{b_i}\big)\Big)$$

$$\left[\!\left[\begin{array}{l}\text{universal coefficient theorem (Thm. 2.3.3) applied to } k\text{-copies of } \mathrm{T}(V),\\ \odot \text{ has right-ordered monomials property, thus } \alpha_{\max}^{(\varepsilon)} \text{ (eq. (2.5.2) for } m=1),\\ \text{is well-defined and terms of quadratic order vanish}\end{array}\right]\!\right].$$

AD **(b)**: We calculate

$$\psi^{\boxplus k}(v_{i_1}\otimes\cdots\otimes v_{i_n})$$

$$= \ldots = \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[k]^{\times n}}} \frac{\partial^k}{\partial t_1 \ldots \partial t_k} \Big(\big((t_1\psi) \odot \cdots \odot (t_k\psi)\big) \big(\iota_{\varepsilon_1}(v_{i_1}) \cdot \cdots \cdot \iota_{\varepsilon_n}(v_{i_n})\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

$$\left[\!\!\left[\begin{array}{l} \text{same steps as for } \textbf{(a)}, \\ \text{set } \forall i \in [k]\colon\ \iota_i = \mathsf{can} \circ \mathsf{inc}_i\colon V \hookrightarrow \mathrm{T}(V) \sqcup \cdots \sqcup \underbrace{\mathrm{T}(V)}_{i\text{-th position}} \sqcup \cdots \sqcup \mathrm{T}(V) \end{array} \right]\!\!\right]$$

$$= 0 \quad [\![\text{ since } n < k \text{ we can apply Lem. 2.5.9 to each summand }]\!].$$

AD **(c)**: The assertion follows by application of equation (2.4.7), **(a)** and **(b)**.
AD **(d)**: The assertion follows by application of equation (2.4.8), **(a)** and **(b)**.
AD **(e)**: From equation (2.4.6) we have

$$[\psi, \tilde{\psi}]_{\boxplus} = \psi \boxplus \tilde{\psi} - \tilde{\psi} \boxplus \psi.$$

Then, equation (2.5.22) follows from **(b)** and equation (2.5.23) has a similar calculation as in the proof of **(a)** for $k = 2$. □

2.5.12 Lemma. Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$, which has the right ordered monomials property. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces and consider $(\bigsqcup_{i=1}^m \mathrm{T}(V_i), \coprod_{i=1}^m \Delta_i, 0)$ as the $\mathbb{N}_0$-graded m-faced dual semigroup with primitive comultiplication $\coprod_{i=1}^m \Delta_i$ (using Convention 2.4.10). Assume that $(v_i^{(j)})_{i\in I_j} \in (V_j)^{I_j}$ is a basis of V_j with index set I_j for each $j \in [m]$ and $\psi, \tilde{\psi} \in \mathrm{Lin}\big(\bigsqcup_{i=1}^m \mathrm{T}(V_i), \mathbb{C}\big)$ and let $\mathsf{can}\colon \mathrm{T}(\bigoplus_{i=1}^m V_i) \longrightarrow \bigsqcup_{i=1}^m \mathrm{T}(V_i)$ be the canonical isomorphism, then

(a) $\forall k \in \mathbb{N} \setminus \{1\},\ \forall n \geq k,\ \forall (\delta_\ell)_{\ell\in[n]} \in [m]^{\times n},\ \forall (i_j)_{j\in[n]} \in \prod_{\ell=1}^n I_{\delta_\ell}\colon$

$$\psi^{\boxplus k}\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big) = \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[k]^{\times n}, \\ |\mathsf{set}\,\varepsilon|=k}} \Big(\alpha_{\max}^{(\mathrm{red}(((\varepsilon_i,\delta_i))_{i\in[n]}))} \prod_{i=1}^k \psi\big(\mathsf{can}(\tilde{v}_{b_i})\big)\Big). \tag{2.5.24}$$

Herein, $\tilde{v}_{b_i}$ is defined as follows. For each $j \in [k]$, $i \in [m]$ let $\iota_j^{(i)}\colon T(V_i) \hookrightarrow \bigsqcup_{j=1}^k \bigsqcup_{i=1}^m \mathrm{T}(V_i)$ denote the canonical homomorphic embeddings. Then, define for each $\varepsilon \in [k]^{\times n}$ with $|\mathsf{set}\,\varepsilon| = k$

$$\tilde{v}_1 \cdot \cdots \cdot \tilde{v}_{\tilde{n}} := \iota_{\varepsilon_1}^{(\delta_1)}(v_{i_1}^{(\delta_1)}) \cdot \cdots \cdot \iota_{\varepsilon_n}^{(\delta_n)}(v_{i_n}^{(\delta_n)}) \tag{2.5.25}$$

such that $(\tilde{v}_{i_j})_{j\in[\tilde{n}]}$ denotes the transition from $(v_{i_j})_{j\in[n]}$ to a tuple, wherein segmentwise repeated occurrences have been multiplied (eq. (2.5.9)). Then, set

$$(\tilde{\varepsilon}_r, \tilde{\delta}_r)_{r\in[\tilde{n}]} := \mathrm{red}\Big(\big((\varepsilon_i, \delta_i)\big)_{i\in[n]}\Big), \tag{2.5.26}$$

$$\forall j \in [k]\colon b_j := \mathsf{block}\Big(\Big\{ \ell \in [\tilde{n}] \,\Big|\, \Big(\mathsf{type}\big((\tilde{\varepsilon}_r, \tilde{\delta}_r)_{r\in[\tilde{n}]}\big)\Big)_\ell = j \Big\}\Big). \tag{2.5.27}$$

Due to Convention 2.5.7 the expression $\tilde{v}_{b_i}$ is well-defined.

(b) $\forall k \in \mathbb{N} \setminus \{1\},\ \forall n < k,\ \forall (\delta_\ell)_{\ell\in[n]} \in [m]^{\times n},\ \forall (i_j)_{j\in[n]} \in \prod_{\ell=1}^n I_{\delta_\ell}\colon$

$$\psi^{\boxplus k}\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big) = 0. \tag{2.5.28}$$

(c) $\forall n \in \mathbb{N},\ \forall (\delta_\ell)_{\ell\in[n]} \in [m]^{\times n},\ \forall (i_j)_{j\in[n]} \in \prod_{\ell=1}^n I_{\delta_\ell}$:

$$(\exp_\odot \psi)\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big) = \sum_{k=1}^n \left(\frac{1}{k!} \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[k]^{\times n}, \\ |\mathsf{set}\,\varepsilon|=k}} \left(\alpha_{\max}^{(\mathrm{red}(((\varepsilon_i,\delta_i))_{i\in[n]}))} \prod_{i=1}^k \psi\big(\mathsf{can}(\tilde v_{b_i})\big) \right) \right). \tag{2.5.29}$$

(d) $\forall n \in \mathbb{N},\ \forall (\delta_\ell)_{\ell\in[n]} \in [m]^{\times n},\ \forall (i_j)_{j\in[n]} \in \prod_{\ell=1}^n I_{\delta_\ell}$:

$$\begin{aligned}(\log_\odot \psi)\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big) &= \psi\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big) \\ &- \sum_{k=1}^n \left(\frac{1}{k!} \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[k]^{\times n}, \\ |\mathsf{set}\,\varepsilon|=k}} \left(\alpha_{\max}^{(\mathrm{red}(((\varepsilon_i,\delta_i))_{i\in[n]}))} \prod_{i=1}^k (\log_\odot \psi)\big(\mathsf{can}(\tilde v_{b_i})\big) \right) \right).\end{aligned} \tag{2.5.30}$$

(e) $\forall \delta \in [m],\ \forall i \in I_\delta$:

$$[\psi, \tilde\psi]_{\boxplus}(v_i) = 0 \tag{2.5.31}$$

and $\forall n \in \mathbb{N},\ \forall (\delta_\ell)_{\ell\in[n]} \in [m]^{\times n},\ \forall (i_j)_{j\in[n]} \in \prod_{\ell=1}^n I_{\delta_\ell}$:

$$\begin{aligned}[\psi, \tilde\psi]_{\boxplus}\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big) &= \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[2]^{\times n}, \\ |\mathsf{set}\,\varepsilon|=2}} \Big(\alpha_{\max}^{(\mathrm{red}(((\varepsilon_i,\delta_i))_{i\in[n]}))} \big(\psi\big(\mathsf{can}(\tilde v_{b_1})\big)\tilde\psi\big(\mathsf{can}(\tilde v_{b_2})\big) \\ &\qquad - \tilde\psi\big(\mathsf{can}(\tilde v_{b_1})\big)\psi\big(\mathsf{can}(\tilde v_{b_2})\big)\big)\Big).\end{aligned} \tag{2.5.32}$$

Proof: Ad **(a)**: The proof is formally the same as the proof of Lemma 2.5.11 **(a)**. We just need to substitute V by $(V_i)_{i\in[m]}$ and the primitive comultiplication Δ by $\coprod_{i=1}^m \Delta_i$. The remaining parts are all similar to the proofs of Lemma 2.5.11 **(b)** – **(e)**. □

2.5.13 Theorem. Let $m \in \mathbb{N}$. Assume that $\odot$, $\tilde\odot$ are u.a.u.-products with right-ordered monomials property in the category $\mathsf{AlgP_m}$. According to the universal coefficient theorem (Thm. 2.3.3) each universal product implies the existence of uniquely determined coefficients $\alpha_{\pi_1,\ldots,\pi_k}^{(\varepsilon)} \in \mathbb{C}$ for $k \in \mathbb{N}$, $\varepsilon \in \mathbb{A}([k]\times[m])$ and $\pi_i \in \mathcal{OM}(X^{(i)})$. Denote the corresponding coefficients for $\odot$ by $\alpha_{\pi_1,\ldots,\pi_k}^{(\varepsilon)}$ and for $\tilde\odot$ by $\tilde\alpha_{\pi_1,\ldots,\pi_k}^{(\varepsilon)}$. If

$\forall k \in \mathbb{N},\ \forall \varepsilon \in \mathbb{A}([k]\times[m])$:

$$\alpha_{\max}^{(\varepsilon)} = \tilde\alpha_{\max}^{(\varepsilon)}, \tag{2.5.33}$$

then $\odot = \tilde\odot$ as bifunctors.

Proof: Since $\odot, \tilde\odot$ are assumed to be associative, it suffices to show that

$\forall\, \varepsilon \in \mathbb{A}([2]\times[m]),\ \forall (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i)_{i\in[2]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times 2}$:

$$\varphi_1 \odot \varphi_2 = \varphi_1 \tilde{\odot} \varphi_2.$$

For the u.a.u.-product $\odot$ we obtain from equation (2.4.22) that

$$\varphi_1 \odot \varphi_2 = \Big(\exp_{\odot}\Big(\mathsf{BCH}_{\boxplus}\big(\log_{\odot}\tilde{\varphi}_1, \log_{\odot}\tilde{\varphi}_2\big)\Big)\Big) \circ \mathsf{inc}_{\mathscr{A}_1 \sqcup \mathscr{A}_2, \bigsqcup_{j=1}^m \mathrm{T}(\mathscr{A}_1^{(j)} \oplus \mathscr{A}_2^{(j)})}, \tag{I}$$

where we refer for the used notation, introduced in Theorem 2.4.12. Since $\odot$ is assumed to have the right-ordered monomials property it follows by Lemma 2.5.12 that each involved operation appearing on the right hand side of equation (I) is uniquely determined by the highest coefficients $\alpha_{\max}^{(\varepsilon)}$ for $\varepsilon \in \mathbb{A}([2]\times[m])$. The same holds for $\tilde{\odot}$ and thus the equality of $\varphi_1 \odot \varphi_2$ and $\varphi_1 \tilde{\odot} \varphi_2$ as linear functionals follows for basis elements of $\bigsqcup_{j=1}^m \mathrm{T}(\mathscr{A}_1^{(j)} \oplus \mathscr{A}_2^{(j)})$ (look at proof of Theorem 2.4.12 for a basis). □

Chapter 3

Symmetric u.a.u.-products induced by partitions

In noncommutative probability moment-cumulant formulas are strongly related to partitions. This can be seen from previous works of Hasebe and Lehner ([HL19]) or Hasebe and Saigo ([HS11]). In this chapter it is our goal to use partitions in order to define a u.a.u.-product. We only focus on the symmetric case. The nonsymmetric case seems quite involved. We refer to Remark 3.3.11 for a discussion on this topic. The idea to mimic the moment-cumulant formula for a positive u.a.u.-product by means of partitions is due to Malte Gerhold. In some sense this approach provides a simplification of the question, originally posed by Michael Schürmann, to find necessary conditions for a somehow generalized ⊞-product from the Definition 2.4.4 **(a)** which lead to a well-defined (positive) u.a.u.-product. Proposition 5.1.6 for the single-faced case and Proposition 5.2.4 for the multi-faced case provide such necessary conditions for the ⊞-product. In this section we try to clarify this approach using partitions and define sufficient conditions which ensure a systematic construction of a symmetric u.a.u.-product induced by partitions.

3.1 Universal class of partitions: single-colored case

This section in its character is very technical. We define necessary maps in order to define our so-called "universal class of partitions" in the single-colored setting. For concrete examples from where we can see the following yet to be defined maps in action, we refer the reader to examples right after Definition 3.1.9.

We understand partitions not only as a set of subsets but as a set of sorted tuples. Therefore, we have the following definition.

3.1.1 Definition (Partition of a subset of natural numbers, block neighboring elements). Let $\emptyset \neq X \subseteq \mathbb{N}$ and $|X| < \infty$.

(a) We say that $\pi = \{b_1, \ldots, b_k\}$ with $k \in \mathbb{N}$ is a *partition* of X if and only if for each $i \in [k]$ the element b_i is a block (Definition 2.5.6 **(b)**) and the set $\{\operatorname{set} b_1, \ldots, \operatorname{set} b_k\}$ is an ordinary partition of the set X. The latter means that the following conditions are fulfilled

- $\forall\, i \in [k]\colon\ \operatorname{set} b_i \neq \emptyset$,
- $\forall\, i, j \in [k]\colon i \neq j \implies \operatorname{set} b_i \cap \operatorname{set} b_j = \emptyset$,
- $X = \bigcup_{i=1}^{k} \operatorname{set} b_i$.

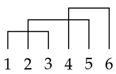
(b) The set of partitions of X is denoted by Part_X. In the case $X = [n]$ for some $n \in \mathbb{N}$ we set $\mathsf{Part}_n := \mathsf{Part}_{[n]}$. In the case $X = \mathbb{N}$ we set $\mathsf{Part} := \mathsf{Part}_{\mathbb{N}}$. The set of all partitions in Part_n for a given $n \in \mathbb{N}$ with k-blocks for $k \in [n]$ is denoted by $\mathsf{Part}_{n,k}$.

(c) Let $n \in \mathbb{N}$. By $\mathbb{1}_n \in \mathsf{Part}_n$ we will denote the unique partition which consists of only one block and call it the *unit partition* of Part_n.

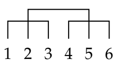
(d) For a given partition $\pi \in \mathsf{Part}_X$ we say that two elements $x_1, x_2 \in X$ are *block neighboring* (w. r. t. $\pi \in \mathsf{Part}_X$) if and only if

- $x_1 \neq x_2$,
- there exists a block $b \in \pi$ such that $x_1, x_2 \in \mathsf{set}(b)$,
- there does not exist any element $x_3 \in X$ such that $x_1 < x_3 < x_2$ or $x_2 < x_3 < x_1$.

3.1.2 Convention. We introduce a graphical notation for a given partition $\pi \in \mathsf{Part}_n$ for some $n \in \mathbb{N}$. We call elements of a block $b \in \pi$ a *leg* because we represent each element in $\mathsf{set}\, b$ by a vertical bar and we connect vertical bars of the same block by a horizontal line at the top of the legs. If these connecting horizontal lines overlap, then we draw them at different height. The relative height between connecting horizontal lines does not matter and can be drawn arbitrarily. As an example, we consider $\pi = \{(1,3),(2,5),(4,6)\}$. The following diagram represents this partition π


. (3.1.1)

Since we do not consider ordered set partitions, it does not matter at which height we connect the legs coming from the same block. Thus, the following partition also represents π


. (3.1.2)

Sometimes we will also drop the bottom row in diagrams and there will be no numbers at all. It will be clear from the context which numbers need to be put in the bottom row such that we can synonymously use a graphical notation for elements of Part.

In preparation for the definition of our so-called "universal class of partitions" we need to introduce some maps, defined on certain subsets of Part. The idea behind the first pair of maps is that we want to "annihilate" and "create" legs in a partitions. This is technically done by the map defined in equation (3.1.6) respectively by the map defined in equation (3.1.9).

3.1.3 Definition.

(a) For any $\ell \in \mathbb{N}_0$ define

$$\mathsf{down}_{\ell+2} \colon \begin{cases} \mathbb{N} \longrightarrow \mathbb{N} \\ i \longmapsto \begin{cases} i & \text{for } i \leq \ell+1 \\ i-1 & \text{for } i \geq \ell+2. \end{cases} \end{cases} \tag{3.1.3}$$

Let $n \in \mathbb{N} \setminus \{1\}$ and $\pi = \{b_1, \ldots, b_k\} \in \mathsf{Part}_n$ be a partition for some $k \in \mathbb{N}$ with the following notations: Each block can be expressed by $b_i = (b_i^1, \ldots, b_i^{j_i}) \in \mathbb{N}^{\times j_i}$ for all $i \in \{1, \ldots, k\}$, where $(j_1, \ldots, j_k) \in \mathbb{N}^{\times k}$. Let $\ell \in \{0, \ldots, n-1\}$. Then, let us assume that for the above partition π there exists an $r \in [k]$ such that $\ell+1, \ell+2 \in \mathsf{set}\, b_r$. If we set

$\lambda := \mathsf{pos}_{b_r}(\ell+1)$, then in particular $\lambda + 1 = \mathsf{pos}_{b_r}(\ell+2)$ (Definition 2.5.6 **(d)**). Then, we define for any such partition π by the preceding notations and any block $b_i \in \pi$

$$\begin{aligned}\mathsf{delete}_{\ell+2,\pi}(b_i) &= \mathsf{delete}_{\ell+2,\pi}((b_i^1,\dots,b_i^{j_i}))\\ &:= \begin{cases}\Big(\mathsf{down}_{\ell+2}(b_i^1),\dots,\mathsf{down}_{\ell+2}(b_i^{j_i})\Big) \in \mathbb{N}^{\times j_i} & \text{for } i \neq r\\ \Big(b_r^1,\dots,b_r^{\lambda},b_r^{(\lambda+2)}-1,\dots,b_r^{j_r}-1\Big) \in \mathbb{N}^{\times(j_i-1)} & \text{for } i = r\end{cases}\end{aligned} \tag{3.1.4}$$

By definition for each choice $i \in [k]$ the expression $\mathsf{delete}_{\ell+2,\pi}(b_i)$ is a block over $\mathbb{N}$. Let

$$\mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)} := \{\, \pi \in \mathsf{Part}_n \mid \exists b \in \pi\colon \ell+1,\ \ell+2 \in \mathsf{set}(b)\,\} \subseteq \mathsf{Part}_n, \tag{3.1.5}$$

then the following assignment is well-defined

$$\mathsf{delete}_{n,\ell+2}\colon \begin{cases}\mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathsf{Part}_{n-1}\\ \pi = \{b_1,\dots,b_k\}\\ \quad\longmapsto \Big\{\mathsf{delete}_{\ell+2,\pi}(b_1),\dots,\mathsf{delete}_{\ell+2,\pi}(b_k)\Big\}.\end{cases} \tag{3.1.6}$$

To put the above prescription in words: by $\mathsf{delete}_{n,\ell+2}\colon \mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathsf{Part}_{n-1}$ we "delete" or "annihilate" the leg $\ell+2$ from the partition $\pi \in \mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)}$.

(b) For $\ell \in \mathbb{N}_0$ we define

$$\mathsf{up}_{\ell+1}\colon \begin{cases}\mathbb{N} \longrightarrow \mathbb{N}\\ i \longmapsto \begin{cases} i & \text{for } i \leq \ell+1\\ i+1 & \text{for } i \geq \ell+2.\end{cases}\end{cases} \tag{3.1.7}$$

Let $n \in \mathbb{N}$ and $\pi = \{b_1,\dots,b_k\} \in \mathsf{Part}_n$ for some $k \in [n]$. Each block b_i of π is a tuple, i. e., $b_i = (b_i^1,\dots,b_i^{j_i}) \in \mathbb{N}^{\times j_i}$ for $i \in \{1,\dots,k\}$ and $(j_1,\dots,j_k) \in \mathbb{N}^{\times k}$. Moreover, assume $\ell \in \{0,\dots,n-1\}$. Then, there exists $r \in [k]$ such that $\ell+1 \in \mathsf{set}\, b_r$. We set $\lambda := \mathsf{pos}_{b_r}(\ell+1)$. By these notations we define for any partition $\pi \in \mathsf{Part}_{n-1}$ and any block $b_i \in \pi$

$$\begin{aligned}\mathsf{double}_{\ell+1,\pi}(b_i) &= \mathsf{double}_{\ell+1,\pi}((b_i^1,\dots,b_i^{j_i}))\\ &:= \begin{cases}\Big(\mathsf{up}_{\ell+1}(b_i^1),\dots,\mathsf{up}_{\ell+1}(b_i^{j_i})\Big) \in \mathbb{N}^{\times j_i} & \text{for } i \neq r\\ \Big(b_r^1,\dots,b_r^{\lambda},b_r^{\lambda}+1,b_r^{(\lambda+1)}+1,\dots,b_r^{j_r}+1\Big) \in \mathbb{N}^{\times(j_r+1)} & \text{for } i = r.\end{cases}\end{aligned} \tag{3.1.8}$$

By definition for each choice $i \in [k]$ the expression $\mathsf{double}_{\ell+1,\pi}(b_i)$ is a block over $\mathbb{N}$. Therefore, the following assignment is well-defined

$$\mathsf{double}_{n,\ell+1}\colon \begin{cases}\mathsf{Part}_n \longrightarrow \mathsf{Part}_{n+1}\\ \pi = \{b_1,\dots,b_k\}\\ \quad\longmapsto \Big\{\mathsf{double}_{\ell+1,\pi}(b_1),\dots,\mathsf{double}_{\ell+1,\pi}(b_k)\Big\}.\end{cases} \tag{3.1.9}$$

To put the above prescription in words; by $\mathsf{double}_{n,\ell+1}\colon \mathsf{Part}_n \longrightarrow \mathsf{Part}_{n+1}$ we can "double" a leg from position $\ell+1$, in other words we create a copy of the leg from position $\ell+1$ right next to it.

The intention behind the notation of $\mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)}$ is to read the symbol $\wedge$ like a logical "and". It shall indicate that the leg $\ell+1$ and $\ell+2$ are in one block of a certain partition.

3.1.4 Lemma.

(a) $\forall n \in \mathbb{N},\ \forall \ell \in \{0,\dots,n-1\} \subseteq \mathbb{N}\colon\ \mathsf{double}_{n,\ell+1}(\mathsf{Part}_n) \subseteq \mathsf{Part}_{n+1}^{(\ell+1)\wedge(\ell+2)}$.

(b) The maps

$$\mathsf{delete}_{n,\ell+2}\colon\ \mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathsf{Part}_{n-1} \tag{3.1.10}$$

$$\mathsf{double}_{n,\ell+1},\colon\ \mathsf{Part}_n \longrightarrow \mathsf{Part}_{n+1}^{(\ell+1)\wedge(\ell+2)} \tag{3.1.11}$$

are well-defined and inverse to each other, i. e.,

$$\forall n \in \mathbb{N},\ \forall \ell \in \{0,\dots,n-1\} \subseteq \mathbb{N}\colon\ \mathsf{delete}_{n+1,\ell+2} \circ \mathsf{double}_{n,\ell+1} = \mathsf{id}_{\mathsf{Part}_n}, \tag{3.1.12}$$

$$\forall n \in \mathbb{N}\setminus\{1\},\ \forall \ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}\colon \quad \mathsf{double}_{n-1,\ell+1} \circ \mathsf{delete}_{n,\ell+2} = \mathsf{id}_{\mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)}}. \tag{3.1.13}$$

(c) $\forall n \in \mathbb{N}\setminus\{1\},\ \forall \ell \in \{0,\dots,n-2\},\ \forall \pi \in \mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)}\colon\ |\pi| = |\mathsf{delete}_{n,\ell+2}(\pi)|$.

(d) $\forall n \in \mathbb{N},\ \forall \ell \in \{0,\dots,n-1\}\forall \pi \in \mathsf{Part}_n\colon\ |\pi| = |\mathsf{double}_{n,\ell+1}(\pi)|$.

PROOF: This is straightforward from the definitions. □

3.1.5 Definition (Induced partition of a block). Assume we are given a block $B = (\beta_1,\dots,\beta_n)$ for some $n \in \mathbb{N}$. Moreover, let $\delta = \{\delta_1,\dots,\delta_k\} \in \mathsf{Part}_n$ be a partition with k blocks for $k \in [n]$. Then, δ induces a partition $\{B_1,\dots,B_k\}$ of the set $\mathsf{set}\,B$ with k blocks, where the i-th block for each $i \in [k]$ is given by

$$B_i := \mathsf{block}\big(\{\, \beta_j \mid j \in \mathsf{set}\,\delta_i \,\}\big) \equiv (\beta_j)_{j\in\mathsf{set}(\delta_i)} \quad [\![\text{ Conv. 2.5.5 **(a)** }]\!]. \tag{3.1.14}$$

We denote this partition by $B \Vdash \delta$ and say that $B \Vdash \delta$ is *the partition of B induced by δ*.

3.1.6 Definition.

(a) Let $n \in \mathbb{N}\setminus\{1\}$, $\pi \in \mathsf{Part}_n$ and $b,\ b' \in \pi$ some blocks in π. If we put

$$n' := |\mathsf{set}(b)| + |\mathsf{set}(b')| \in \mathbb{N}, \tag{3.1.15a}$$

$$\tilde{b} := \mathsf{block}\Big(\mathsf{set}(b) \cup \mathsf{set}(b')\Big) \in \mathbb{N}^{n'}, \tag{3.1.15b}$$

then we define

$$\mathsf{release}_{n,b,b'}(\pi) := \Big\{\big(\mathsf{pos}_{\tilde{b}}(b_1),\dots,\mathsf{pos}_{\tilde{b}}(b_r)\big), \big(\mathsf{pos}_{\tilde{b}}(b'_1),\dots,\mathsf{pos}_{\tilde{b}}(b'_{r'})\big)\Big\} \in \mathsf{Part}_{n'}. \tag{3.1.16}$$

We want to define the following subsets of Part_n. Let $\mathbb{N}\setminus\{1\}$, $k \in [n]$ and $\ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}_0$ and set

$$\mathsf{Part}_{n,k}^{(\ell+1)\wedge(\ell+2)} := \{\, \pi \in \mathsf{Part}_n \mid |\pi| = k,\ \exists b \in \pi\colon\ \ell+1,\ \ell+2 \in \mathsf{set}(b) \,\} \tag{3.1.17}$$

(set of all partitions, where $\ell+1$ and $\ell+2$ belong to one block),

$$\mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} := \left\{ \pi \in \mathsf{Part}_n \;\middle|\; \begin{array}{l} |\pi| = k,\ \exists b,\, b' \in \pi: \\ b \neq b',\ \ell+1 \in \mathsf{set}(b),\ \ell+2 \in \mathsf{set}(b') \end{array} \right\} \tag{3.1.18}$$

(set of all partitions, where $\ell+1$ and $\ell+2$ are in different blocks),

$$\mathrm{sub}\big(\mathsf{Part}_{n,k}^{(\ell+1)\wedge(\ell+2)}\big) := \left\{ (\pi,\tilde{\pi}) \;\middle|\; \begin{array}{l} \pi \in \mathsf{Part}_{n,k}^{(\ell+1)\wedge(\ell+2)}, \\ \exists \hat{b} \in \pi\colon \ell+1, \ell+2 \in \mathsf{set}(\hat{b}), \\ \tilde{\pi} \in \mathsf{Part}_{|\mathsf{set}(\hat{b})|},\ |\tilde{\pi}| = 2, \\ \nexists \beta \in (\hat{b} \Vdash \tilde{\pi})\colon \ell+1,\ \ell+2 \in \mathsf{set}(\beta) \end{array} \right\} \tag{3.1.19}$$

(set of all ordered pairs of partitions, where the first partition is a partition such that $\ell+1$ and $\ell+2$ are in one block $\hat{b}$ and the second partition is a two-block partition of the set $\mathsf{set}(\hat{b})$ such that $\ell+1$ and $\ell+2$ are in different blocks).

(b) We define for $\mathbb{N}\setminus\{1\}$, $k \in \{2,\ldots,n\} \subseteq \mathbb{N}$ and $\ell \in \{0,\ldots,n-2\} \subseteq \mathbb{N}_0$

$$\mathsf{UMem}_{n,k}^{\ell+1}\colon \begin{cases} \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} \longrightarrow \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \\ \pi = \{b_1,\ldots,\underbrace{b}_{\sim\ell+1\in\mathsf{set}\,b},\ldots,\underbrace{b'}_{\sim\ell+2\in\mathsf{set}\,b'},\ldots,b_k\} \\ \quad\longmapsto \Big(\big\{b_1,\ldots,\mathsf{block}\big(\mathsf{set}(b)\cup\mathsf{set}(b')\big),\ldots,b_k\big\}, \\ \qquad\qquad \mathsf{release}_{n,b,b'}(\pi)\Big) \end{cases} \tag{3.1.20}$$

To put the above description in words: By $\mathsf{UMem}_{n,k}^{\ell+1}\colon \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} \longrightarrow \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big)$ we can "unify" the legs $\ell+1$ and $\ell+2$, which belong to distinct blocks b and b' and "memorize" the induced two-block partition by b and b'.

(c) Let $\mathbb{N}\setminus\{1\}$, $k \in \{2,\ldots,n\} \subseteq \mathbb{N}$ and $\ell \in \{0,\ldots,n-2\} \subseteq \mathbb{N}_0$

$$\mathsf{split}_{n,k-1}^{\ell+1}\colon \begin{cases} \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \longrightarrow \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} \\ (\pi,\tilde{\pi}) = \Big(\{b_1,\ldots,\underbrace{b_r}_{\sim\ell+1,\ell+2\in\mathsf{set}\,b_r},\ldots,b_{k-1}\},\{\beta_1,\beta_2\}\Big) \\ \quad\longmapsto \Bigg(\bigcup_{i\in[k-1]\setminus\{r\}} \{b_i\}\Bigg) \cup \big(b_r \Vdash \{\beta_1,\beta_2\}\big). \end{cases} \tag{3.1.21}$$

To put the above prescription in words: by $\mathsf{split}_{n,k-1}^{\ell+1}\colon \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \longrightarrow \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}$ we can "split" the partition π by $\tilde{\pi}$ which means we split the block neighboring legs $\ell+1$ and $\ell+2$ into 2 distinct blocks, which is induced by the two-block partition $\tilde{\pi}$.

The intention behind the notation $\mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}$ is to indicate by the symbol $\vee$ that the legs $\ell+1$ and $\ell+1$ need to be in different blocks of a certain partition. We admit that this might not be the best notation but at least it is different to the symbol $\wedge$.

3.1.7 Lemma. For $n \in \mathbb{N} \setminus \{1\}$, $k \in \{2, \dots, n\} \subseteq \mathbb{N}$ and $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$ the maps

$$\mathsf{UMem}_{n,k}^{\ell+1} \colon \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} \longrightarrow \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big), \tag{3.1.22}$$

$$\mathsf{split}_{n,k-1}^{\ell+1} \colon \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \longrightarrow \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}. \tag{3.1.23}$$

are well-defined and inverse to each other, i. e.,

$$\mathsf{split}_{n,k-1}^{\ell+1} \circ \mathsf{UMem}_{n,k}^{\ell+1} = \mathrm{id}_{\mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}} \tag{3.1.24}$$

and

$$\mathsf{UMem}_{n,k}^{\ell+1} \circ \mathsf{split}_{n,k-1}^{\ell+1} = \mathrm{id}_{\mathrm{sub}\left(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\right)}. \tag{3.1.25}$$

PROOF: The proof is a straightforward calculation. □

3.1.8 Convention. We set for $n \in \mathbb{N} \setminus \{1\}$, $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$, $k \in [n]$ and any subset $\mathcal{P} \subseteq \mathsf{Part}$

$$\mathcal{P}_n := \mathsf{Part}_n \cap \mathcal{P}, \tag{3.1.26}$$

$$\mathcal{P}_{n,k} := \{ \pi \in \mathcal{P}_n \mid |\pi| = k \}, \tag{3.1.27}$$

$$\mathcal{P}_n^{(\ell+1)\wedge(\ell+2)} := \mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)} \cap \mathcal{P}, \tag{3.1.28}$$

$$\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} := \mathsf{Part}_{n,k}^{(\ell+1)\wedge(\ell+2)} \cap \mathcal{P}, \tag{3.1.29}$$

$$\mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} := \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} \cap \mathcal{P}, \tag{3.1.30}$$

$$\mathrm{sub}\big(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) := \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \cap (\mathcal{P} \times \mathcal{P}). \tag{3.1.31}$$

3.1.9 Definition (Universal class of partitions). Let $\mathcal{P} \subseteq \mathsf{Part}$. We say that $\mathcal{P}$ is a *universal class of partitions* (abbreviated by *u.c.p.*) if and only if the following properties are satisfied:

(a) $\mathbb{1}_1 \in \mathcal{P}_1$,

(b) $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0 \colon \Big(\pi \in \mathcal{P}_n^{(\ell+1)\wedge(\ell+2)} \implies \mathsf{delete}_{n,\ell+2}(\pi) \in \mathcal{P}_{n-1}\Big)$,

(c) $\forall n \in \mathbb{N}$, $\forall \ell \in \{0, \dots, n-1\} \subseteq \mathbb{N}_0 \colon \Big(\pi \in \mathcal{P}_n \implies \mathsf{double}_{n,\ell+1}(\pi) \in \mathcal{P}_{n+1}\Big)$,

(d) $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall k \in [n]$, $\forall \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$:

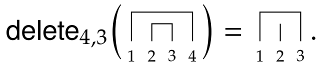
$$\Big(\pi \in \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \implies \mathsf{UMem}_{n,k}^{\ell+1}(\pi) \in (\mathcal{P}_n \times \mathcal{P})\Big), \tag{3.1.32}$$

(e) $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall k \in [n]$, $\forall \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$:

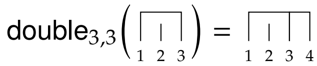
$$\Big((\pi, \tilde{\pi}) \in \mathrm{sub}\big(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big)\Big) \implies \mathsf{split}_{n,k-1}^{\ell+1}(\pi, \tilde{\pi}) \in \mathcal{P}_n\Big). \tag{3.1.33}$$

We want to discuss the above axioms and try to give a rough explanation what might be the idea behind these.

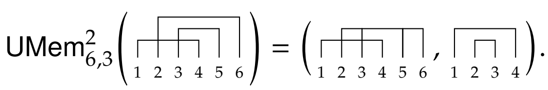
- By demanding the existence of axiom Definition 3.1.9 **(b)** we are able to replace an interval of two elements by a leg at the position $\ell + 1$. We do this by deleting the element $\ell + 2$ from the partition π and subtract 1 from elements greater than $\ell + 2$. For example


(3.1.34)

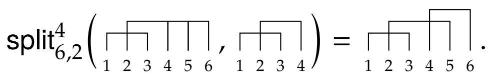
- In contrast, the axiom formulated in Definition 3.1.9 **(c)** enables us to insert or implement an element $\ell + 2$ right after $\ell + 1$ into the partition π and add 1 for elements greater than $\ell + 1$. Hence, we double the original element $\ell + 1$. We could also say that this reflects the idea, that we can replace a leg by an interval of two elements. For example


(3.1.35)

- The meaning of axiom Definition 3.1.9 **(d)** is twofold: On one hand $\mathsf{release}_{b,b'}(\pi)$ (this is the memory part of UMem) denotes a partition regarded as an element of $\mathcal{P}$ constituted by blocks b and b' which have been *released* from the partition π and on the other hand $\{b_1, \ldots, \mathsf{block}(\mathsf{set}(b) \cup \mathsf{set}(b')), \ldots, b_k\}$ denotes a partition in $\mathcal{P}$, where the blocks b and b' have been *unified* to one single block. For example


(3.1.36)

- The last axiom Definition 3.1.9 **(d)** describes the possibility to replace a single block in a partition by an appropriate two-block partition which is distinguished by the property that it split the legs $\ell + 1$ and $\ell + 2$ in the original block.


(3.1.37)

Obviously, Part is a universal class of partitions. We get to know more examples of universal classes of partitions in Section 4.1.

3.1.10 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Then,

(a) $\forall n \in \mathbb{N} \colon \mathbb{1}_n \in \mathcal{P}_n$.

Let $n \in \mathbb{N} \setminus \{1\}$, $\ell \in \{0, \ldots, n-2\} \subseteq \mathbb{N}$ and $k \in [n]$, then

(b) $\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} = \emptyset \iff \mathcal{P}_{n-1,k} = \emptyset$.

(c) $\mathrm{sub}\big(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) = \emptyset \iff \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} = \emptyset$.

Proof: Ad **(a)**: We show this assertion by induction over $n \in \mathbb{N}$. For $n = 1$ there is nothing to show, since by Definition 3.1.9 **(a)** we have $\mathbb{1}_1 \in \mathcal{P}$. Assume $\mathbb{1}_n \in \mathcal{P}_n$ for some $n \in \mathbb{N}$ as induction hypothesis. We have to show the induction step $n \to n + 1$. Set $\ell := 0$, then Definition 3.1.9 **(c)** implies that $\mathbb{1}_{n+1} = \mathsf{double}_{n,\ell+1}(\mathbb{1}_n) \in \mathcal{P}_{n+1}$.
Ad **(b)**: Consider the implication $\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} = \emptyset \implies \mathcal{P}_{n-1,k} = \emptyset$. Assume $\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} = \emptyset$ and $\mathcal{P}_{n-1,k} \neq \emptyset$. Hence, there exists at least one partition $\pi \in \mathcal{P}_{n-1,k} \subseteq \mathcal{P}_{n-1}$. According to Definition 3.1.9 **(c)** we can conclude that $\mathsf{double}_{n-1,\ell+1}(\pi) \in \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \subseteq \mathcal{P}_n$. Hence, we have shown that $\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \neq \emptyset$ which contradicts our assumption.

For the other direction we have to show $\mathcal{P}_{n-1,k} = \emptyset \implies \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} = \emptyset$. We show this by proof of contradiction, hence we assume that $\mathcal{P}_{n-1,k} = \emptyset$ and $\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \neq \emptyset$. Then, there exists at least one partition $\pi \in \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}$. By Definition 3.1.9 **(b)** we can conclude that $\mathsf{delete}_{n,\ell+2}(\pi) \in \mathcal{P}_{n-1,k} \subseteq \mathcal{P}_{n-1}$. But this means that $\mathcal{P}_{n-1,k} \neq \emptyset$ which contradicts our assumption. AD **(c)**: Consider the direction $\mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) = \emptyset \implies \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} = \emptyset$. We assume $\mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) = \emptyset$ and $\mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \neq \emptyset$, i. e., $\exists \pi \in \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}$. According to Definition 3.1.9 **(d)** we have that $\mathsf{UMem}_{n,k}^{\ell+1}(\pi) \in \mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)})$. We arrive at $\mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) \neq \emptyset$ which contradicts our assumption.

For the direction $\mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} = \emptyset \implies \mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) = \emptyset$, we assume that $(\mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} = \emptyset$ and $\mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) \neq \emptyset$, i. e., there exists at least one element $(\pi, \tilde{\pi}\} \in \mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)})$. By Definition 3.1.9 **(e)** we can conclude that $\mathsf{split}_{n,k-1}^{\ell+1}(\pi, \tilde{\pi}) \in \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}$. This means that $\mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \neq \emptyset$ which contradicts our assumption. □

3.1.11 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let $n \in \mathbb{N} \setminus \{1\}$, $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$ and $k \in [n]$. Then:

(a) If we set $\mathsf{delete}_{n,\ell+2}(\emptyset) := \emptyset$ and $\mathsf{double}_{n-1,\ell+1}(\emptyset) := \emptyset$, then the following maps

$$\mathsf{delete}_{n,\ell+2} \restriction \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \colon \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathcal{P}_{n-1,k}, \tag{3.1.38}$$

$$\mathsf{double}_{n-1,\ell+1} \restriction \mathcal{P}_{n-1,k} \colon \mathcal{P}_{n-1,k} \longrightarrow \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \tag{3.1.39}$$

are well-defined and are inverse to each other.

(b) If we set $\mathsf{UMem}_{n,k}^{\ell+1}(\emptyset) := \emptyset$ and $\mathsf{split}_{n,k-1}^{\ell+1}(\emptyset) := \emptyset$, then the following maps

$$\mathsf{UMem}_{n,k}^{\ell+1} \restriction \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \colon \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \longrightarrow \mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}), \tag{3.1.40}$$

$$\mathsf{split}_{n,k-1}^{\ell+1} \restriction \mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) \colon \mathrm{sub}(\mathcal{P}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}) \longrightarrow \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \tag{3.1.41}$$

are well-defined and inverse to each other.

PROOF: AD **(a)**: The well-definedness for these maps follows from Definition 3.1.9 and Lemma 3.1.10 **(b)**. That the two maps are inverse to each other is an implication from Lemma 3.1.4 **(b)**. AD **(b)**: That these maps are well-defined follows from Definition 3.1.9 and Lemma 3.1.10 **(c)**. The two maps are inverse to each other is an implication from Lemma 3.1.7. □

3.2 Partition induced exponential and logarithm: single-colored case

We want to construct a symmetric u.a.u.-product in the single-faced case by using partitions which are elements in a universal class of partitions. But how can we do this? We need a leitmotiv for an ansatz. For an ansatz we could use equation (2.4.22), where a universal product $\odot$ is expressed by a BCH-formula for linear functionals on a certain tensor algebra. Our partition induced universal product shall have the right-ordered monomials property and furthermore its highest-coefficients shall all be 1. Thus, we make a more refined ansatz. We consider equation (2.4.22) in the case when $\odot$ is positive and symmetric. Due to Lemma 2.5.11 this means $\exp_\odot \cdot$, $\log_\odot \cdot$ and $[\cdot,\cdot]_\boxplus$ are all uniquely determined by the highest coefficients of $\odot$. To make things a little bit more precise; If we look at equation (2.5.24), then our philosophy will be to replace any appearances of nonzero highest coefficients by 1 and to replace the sum which runs over all tuples $\varepsilon \in [k]$ with $|\varepsilon| = k$ by a sum which runs over partitions π coming from a certain universal class of partitions. Symmetry of $\odot$ implies that we do not have any occurrences of the factor $\frac{1}{k!}$ in the series of $\exp_\odot$ and $\log_\odot$. We will prove this result (Lemma 5.1.12) later in a different context.

So, starting with a yet to be defined universal class of partitions $\mathcal{P}$ and trying to define a universal product, could mean that we first need to define an *exponential* $\exp_{\mathcal{P}}$ and a *logarithm* $\log_{\mathcal{P}}$ for functionals on a tensor product $\mathrm{T}(V)$ for a vector space V. We do not need an equivalent for a Lie bracket because we want our partition induced universal product to be symmetric and our leitmotiv from equation (2.4.22) tells us in this case that the Lie bracket is trivial.

3.2.1 Convention. Let V, W be some vector spaces and assume there exists a canonical linear injection from V to W, then we will denote such a canonical map by $\mathsf{inc}_{V,W}\colon V \hookrightarrow W$. In the case of algebras $\mathscr{A}$, $\mathscr{B}$, we shall denote a canonical homomorphic embedding from $\mathscr{A}$ into $\mathscr{B}$, if it exists, by $\iota_{\mathscr{A},\mathscr{B}}\colon \mathscr{A} \hookrightarrow \mathscr{B}$.

In the following we want to define an "exponential" and a "logarithm" map induced by the universal class of partitions. These are just symbols and some names, where everything is modeled in the way it has been done in Lemma 2.5.11 but the combinatorics has been put into nonzero highest coefficients of value 1.

In the following definition we consequently make use of Convention 2.5.7 and assume all occurring vector spaces to be over $\mathbb{C}$.

3.2.2 Definition. Let $\mathcal{P}$ be a universal class of partitions. Let V be a vector space and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$.

(a) We define for any $n \in \mathbb{N}$

$$\exp_{\mathcal{P}_n} \varphi\colon \begin{cases} V \times \cdots \times V \longrightarrow \mathbb{C} \\ (x_1, \ldots, x_n) \longmapsto \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \varphi(x_b). \end{cases} \tag{3.2.1}$$

(b) Let $n \in \mathbb{N}$. We recursively define a map $\log_{\mathcal{P}_n} \varphi\colon V^{\times n} \longrightarrow \mathbb{C}$ by setting for any $x \in V$ and any n-tuple $(x_i)_{i \in [n]} \in V^{\times n}$

$$(\log_{\mathcal{P}_1} \varphi)(x) = \varphi(x) \tag{3.2.2a}$$

$$(\log_{\mathcal{P}_n} \varphi)(x_1, \ldots, x_n), = \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_n,\\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} (\log_{\mathcal{P}_{|b|}} \varphi)(x_b). \tag{3.2.2b}$$

3.2.3 Theorem (Principle of strong induction [Grin21a, Thm. 2.60]). Let $g \in \mathbb{Z}$. For each $n \in \mathbb{Z}_{\geq g}$, let $\mathcal{A}(n)$ be logical statement. Assume the following

Assumption 1: If $m \in \mathbb{Z}_{\geq g}$ is such that

$$\big(\mathcal{A}(n) \text{ holds for every } n \in \mathbb{Z}_{\geq g} \text{ satisfying } n < m\big), \tag{3.2.3}$$

then $\mathcal{A}(m)$ holds.

Then, $\mathcal{A}(n)$ holds for each $n \in \mathbb{Z}_{\geq g}$.

3.2.4 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Assume V is a vector space and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$.

(a) The map $\exp_{\mathcal{P}_n} \varphi \colon V^{\times n} \longrightarrow \mathbb{C}$ is multilinear for all $n \in \mathbb{N}$.

(b) The map $\log_{\mathcal{P}_\mathrm{n}} \varphi \colon V^{\times n} \longrightarrow \mathbb{C}$ is multilinear for all $n \in \mathbb{N}$.

Proof: Ad **(a)**: We show linearity in the i-th entry of an n-tuple $(x_1, \ldots, x_n) \in V^{\times n}$ for some $i \in [n]$. Therefore assume $x := (x_1, \ldots, x_i + r_i x_i', \ldots, x_n) \in V^{\times n}$ with $r_i \in \mathbb{C}$. Then, we calculate (using Convention 2.5.5 **(a)**)

$$
\begin{aligned}
&(\exp_{\mathcal{P}_n} \varphi)(x_1, \ldots, x_i + r_i x_i, \ldots, x_n) \\
&\quad = \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \varphi(x_b) \quad [\![\text{ def. of } \exp_{\mathcal{P}_n} \varphi \text{ in eq. (3.2.1) }]\!] \\
&\quad = \sum_{\pi \in \mathcal{P}_n} \Big(\prod_{\substack{b \in \pi \\ i \notin \mathrm{set}\, b}} \varphi(x_b) \Big) \varphi\big(\cdots \otimes (x_i + r_i x_i') \otimes \cdots \big) \\
&\quad = \sum_{\pi \in \mathcal{P}_n} \Big(\prod_{\substack{b \in \pi \\ i \notin \mathrm{set}\, b}} \varphi(x_b) \Big) \Big(\varphi\big(\cdots \otimes x_i \otimes \cdots + r_i \cdots \otimes x_i' \otimes \cdots \big) \Big) \quad [\![\text{ linearity of } \otimes]\!] \\
&\quad = \sum_{\pi \in \mathcal{P}_n} \Big(\prod_{\substack{b \in \pi \\ i \notin \mathrm{set}\, b}} \varphi(x_b) \Big) \Big(\varphi\big(\cdots \otimes x_i \otimes \cdots \big) + r_i \varphi\big(\cdots \otimes x_i' \otimes \cdots \big) \Big) \quad [\![\varphi \text{ is } \mathbb{C}\text{-linear }]\!] \\
&\quad = \sum_{\pi \in \mathcal{P}_n} \Big(\prod_{\substack{b \in \pi \\ i \notin \mathrm{set}\, b}} \varphi(x_b) \Big) \varphi\big(\cdots \otimes x_i \otimes \cdots \big) + r_i \sum_{\pi \in \mathcal{P}_n} \Big(\prod_{\substack{b \in \pi \\ i \notin \mathrm{set}\, b}} \varphi(x_b) \Big) \varphi\big(\cdots \otimes x_i' \otimes \cdots \big) \\
&\quad = (\exp_{\mathcal{P}_n} \varphi)(x_1, \ldots, x_i, \ldots, x_n) + r_i (\exp_{\mathcal{P}_n} \varphi)(x_1, \ldots, x_i', \ldots, x_n).
\end{aligned}
$$

This proves that $\exp_{\mathcal{P}_n} \varphi$ is multilinear.

Ad **(b)**: We prove this assertion by the principle of strong induction starting at 1. Let $m \in \mathbb{N}$. Assume that $\log_{\mathcal{P}_\mathrm{n}} \varphi \colon V^{\times n} \longrightarrow \mathbb{C}$ is multilinear for every $n \in \mathbb{N}$ satisfying $n < m$. We have to show that it is multilinear for $n = m$. Assume $m = 1$, then $\log_{\mathcal{P}_1} \varphi$ is linear since by equation (3.2.2) we have $\log_{\mathcal{P}_1} \varphi = \varphi$ and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ by assumption.

Now assume $m \in \mathbb{N} \setminus \{1\}$ and $\log_{\mathcal{P}_\mathrm{n}} \varphi \colon V^{\times n} \longrightarrow \mathbb{C}$ is multilinear for every $n \in \mathbb{N}$ satisfying $n < m$. We have to show that it is multilinear for $n = m$. We show linearity in the i-th entry of an arbitrary m-tuple for some $i \in [m]$. We assume $(\chi_j)_{j \in [m]} := (x_1, \ldots, x_i + r_i x_i', \ldots, x_m) \in V^{\times m}$ with $r_i \in \mathbb{C}$ and calculate

$$
\begin{aligned}
&(\log_{\mathcal{P}_\mathrm{m}} \varphi)(x_1, \ldots, x_i + r_i x_i', \ldots, x_m) \\
&\quad = \varphi\big(x_1 \otimes \cdots \otimes (x_i + r_i x_i') \otimes \cdots \otimes x_m\big) - \sum_{\substack{\pi \in \mathcal{P}_m \\ \pi \neq \mathbb{1}_m}} \prod_{b \in \pi} (\log_{\mathcal{P}_{|\mathrm{b}|}} \varphi)\big((\chi_b)_{j \in \mathrm{set}\, b}\big) \\
&\qquad [\![\text{ def. of } \log_{\mathcal{P}_\mathrm{m}} \varphi \text{ in eq. (3.2.2) }]\!] \\
&\quad = \varphi(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_m) + r_i \varphi(x_1 \otimes \cdots \otimes x_i' \otimes \cdots \otimes x_m)
\end{aligned}
$$

$$- \sum_{\substack{\pi\in\mathcal{P}_m\\ \pi\neq\mathbb{1}_m}} \prod_{b\in\pi} (\log_{\mathcal{P}_{|b|}} \varphi)\big((x_j)_{j\in\mathsf{set}\, b}\big) \quad [\![\, \varphi \in \mathrm{Lin}(\mathrm{T}(V,\mathbb{C}))\,]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_m) + r_i \varphi(x_1 \otimes \cdots \otimes x_i' \otimes \cdots \otimes x_m)$$

$$- \sum_{\substack{\pi\in\mathcal{P}_m\\ \pi\neq\mathbb{1}_m}} \Big(\prod_{\substack{b\in\pi\\ i\notin \mathsf{set}\, b}} (\log_{\mathcal{P}_{|b|}} \varphi)\big((x_j)_{j\in\mathsf{set}\, b}\big)\Big)(\log_{\mathcal{P}_{|b|}} \varphi)\big(\ldots,(x_i + r_i x_i'),\ldots\big)$$

$$[\![\, \forall \pi \in \mathcal{P}_m,\ \exists \hat{b} \in \pi\colon i \in \mathsf{set}\, \hat{b}\,]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_m) + r_i \varphi(x_1 \otimes \cdots \otimes x_i' \otimes \cdots \otimes x_m)$$

$$- \sum_{\substack{\pi\in\mathcal{P}_m\\ \pi\neq\mathbb{1}_m}} \Big(\prod_{\substack{b\in\pi\\ i\notin \mathsf{set}\, b}} (\log_{\mathcal{P}_{|b|}} \varphi)\big((x_j)_{j\in\mathsf{set}\, b}\big)\Big)(\log_{\mathcal{P}_{|b|}} \varphi)\big(\ldots,x_i,\ldots\big)$$

$$- r_i \sum_{\substack{\pi\in\mathcal{P}_m\\ \pi\neq\mathbb{1}_m}} \Big(\prod_{\substack{b\in\pi\\ i\notin \mathsf{set}\, b}} (\log_{\mathcal{P}_{|b|}} \varphi)\big((x_j)_{j\in\mathsf{set}\, b}\big)\Big)(\log_{\mathcal{P}_{|b|}} \varphi)\big(\ldots,x_i',\ldots\big)$$

$$[\![\, \text{IH applied, since } \forall \pi \in \mathcal{P}_m \setminus \{\mathbb{1}_m\},\ \forall b \in \pi\colon |b| < m\,]\!]$$

$$= (\log_{\mathcal{P}_\mathrm{m}} \varphi)(x_1,\ldots,x_i\ldots,x_m) + r_i(\log_{\mathcal{P}_\mathrm{m}} \varphi)(x_1,\ldots,x_i',\ldots,x_m).$$

This shows $\log_{\mathcal{P}_\mathrm{n}} \varphi$ is multilinear for $n = m$. By strong induction we can conclude that $\log_{\mathcal{P}_\mathrm{n}} \varphi$ is multilinear for any $n \in \mathbb{N}$. □

3.2.5 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Assume V is a vector space and $\varphi \in \mathrm{Lin}(\mathrm{T}(V),\mathbb{C})$.

- **(a)** There exists a unique $\mathbb{C}$-linear map $\exp_{\mathcal{P}} \varphi\colon\ \mathrm{T}(V) \longrightarrow \mathbb{C}$, such that $(\exp_{\mathcal{P}} \varphi) \circ \mathsf{inc}_{V,V^{\otimes n}} = \mathcal{T}(\exp_{\mathcal{P}_n} \varphi)$ for all $n \in \mathbb{N}$.
- **(b)** There exists a unique $\mathbb{C}$-linear map $\log_{\mathcal{P}} \varphi\colon\ \mathrm{T}(V) \longrightarrow \mathbb{C}$, such that $(\log_{\mathcal{P}} \varphi) \circ \mathsf{inc}_{V,V^{\otimes n}} = \mathcal{T}(\log_{\mathcal{P}_\mathrm{n}} \varphi)$ for all $n \in \mathbb{N}$.

Proof: Ad **(a)**: Because of Lemma 3.2.4 **(a)**, where we have shown that $\exp_{\mathcal{P}_n} \varphi\colon V^{\times n} \longrightarrow \mathbb{C}$ is multilinear and by the universal mapping property for $\bigotimes_{i\in[n]} V = V^{\otimes n}$ we obtain a $\mathbb{C}$-linear map $\mathcal{T}(\exp_{\mathcal{P}_n} \varphi)\colon V^{\otimes n} \longrightarrow \mathbb{C}$ such that $\mathcal{T}(\exp_{\mathcal{P}_n} \varphi) \circ \mathsf{inc}_{V,V^{\times n}} = \exp_{\mathcal{P}_n} \varphi$. Since $\mathrm{T}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$, we have by the universal mapping property for the direct sum that there exists a unique $\mathbb{C}$-linear map $\exp_{\mathcal{P}} \varphi\colon\ \mathrm{T}(V) \longrightarrow \mathbb{C}$, such that $(\exp_{\mathcal{P}} \varphi) \circ \mathsf{inc}_{V,V^{\otimes n}} = \exp_{\mathcal{P}_n} \varphi$. The statement of **(b)** has a similar reasoning as **(a)**. □

3.2.6 Remark. Let $\mathcal{P}$ be a universal class of partitions. As a consequence of Lemma 3.2.5 and Definition 3.2.2, we have (using Convention 2.5.7) for any $\varphi \in \mathrm{Lin}(\mathrm{T}(V),\mathbb{C})$ and $k \in \mathbb{N}$ that the following maps are elements of $\mathrm{Lin}(\mathrm{T}(V),\mathbb{C})$

$$\exp_{\mathcal{P}} \varphi\colon \begin{cases} \mathrm{T}(V) \longrightarrow \mathbb{C} \\ x_1 \otimes \cdots \otimes x_n \longmapsto \sum_{\pi\in\mathcal{P}_n} \prod_{b\in\pi} \varphi(x_b), \end{cases} \tag{3.2.4}$$

$$\log_{\mathcal{P}} \varphi \colon \begin{cases} \mathrm{T}(V) \longrightarrow \mathbb{C} \\ x_1 \otimes \cdots \otimes x_n \longmapsto \begin{cases} \varphi(x_1) & \text{for } n = 1 \\ \varphi(x_1 \otimes \cdots \otimes x_n) - \sum\limits_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod\limits_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) & \text{else.} \end{cases} \end{cases} \tag{3.2.5}$$

3.2.7 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let V be a vector space and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$. Then,

$$\exp_{\mathcal{P}}\big(\log_{\mathcal{P}} \varphi\big) = \varphi \quad \text{and} \quad \log_{\mathcal{P}}\big(\exp_{\mathcal{P}} \varphi\big) = \varphi. \tag{3.2.6}$$

Proof: Since the tensor algebra $\mathrm{T}(V)$ is generated by elements of V and all maps of consideration are linear maps, it suffices to prove the statements for arbitrary pure tensors $x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \subseteq \mathrm{T}(V)$, where $n \in \mathbb{N}$ and $(x_i)_{i \in [n]} \in V^{\times n}$.

$$\begin{aligned}
&\big(\exp_{\mathcal{P}}(\log_{\mathcal{P}} \varphi)\big)(x_1 \otimes \cdots \otimes x_n) \\
&= \big(\mathcal{T}(\exp_{\mathcal{P}_n})(\log_{\mathcal{P}} \varphi)\big)(x_1 \otimes \cdots \otimes x_n) \quad [\![\text{ def. of } \exp_{\mathcal{P}} \varphi \text{ in Lemma 3.2.5 } \textbf{(a)}]\!] \\
&= \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ UMP \& (3.2.1) }]\!] \\
&= (\log_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) + \sum_{\pi \in \mathcal{P}_n \setminus \{\mathbb{1}_n\}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\mathbb{1}_n \in \mathcal{P} \text{ by Lemma 3.1.10 } \textbf{(a)}]\!] \\
&= (\log_{\mathcal{P}_\mathrm{n}} \varphi)(x_1 \otimes \cdots \otimes x_n) + \sum_{\pi \in \mathcal{P}_n \setminus \{\mathbb{1}_n\}} \prod_{b \in \pi} (\log_{\mathcal{P}_{|\mathrm{b}|}} \varphi)(x_b) \quad [\![\text{ def. of } \log_{\mathcal{P}} \varphi \text{ in eq. (3.2.2) }]\!] \\
&= \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} (\log_{\mathcal{P}_{|\mathrm{b}|}} \varphi)(x_b) + \sum_{\pi \in \mathcal{P}_n \setminus \{\mathbb{1}_n\}} \prod_{b \in \pi} (\log_{\mathcal{P}_{|\mathrm{b}|}} \varphi)(x_b) \\
&\qquad [\![\text{ def. of } \log_{\mathcal{P}} \text{ in eq. (3.2.5) }]\!] \\
&= \varphi(x_1 \otimes \cdots \otimes x_n).
\end{aligned}$$

Since the map $\log_{\mathcal{P}} \varphi$ is recursively defined, we prove the equation $\log_{\mathcal{P}}\big(\exp_{\mathcal{P}} \varphi\big) = \varphi$ by strong induction over the "length" $n \in \mathbb{N}$ of a pure tensor $x_1 \otimes \cdots \otimes x_n \in \mathrm{T}(V)$. For the case $n = 1$, we calculate

$$\begin{aligned}
\big(\log_{\mathcal{P}}(\exp_{\mathcal{P}} \varphi)\big)(x_1) &= (\exp_{\mathcal{P}} \varphi)(x_1) \quad [\![\text{ def. of } \log_{\mathcal{P}} \varphi \text{ in eq. (3.2.5) }]\!] \\
&= \sum_{\pi \in \mathcal{P}_1} \prod_{b \in \pi} \varphi(x_b) \quad [\![\text{ def. of } \exp_{\mathcal{P}} \varphi \text{ in eq. (3.2.4) }]\!] \\
&= \varphi(x_1) \quad [\![\text{ by Definition 3.1.9 } \textbf{(a)} \ \mathbb{1}_1 \in \mathcal{P}_1]\!] \\
&= \varphi(x_1).
\end{aligned}$$

Let $n \in \mathbb{N} \setminus \{1\}$ and assume that

$$\Big(\log_{\mathcal{P}}\big(\exp_{\mathcal{P}} \varphi\big)\Big)(x_1 \otimes \cdots \otimes x_m) = \varphi(x_1 \otimes \cdots \otimes x_m) \tag{I}$$

holds for all $m \in \mathbb{N}$ such that $m < n$. We have to show that equation (I) holds for $n = m$. We calculate

$$
\begin{aligned}
&\big(\log_{\mathcal{P}}(\exp_{\mathcal{P}} \varphi)\big)(x_1 \otimes \cdots \otimes x_n) \\
&= (\exp_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} \big(\log_{\mathcal{P}}(\exp_{\mathcal{P}} \varphi)\big)(x_b) \quad [\![\text{ def. of } \log_{\mathcal{P}} \text{ in eq. (3.2.5) }]\!] \\
&= (\exp_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} \varphi(x_b) \quad [\![\text{ IH applied since } \forall \pi \in \mathcal{P}_n \colon |b| < n]\!] \\
&= \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \varphi(x_b) - \sum_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} \varphi(x_b) \quad [\![\text{ def. of } \exp_{\mathcal{P}} \text{ in eq. (3.2.4) }]\!] \\
&= \varphi(x_1 \otimes \cdots \otimes x_n) + \sum_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} \varphi(x_b) - \sum_{\substack{\pi \in \mathcal{P}_n \\ \pi \neq \mathbb{1}_n}} \prod_{b \in \pi} \varphi(x_b) \quad [\![\text{ by Lemma 3.1.10 } \textbf{(a)} \ \mathbb{1}_n \in \mathcal{P}_n]\!] \\
&= \varphi(x_1 \otimes \cdots \otimes x_n).
\end{aligned}
$$

□

3.2.8 Definition. Let $\mathcal{P}$ be a universal class of partitions. Let V be a vector space. Then, we denote by $\circledcirc_{\mathcal{P}}$ the following binary operation on $\mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$

$$
\circledcirc_{\mathcal{P}} \colon \begin{cases} \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \times \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \\ (\varphi, \psi) \longmapsto \exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi) + \log_{\mathcal{P}}(\psi)\big). \end{cases} \tag{3.2.7}
$$

We collect some properties of $\exp_{\mathcal{P}}$, $\log_{\mathcal{P}}$ and $\circledcirc_{\mathcal{P}}$ which will be of interest in the following sections.

3.2.9 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let V be a vector space. Then, the binary mapping $\circledcirc_{\mathcal{P}} \colon \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \times \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ is associative and commutative.

Proof: Proving commutativity is easily done and skip this. For associativity we have to show that for any $\varphi_i \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ for $i \in [3]$

$$
\big((\varphi_1 \circledcirc_{\mathcal{P}} \varphi_2) \circledcirc_{\mathcal{P}} \varphi_3\big) = \big(\varphi_1 \circledcirc_{\mathcal{P}} (\varphi_2 \circledcirc_{\mathcal{P}} \varphi_3)\big).
$$

For this, we can calculate

$$
\begin{aligned}
&\big((\varphi_1 \circledcirc_{\mathcal{P}} \varphi_2) \circledcirc_{\mathcal{P}} \varphi_3\big) \\
&\quad = \exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi_1 \circledcirc_{\mathcal{P}} \varphi_2) + \log_{\mathcal{P}}(\varphi_3)\big) \quad [\![\text{ definition of } \circledcirc_{\mathcal{P}} \text{ in eq. (3.2.7) }]\!] \\
&\quad = \exp_{\mathcal{P}}\Big(\log_{\mathcal{P}}\Big(\exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi_1) + \log_{\mathcal{P}}(\varphi_2)\big)\Big) + \log_{\mathcal{P}}(\varphi_3)\Big) \quad [\![\text{ Lemma 3.2.7 }]\!] \\
&\quad = \exp_{\mathcal{P}}\Big(\big(\log_{\mathcal{P}}(\varphi_1) + \log_{\mathcal{P}}(\varphi_2)\big) + \log_{\mathcal{P}}(\varphi_3)\Big) \\
&\quad = \exp_{\mathcal{P}}\Big(\log_{\mathcal{P}}(\varphi_1) + \big(\log_{\mathcal{P}}(\varphi_2) + \log_{\mathcal{P}}(\varphi_3)\big)\Big) \\
&\qquad [\![+ \text{ is associative on } \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})]\!]
\end{aligned}
$$

$$= \exp_{\mathcal{P}}\Big(\log_{\mathcal{P}}(\varphi_1) + \log_{\mathcal{P}}\big(\exp_{\mathcal{P}}(\log_{\mathcal{P}}(\varphi_2) + \log_{\mathcal{P}}(\varphi_3))\big)\Big) \quad [\![\text{ Lemma 3.2.7 }]\!]$$
$$= \exp_{\mathcal{P}}(\log_{\mathcal{P}}(\varphi_1) + \log_{\mathcal{P}}(\varphi_2 \circledcirc_{\mathcal{P}} \varphi_3))$$
$$= (\varphi_1 \circledcirc_{\mathcal{P}} (\varphi_2 \circledcirc_{\mathcal{P}} \varphi_3)). \qquad \square$$

Now, that we know that $\circledcirc_{\mathcal{P}}$ is associative, we can omit putting brackets and the following statement makes sense.

3.2.10 Corollary. Assume that all the prerequisite from Lemma 3.2.9 hold. Then, for the binary mapping $\circledcirc_{\mathcal{P}}\colon \operatorname{Lin}(\mathrm{T}(V), \mathbb{C}) \times \operatorname{Lin}(\mathrm{T}(V), \mathbb{C}) \longrightarrow \operatorname{Lin}(\mathrm{T}(V), \mathbb{C})$

$\forall k \in \mathbb{N} \setminus \{1\},\ \forall (\varphi_i)_{i\in[k]} \in \big(\operatorname{Lin}(\mathrm{T}(V), \mathbb{C})\big)^{\times k}$:

$$\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_k = \exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi_1) + \cdots + \log_{\mathcal{P}}(\varphi_k)\big) \tag{3.2.8}$$

holds.

Proof: A simple inductive argument on $k \in \mathbb{N} \setminus \{1\}$ shows the equation. The induction step is similar to the proof for associativity of $\circledcirc_{\mathcal{P}}$. Therefore, we omit the proof. □

3.2.11 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let V, V' be two vector spaces, $f \in \operatorname{Lin}(V, V')$, $\varphi \in \operatorname{Lin}(\mathrm{T}(V'), \mathbb{C})$ and $(\varphi_i)_{i\in[k]} \in \big(\operatorname{Lin}(\mathrm{T}(V'), \mathbb{C})\big)^{\times k}$ for some $k \in \mathbb{N} \setminus \{1\}$. Then,

(a) $\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big) = (\log_{\mathcal{P}} \varphi) \circ \mathrm{T}(f)$,

(b) $\exp_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big) = (\exp_{\mathcal{P}} \varphi) \circ \mathrm{T}(f)$,

(c) $\big(\varphi_1 \circ \mathrm{T}(f)\big) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \big((\varphi_k \circ \mathrm{T}(f)\big) = (\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_k) \circ \mathrm{T}(f)$.

Proof: Since the tensor algebra $\mathrm{T}(V)$ is generated by elements of V and all maps of consideration are linear maps, it suffices to prove the statements for arbitrary pure tensors $x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \subseteq \mathrm{T}(V)$, where $n \in \mathbb{N}$ and $(x_i)_{i\in[n]} \in V^{\times n}$.

Ad **(a)**: We show the statement by strong induction over $n \in \mathbb{N}$. According to Theorem 3.2.3 we have to show equation (3.2.3). Let $n = 1$, then we need to show that the assertion is true for $n = 1$. We calculate for any $x_1 \in V$

$$\begin{aligned}
\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big)(x) &= \big(\varphi \circ \mathrm{T}(f)\big)(x_1) && [\![\text{ def. of } \log_{\mathcal{P}} \cdot \text{ in eq. (3.2.5) on } \mathrm{T}(V)]\!] \\
&= \varphi\big(f(x_1)\big) && [\![\text{ UMP of } \mathrm{T}(V)]\!] \\
&= \varphi(y) && [\![f \in \operatorname{Lin}(V, V') \implies f(x_1) =: y \in V']\!] \\
&= (\log_{\mathcal{P}} \varphi)(y) && [\![\text{ def. of } \log_{\mathcal{P}} \cdot \text{ in eq. (3.2.5) on } \mathrm{T}(V')]\!] \\
&= \big((\log_{\mathcal{P}} \varphi) \circ f\big)(x_1).
\end{aligned}$$

Now, let $\mathbb{N} \ni n > 1$ and we assume that the expression of **(a)** is true for all pure tensors of length $\ell \in [n-1]$. Then we calculate for any n-tuple $(x_i)_{i\in[n]} \in V^{\times n}$

$$\Big(\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big)\Big)(x_1 \otimes \cdots \otimes x_n)$$
$$= \big(\varphi \circ \mathrm{T}(f)\big)(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi\in\mathcal{P}_n \\ |\pi|\geq 2}} \prod_{b\in\pi} \Big(\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big)\Big)(x_b)$$

$$\llbracket \text{ def. of } \log_{\mathcal{P}} \text{ in eq. (3.2.5) on } \mathrm{T}(V) \rrbracket$$

$$= \varphi\big(\mathrm{T}(f)(x_1) \otimes \cdots \otimes \mathrm{T}(f)(x_n)\big) - \sum_{\substack{\pi\in\mathcal{P}_n \\ |\pi|\geq 2}} \prod_{b\in\pi} (\log_{\mathcal{P}}(\varphi)\big(\mathrm{T}(f)(x_b)\big)$$

$$\llbracket\, \mathrm{T}(f) \in \mathrm{Alg}(\mathrm{T}(V), \mathrm{T}(V')), \text{IH applied to } \ell = |b| \text{ and } (x_j)_{j\in \mathrm{set}\, b} \in V^{\times |b|} \,\rrbracket$$

$$= \varphi\big(f(x_1) \otimes \cdots \otimes f(x_n)\big) - \sum_{\substack{\pi\in\mathcal{P}_n \\ |\pi|\geq 2}} \prod_{b\in\pi} (\log_{\mathcal{P}}(\varphi)\Big(\big(f(x)\big)_b\Big)$$

$$\llbracket\, \mathrm{T}(f) \in \mathrm{Alg}(\mathrm{T}(V), \mathrm{T}(V')) \;\&\; \text{UMP of } \mathrm{T}(V) \,\rrbracket$$

$$= \varphi\big(y_1 \otimes \cdots \otimes y_n\big) - \sum_{\substack{\pi\in\mathcal{P}_n \\ |\pi|\geq 2}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(y_b)$$

$$\llbracket\, f \in \mathrm{Lin}(V, V') \implies \forall i \in [n]\colon f(x_i) =: y_i \in V' \,\rrbracket$$

$$= (\log_{\mathcal{P}} \varphi)(y_1 \otimes \cdots \otimes y_n) \quad \llbracket \text{ def. of } \log_{\mathcal{P}} \text{ in eq. (3.2.5) on } \mathrm{T}(V') \rrbracket$$

$$= (\log_{\mathcal{P}} \varphi)\big(f(x_1) \otimes \cdots \otimes f(x_n)\big)$$

$$= \big((\log_{\mathcal{P}} \varphi) \circ \mathrm{T}(f)\big)(x_1 \otimes \cdots \otimes x_n).$$

Ad **(b)**: This is proven in a similar way as for **(a)**, just by application of the definition of $\exp_{\mathcal{P}} \varphi$ in equation (3.2.4).
Ad **(c)**: We prove the statement by induction over $k \in \mathbb{N} \setminus \{1\}$. For the induction base $k = 2$ we calculate for $\varphi_1,\ \varphi_2 \in \mathrm{Lin}(\mathrm{T}(V'), \mathbb{C})$, $f \in \mathrm{Lin}(V, V')$

$$\big(\varphi_1 \circ \mathrm{T}(f)\big) \circledcirc_{\mathcal{P}} \big(\varphi_2 \circ \mathrm{T}(f)\big)$$

$$= \exp_{\mathcal{P}}\Big(\log_{\mathcal{P}}\big(\varphi_1 \circ \mathrm{T}(f)\big) + \log_{\mathcal{P}}\big(\varphi_2 \circ \mathrm{T}(f)\big)\Big) \quad \llbracket \text{ def. of } \circledcirc_{\mathcal{P}} \text{ in eq. (3.2.7) on } \mathrm{T}(V) \rrbracket$$

$$= \exp_{\mathcal{P}}\Big(\big((\log_{\mathcal{P}} \varphi_1) \circ \mathrm{T}(f)\big) + \big((\log_{\mathcal{P}} \varphi_2) \circ \mathrm{T}(f)\big)\Big) \quad \llbracket \text{ assertion of } \textbf{(a)} \rrbracket$$

$$= \exp_{\mathcal{P}}\Big(\big((\log_{\mathcal{P}} \varphi_1) + (\log_{\mathcal{P}} \varphi_2)\big) \circ \mathrm{T}(f)\Big)$$

$$= \Big(\exp_{\mathcal{P}}\big((\log_{\mathcal{P}} \varphi_1) + (\log_{\mathcal{P}} \varphi_2)\big)\Big) \circ \mathrm{T}(f) \quad \llbracket \text{ assertion of } \textbf{(b)} \rrbracket$$

$$= (\varphi_1 \circledcirc_{\mathcal{P}} \varphi_2) \circ \mathrm{T}(f) \quad \llbracket \text{ def. of } \circledcirc_{\mathcal{P}} \text{ in eq. (3.2.7) on } \mathrm{T}(V') \rrbracket.$$

Now, the induction step $k \to k+1$. Assume $k \in \mathbb{N} \setminus \{0, 1\}$ and the assertion holds for such all $k' \leq k$. Then, we can calculate

$$(\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_{k+1}) \circ \mathrm{T}(f)$$

$$= \big((\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_k) \circledcirc_{\mathcal{P}} \varphi_{k+1}\big) \circ \mathrm{T}(f) \quad \llbracket\, \circledcirc_{\mathcal{P}} \text{ is associative by Lemma 3.2.9} \,\rrbracket$$

$$= \big((\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_k) \circ \mathrm{T}(f)\big) \circledcirc_{\mathcal{P}} (\varphi_{k+1} \circ \mathrm{T}(f))$$

$$\llbracket \text{ induction hypothesis applied for } k' = 2 \rrbracket$$

$$= \big((\varphi_1 \circ \mathrm{T}(f)) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{T}(f))\big) \circ (\varphi_{k+1} \circ \mathrm{T}(f)$$

$[\![$ induction hypothesis applied for $k' = k$ $]\!]$.

By associativity of $\circledcirc_{\mathcal{P}}$ the induction step follows. □

3.2.12 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let V be a vector space, $\mathscr{A}$ be an algebra, $f \in \mathrm{Lin}(V, \mathscr{A})$ and $\varphi \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$. Then, $\forall n \in \mathbb{N},\ \forall (x_i)_{i\in[n]} \in V^{\times n}$ we have that

$$\Big(\big(\exists j \in [n] \colon x_j \in \ker f\big) \implies x_1 \otimes \cdots \otimes x_n \in \ker\Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)\Big). \tag{3.2.9}$$

Proof: Since $\varphi \circ \mathcal{T}(f) \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ the expression $\log_{\mathcal{P}}(\varphi \circ \mathcal{T}(f))$ is well defined. We prove the assertion by strong induction over $n \in \mathbb{N}$. Assume $n = 1$ and $x_1 \in \ker f$, then

$$\begin{aligned}
\Big(\log_{\mathcal{P}}(\varphi \circ f)\Big)(x_1) &= \big(\varphi \circ \mathcal{T}(f)\big)(x_1) && [\![\text{ def. of } \log_{\mathcal{P}} \text{ in eq. (3.2.2) }]\!] \\
&= \varphi(\mathcal{T}(f)(x_1)) \\
&= \varphi(f(x_1)) && [\![\text{ UMP of } \mathrm{T}(V)]\!] \\
&= 0 && [\![\ x_1 \in \ker f \text{ and } \varphi \in \mathrm{Lin}(\mathrm{T}(V'), \mathbb{C})\ ]\!].
\end{aligned}$$

Now, let $n \in \mathbb{N}$ such that $n > 1$. We assume that equation (3.2.9) holds for all $m \in [n-1]$. We have to show that this equation also holds for $m = n$. So let us assume $j \in [n]$ such that $x_j \in \ker f$. Then, we calculate

$$\begin{aligned}
&\Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_1 \otimes \cdots \otimes x_n) \\
&= \big(\varphi \circ \mathcal{T}(f))\big)(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_n \\ |\pi| \geq 2}} \prod_{b \in \pi} \Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_b) \\
&\qquad [\![\text{ property of } \log_{\mathcal{P}} \text{ from eq. (3.2.5) }]\!] \\
&= \varphi\big(\mathcal{T}(f)(x_1) \otimes \cdots \otimes \mathcal{T}(f)(x_j) \otimes \cdots \otimes \mathcal{T}(f)(x_n)\big) - \sum_{\substack{\pi \in \mathcal{P}_n \\ |\pi| \geq 2}} \prod_{b \in \pi} \Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_b) \\
&\qquad [\![\ \mathcal{T}(f) \in \mathrm{Alg}(\mathrm{T}(V), \mathscr{A})\ ]\!] \\
&= \varphi\big(f(x_1) \otimes \cdots \otimes f(x_j) \otimes \cdots \otimes f(x_n)\big) - \sum_{\substack{\pi \in \mathcal{P}_n \\ |\pi| \geq 2}} \prod_{b \in \pi} \Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_b) \quad [\![\text{ UMP of } \mathrm{T}(V)]\!] \\
&= -\sum_{\substack{\pi \in \mathcal{P}_n \\ |\pi| \geq 2}} \prod_{b \in \pi} \Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_b) \quad [\![\ x_j \in \ker f\ ]\!] \\
&= -\sum_{\substack{\pi \in \mathcal{P}_n \\ |\pi| \geq 2}} \prod_{\substack{b \in \pi \\ b \neq \hat{b}}} \Big(\Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_b) \cdot \underbrace{\Big(\log_{\mathcal{P}}\big(\varphi \circ \mathcal{T}(f)\big)\Big)(x_{\hat{b}})}_{=0}\Big) \\
&\qquad [\![\text{ IH applied since } \forall \pi \in \mathcal{P}_n,\ \exists \hat{b} \in \pi \colon j \in \mathsf{set}\,\hat{b} \text{ and } |\hat{b}| \leq n-1\ ]\!]
\end{aligned}$$

$= 0.$ □

The next assertion seems trivial but is essential for the proof of Lemma 3.2.15.

3.2.13 Theorem ([Grin21a, Thm. 2.132]). Let S and T be two finite sets. Let $f : S \longrightarrow T$ be a bijective map. Let a_t be an element of $\mathbb{A} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ for each $t \in T$. Then,

$$\sum_{t\in T} a_t = \sum_{s\in S} a_{f(s)}. \tag{3.2.10}$$

3.2.14 Convention. Let V be a vector space, $n \in \mathbb{N}$, $(x_i)_{i\in[n]} \in V^{\times n}$, $\pi \in \mathsf{Part}_n$, $b \in \pi$ and $\beta \in \mathsf{Part}_{|b|}$. Then, we set

$$\left(x_b\right)_\beta := \left((x_i)_{i\in\mathsf{set}(b)}\right)_\beta \in \mathrm{T}(V), \tag{3.2.11}$$

where the right hand side is defined by Convention 2.5.5 **(a)** and Convention 2.5.7.

The next lemma makes a statement about the behavior of our "partition induced cumulants" $\log_{\mathcal{P}} \varphi$ evaluated on certain elements in the tensor algebra and the relation between them.

3.2.15 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let V be a vector space. Furthermore, let $n \in \mathbb{N} \setminus \{1\}$, $(x_i)_{i\in[n]} \in V^{\times n}$ and $(y_i)_{i\in[n-1]} \in V^{\times(n-1)}$ be some tuples with entries in V with the property that there exists an $\ell \in \{0, 1, \dots, n-2\}$ such that

$$\forall\, i \in \{0, 1, \dots, \ell\}\colon x_i = y_i, \tag{3.2.12a}$$

$$\forall\, i \in \{\ell+3, \dots, n\}\colon x_i = y_{i-1}. \tag{3.2.12b}$$

Define the set

$$B_n := \left\{ (i_j)_{j\in[m]} \in [n]^{\times m} \;\middle|\; \begin{array}{l} m \in [n],\ \ell+1,\ \ell+2 \in \{i_1, \dots, i_m\}, \\ \forall\, j \in [m-1]\colon i_j < i_{j+1} \end{array} \right\} \tag{3.2.13}$$

Let $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ and assume that the mapping φ satisfies the property

$\forall\, m \in [n],\ \forall (i_j)_{j\in[m]} \in B_n:$

$$\varphi(x_{i_1} \otimes \cdots \otimes x_{\ell+1} \otimes x_{\ell+2} \otimes \cdots x_{i_m}) = \varphi(y_{i_1} \otimes \cdots \otimes y_{\ell+1} \otimes \cdots \otimes y_{i_m-1}). \tag{3.2.14}$$

Then, the following equation holds for the above choices

$$(\log_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) = (\log_{\mathcal{P}} \varphi)(y_1 \otimes \cdots \otimes y_{n-1}) - \sum_{\substack{\{b_1,b_2\}\in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} (\log_{\mathcal{P}} \varphi)(x_{b_1})(\log_{\mathcal{P}} \varphi)(x_{b_2}). \tag{3.2.15}$$

Proof: We prove the statement of equation (3.2.15) by a strong induction over $n \in \mathbb{N} \setminus \{1\}$ as stated in Theorem 3.2.3. Consider the parts of equation (3.2.12) for the choices of $n = 2$ and $\ell = 0$. In the first step we claim that

$$\varphi(x_{\mathbb{1}_2}) = \varphi(y_{\mathbb{1}_1}). \tag{I}$$

But the above equation is just a reformulation of equation (3.2.14) for $n = 2$ and $\ell = 0$.

Now, we proceed with the verification of equation (3.2.15) for the case $n = 2$ and $\ell = 0$ and we calculate

$$(\log_{\mathcal{P}} \varphi)(x_1 \otimes x_2)$$

$$= \varphi(x_1 \otimes x_2) - \sum_{\substack{\pi \in \mathcal{P}_2 \\ \pi \neq \mathbb{1}_2}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ def. of } \log_{\mathcal{P}} \varphi \text{ in equation (3.2.5) }]\!]$$

$$= \varphi(y_1) - \sum_{\substack{\pi \in \mathcal{P}_2 \\ \pi \neq \mathbb{1}_2}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ eq. (I) }]\!]$$

$$= \varphi(y_1) - \sum_{\substack{\pi \in \mathcal{P}_2 \\ \pi \neq \mathbb{1}_2}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\mathcal{P}_{2,2}^{1 \vee 2} = \mathcal{P}_2 \setminus \{\mathbb{1}_2\} \text{ and Convention 3.1.8 }]\!]$$

$$= (\log_{\mathcal{P}} \varphi)(y_1) - \sum_{\substack{\{\pi_1, \pi_2\} \in \\ \mathcal{P}_{2,2}^{1 \vee 2}}} (\log_{\mathcal{P}} \varphi)(x_{\pi_1})(\log_{\mathcal{P}} \varphi)(x_{\pi_2}) \quad [\![\text{ equation (3.2.2) }]\!].$$

We want to show equation (3.2.15) for any $n \in \mathbb{N} \setminus \{1, 2\}$ under the assumption that equation (3.2.15) holds for all $m \in [n-1]$. Let $x_1 \otimes \cdots \otimes x_n \in \mathrm{T}(V)$ and $y_1 \otimes \cdots \otimes_{n-1} \in \mathrm{T}(V)$ be some pure tensors such that there exists an $\ell \in \{0, \ldots, n-2\}$, where the parts of equation (3.2.12) and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ is chosen in such way such that equation (3.2.14) is satisfied. In the first step we claim that for this $\ell \in \{0, \ldots, n-2\}$ and the map φ the following equation holds

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \tag{II}$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}}} \left(\Big(\prod_{\substack{b \in \pi \\ b \neq \hat{b}}} (\log_{\mathcal{P}} \varphi)(y_{\mathsf{delete}_{\ell+2,\pi}(b)}) \Big) (\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(\hat{b})}\big) \right)$$

$$- \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}}} \left(\Big(\prod_{\substack{b \in \pi \\ b \neq \hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b) \Big) \Big(\sum_{\substack{\{\sigma_1, \sigma_2\} \in \\ \mathcal{P}_{|\hat{b}|,2}^{\mathsf{pos}_{\hat{b}}(\ell+1) \vee \mathsf{pos}_{\hat{b}}(\ell+2)}}} (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big)(\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big) \Big) \right),$$

where $\hat{b}$ denotes the unique block for a partition $\pi \in \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}$ such that $\ell+1, \ell+2 \in \mathsf{set}\, \hat{b}$. For the proof of this claim consider the following calculation

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}}} \left(\Big(\prod_{\substack{b \in \pi \\ b \neq \hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b) \Big) \cdot (\log_{\mathcal{P}} \varphi)(x_{\hat{b}}) \right)$$

$$[\![\text{ by definition of } \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)} \text{ in eq. (3.1.29) } \exists \hat{b} \in \pi \colon \ell+1, \ell+2 \in \hat{b}]\!]$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\prod_{\substack{b\in\pi\\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$\cdot \Bigg((\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(\hat{b})}\big) - \sum_{\substack{\{\sigma_1,\sigma_2\}\in\\ \mathcal{P}_{|\hat{b}|,2}^{\mathsf{pos}_{\hat{b}}(\ell+1)\vee\mathsf{pos}_{\hat{b}}(\ell+2)}}} (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big)(\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big)\Bigg)\Bigg)$$

⟦ IH applied to $(\log_{\mathcal{P}} \varphi)(x_{\hat{b}})$ ⟧.

The last step in the above equation needs some justification. We want to apply the induction hypothesis to $|\hat{b}|$ instead of n, to $(\mathsf{pos}_{\hat{b}}(\ell+1))-1$ instead of ℓ, to $(x_i)_{i\in\mathsf{set}\,\hat{b}}$ and $(y_i)_{i\in\mathsf{delete}_{\ell+2,\pi}(\hat{b})}$ instead of $(x_i)_{i\in[n]}$ respectively $(y_i)_{i\in[n-1]}$. We have to show that for these index choices equation (3.2.12) and equation (3.2.14) are satisfied. But this is the case because any subsequence of $(x_i)_{i\in\mathsf{set}\,\hat{b}}$ can be regarded as a subsequence of $(x_i)_{i\in[n]}$. Since $|\hat{b}| \leq n$ and our assumption that equation (3.2.15) holds for all $m \in \{2,\dots,n-1\} \subseteq \mathbb{N}$, we are allowed to apply the induction hypothesis.

Moreover, as a consequence by the definition of the map $\mathsf{delete}_{n,\ell+2}$ in equation (3.1.6) and by the assumption of equation (3.2.12) and equation (3.2.14) we have

$$\begin{aligned} &\forall k \in \{1,\dots,n-1\}\ \forall \pi \in \mathsf{Part}_{n,k}^{(\ell+1)\wedge(\ell+2)}\ \forall b \in \pi \setminus \{\hat{b}\}: \\ &\qquad (\log_{\mathcal{P}} \varphi)(x_b) = (\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(b)}\big), \end{aligned} \tag{III}$$

where $\hat{b}$ denotes the unique block in a partition π such that $\ell+1, \ell+2 \in \mathsf{set}\,\hat{b}$.

Now, finally we obtain

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Bigg(\prod_{\substack{b\in\pi\\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b)\Bigg)(\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(\hat{b})}\big)\Bigg)$$

$$- \sum_{k=2}^{n-1} \sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Bigg(\prod_{\substack{b\in\pi\\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b)\Bigg)\Bigg(\sum_{\substack{\{\sigma_1,\sigma_2\}\in\\ \mathcal{P}_{|\hat{b}|,2}^{\mathsf{pos}_{\hat{b}}(\ell+1)\vee\mathsf{pos}_{\hat{b}}(\ell+2)}}} (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big)(\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big)\Bigg)\Bigg)$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Bigg(\prod_{\substack{b\in\pi\\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(b)}\big)\Bigg)(\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(\hat{b})}\big)\Bigg)$$

$$- \sum_{k=2}^{n-1} \sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Bigg(\prod_{\substack{b\in\pi\\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b)\Bigg)\Bigg(\sum_{\substack{\{\sigma_1,\sigma_2\}\in\\ \mathcal{P}_{|\hat{b}|,2}^{\mathsf{pos}_{\hat{b}}(\ell+1)\vee\mathsf{pos}_{\hat{b}}(\ell+2)}}} (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big)(\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big)\Bigg)\Bigg)$$

$[\![$ eq. (III) $]\!]$.

This proves the statement of equation (II). Now we are going to further investigate the summands of equation (II) and provide the following claim for the first summand of the right hand side, namely

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Big(\prod_{\substack{b\in\pi \\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(y_{\mathsf{delete}_{\ell+2,\pi}(b)})\Big) (\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(\hat{b})}\big) \Bigg)$$
$$= \sum_{k=2}^{n-1} \sum_{\pi\in\mathcal{P}_{n-1,k}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(y_b). \quad \text{(IV)}$$

For the proof of this claim consider the following calculation

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Big(\prod_{\substack{b\in\pi \\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(y_{\mathsf{delete}_{\ell+2,\pi}(b)})\Big) (\log_{\mathcal{P}} \varphi)\big(y_{\mathsf{delete}_{\ell+2,\pi}(\hat{b})}\big) \Bigg)$$
$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(y_{\mathsf{delete}_{\ell+2,\pi}(b)})$$
$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi'\in \\ \mathcal{P}_{n-1,k}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(y_{(\mathsf{double}_{n-1,\ell+1}\,\circ\,\mathsf{delete}_{n,\ell+2})(b)})$$
$$\left[\!\!\left[\begin{array}{l} \text{def. of } \mathsf{double}_{n-1,\ell+1} \restriction \mathcal{P}_{n-1,k} \colon \mathcal{P}_{n-1,k} \longrightarrow \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \text{ in eq. (3.1.39)} \\ \mathsf{double}_{n-1,\ell+1} \restriction \mathcal{P}_{n-1,k} \text{ is bijective by Lemma 3.1.11 \textbf{(a)}, Thm. 3.2.13} \end{array}\right]\!\!\right]$$
$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n-1,k}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(y_b) \quad [\![\text{ Lemma 3.1.11 \textbf{(a)} }]\!].$$

This proves the statement of equation (IV). Furthermore, the following assertion holds for the second summand of the right hand side of equation (II), namely

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Big(\prod_{\substack{b\in\pi \\ b\neq\hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b)\Big) \Big(\sum_{\substack{\{\sigma_1,\sigma_2\}\in \\ \mathcal{P}_{|\hat{b}|,2}^{\mathsf{pos}_{\hat{b}}(\ell+1)\vee\mathsf{pos}_{\hat{b}}(\ell+2)}}} (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big) (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big) \Big) \Bigg) \quad \text{(V)}$$
$$= \sum_{k=3}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b).$$

For the proof of this claim, consider the following calculation

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \left(\Big(\prod_{\substack{b\in\pi \\ b\neq \hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b) \Big) \Big(\sum_{\substack{\{\sigma_1,\sigma_2\}\in \\ \mathcal{P}_{|\hat{b}|,2}^{\mathrm{pos}_{\hat{b}}(\ell+1)\vee \mathrm{pos}_{\hat{b}}(\ell+2)}}} (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big)(\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big) \Big) \right)$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{(\pi,\{\sigma_1,\sigma_2\}) \\ \in \mathrm{sub}\left(\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\right)}} \Big(\prod_{\substack{b\in\pi \\ b\neq \hat{b}}} (\log_{\mathcal{P}} \varphi)(x_b) \Big) (\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_1}\big)(\log_{\mathcal{P}} \varphi)\big((x_{\hat{b}})_{\sigma_2}\big)$$

$[\![$ def. of $\mathrm{sub}\big(\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\big)$ in eq. (3.1.31) $]\!]$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \\ \in \mathcal{P}_{n,k+1}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$[\![$ $\mathsf{UMem}_{n,k+1}^{\ell+1} \restriction \mathcal{P}_{n,k+1}^{(\ell+1)\vee(\ell+2)} : \mathcal{P}_{n,k+1}^{(\ell+1)\vee(\ell+2)} \longrightarrow \mathrm{sub}\big(\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\big)$ is bijective by Lemma 3.1.11 **(b)**, Thm. 3.2.13 $]\!]$

$$= \sum_{k=3}^{n} \sum_{\substack{\pi \\ \in \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b).$$

This finishes the proof of the assertion of equation (V). Now we can plug in the results of equation (IV) and (V) into equation (II) and obtain

$$\sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b) = \sum_{\substack{\pi\in\mathcal{P}_{n-1} \\ \pi\neq \mathbb{1}_{n-1}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(y_b) - \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b) \tag{VI}$$

We make one final auxiliary claim

$$\varphi(x_1 \otimes \cdots \otimes x_n) = \varphi(y_1 \otimes \cdots \otimes y_{n-1}). \tag{VII}$$

This is a immediate consequence from equation (3.2.14).

Now, we are ready to finish the proof of the statement of equation (3.2.15) for arbitrary $n \in \mathbb{N} \setminus \{1, 2\}$ by strong induction. For this, we consider the following calculation

$$(\log_{\mathcal{P}}\varphi)(x_1 \otimes \cdots \otimes x_n)$$

$$= \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi\in\mathcal{P}_n \\ \pi\neq \mathbb{1}_n}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$= \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{k=2}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b) - \sum_{k=2}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$[\![$ $\bigcup_{k=2}^{n} (\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)} \cup \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}) = \mathcal{P}_n \setminus \{\mathbb{1}_n\}$ $]\!]$

$$= \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) - \sum_{\substack{\pi \in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$- \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\, \mathcal{P}_{n,n}^{(\ell+1)\wedge(\ell+2)} = \emptyset \,]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_{n-1} \\ \pi \neq \mathbb{1}_{n-1}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(y_b) + \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$- \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) - \sum_{\substack{\pi \in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ eq. (VI) }]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_{n-1} \\ \pi \neq \mathbb{1}_{n-1}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(y_b) - \sum_{\substack{\pi \in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$= \varphi(y_1 \otimes \cdots \otimes y_{n-1}) - \sum_{\substack{\pi \in \mathcal{P}_{n-1} \\ \pi \neq \mathbb{1}_{n-1}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(y_b) - \sum_{\substack{\pi \in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ eq. (VII) }]\!]$$

$$= (\log_{\mathcal{P}} \varphi)(y_1 \otimes \cdots \otimes y_{n-1}) - \sum_{\substack{\pi \in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ def. of } \log_{\mathcal{P}} \phi \text{ in eq. (3.2.5) }]\!]. \quad \square$$

The next lemma tells us that the logarithm $\log_{\mathcal{P}} \varphi$ w. r. t. a universal class of partitions $\mathcal{P}$ is uniquely determined by the application of φ to right-ordered monomials and some uniquely determined coefficients γ_π for all $\pi \in \mathsf{Part}$. Notice that we do not make any further statements about the properties of the coefficients γ_π. The essence of the following lemma is only to say that in the calculation of the $\log_{\mathcal{P}} \varphi$ only right-ordered monomials appear.

3.2.16 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let V be vector space and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$. Then, we have

$\forall n \in \mathbb{N},\ \forall (x_i)_{i \in [n]} \in V^{\times n},\ \forall \pi \in \mathsf{Part}_n,\ \exists \gamma_\pi \in \mathbb{C}:$

$$(\log_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) = \sum_{\pi \in \mathsf{Part}_n} \gamma_\pi \prod_{b \in \pi} \varphi(x_b) \tag{3.2.16}$$

Proof: The induction base $n = 1$ follows from equation (3.2.5). For the induction step $n \to n+1$ we have

$$(\log_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_{n+1})$$

$$= \varphi(x_1 \otimes \cdots \otimes x_{n+1}) - \sum_{\substack{\pi \in \mathcal{P}_{n+1} \\ |\pi| \geq 2}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b) \quad [\![\text{ eq. (3.2.5) }]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_{n+1}) - \sum_{\substack{\pi \in \mathsf{Part}_{n+1} \\ |\pi| \geq 2}} \gamma'_\pi \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(x_b)$$

$$[\![\, \forall \pi \in \mathsf{Part}_{n+1} \colon \gamma'_\pi := 1, \text{ for } \pi \in \mathcal{P}, \gamma'_\pi := 0, \text{ else} \,]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_{n+1}) - \sum_{\substack{\pi \in \mathsf{Part}_{n+1} \\ |\pi| \geq 2}} \gamma'_\pi \prod_{b \in \pi} \Big(\sum_{\sigma \in \mathsf{Part}_{|\mathsf{set}\, b|}} \gamma_\sigma \prod_{b' \in \sigma} \varphi(x_{b'}) \Big)$$

$$[\![\, \text{induction hypothesis applied} \,]\!]$$

$$= \varphi(x_1 \otimes \cdots \otimes x_{n+1}) - \sum_{k=2}^{n+1} \sum_{\substack{\pi \in \mathsf{Part}_{n+1} \\ |\pi| = k}} \gamma'_\pi \prod_{b \in \pi} \Big(\sum_{\sigma \in \mathsf{Part}_{|\mathsf{set}\, b|}} \gamma_\sigma \prod_{b' \in \sigma} \varphi(x_{b'}) \Big) \tag{I}$$

$$= \sum_{\pi \in \mathsf{Part}_{n+1}} \gamma_\pi \prod_{b \in \pi} \varphi(x_b).$$

For the last step we have used that the map

$$f_k \colon \begin{cases} \mathsf{Part}_{n+1,k} \longrightarrow \mathsf{Part}_{|\mathsf{set}\, b_1|} \times \cdots \times \mathsf{Part}_{|\mathsf{set}\, b_k|} \\ \{b_1, \ldots, b_k\} \longmapsto \big(\{b_1\}, \ldots, \{b_k\}\big) \end{cases}$$

is a bijection for all $k \in [n+1]$, i. e., we can sort all terms appearing in equation (I) by $\prod_{b \in \pi} \varphi(x_b)$ for any $\pi \in \mathsf{Part}_{n+1}$. The coefficient in front of them is denoted by γ_π. □

3.3 Partition induced universal product: single-colored case

We continue our quest to find a partition induced universal product by using a universal class of partitions. So far, we have a partition induced exponential $\exp_{\mathcal{P}}$, a partition induced logarithm $\log_{\mathcal{P}}$ and an operation $\circledcirc_{\mathcal{P}}$ living on a certain tensor algebra. The leitmotiv for the construction of these building pieces was to mimic the formula of equation (2.4.22) for a positive and symmetric u.a.u.-product in the single-faced case. Any universal product needs to be defined on the free product $\mathscr{A}_1 \sqcup \mathscr{A}_2$ for algebras $\mathscr{A}_1$ and $\mathscr{A}_2$. By Theorem 2.1.2 we know that $\mathscr{A}_1 \sqcup \mathscr{A}_2 \cong \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I$. Maybe we can lift $\varphi_1 \circledcirc_{\mathcal{P}} \varphi_2$ to the quotient algebra $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I$? In this sense, it is our goal to define a universal product $\varphi_1 \odot \varphi_2$ as a linear map on $\mathscr{A}_1 \sqcup \mathscr{A}_2$. We explicitly do this on $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I$ and therefore we shall apply the results of Section 3.2 to the case $V = \mathscr{A}_1 \oplus \mathscr{A}_2$. Lemma 3.3.2 then shows we actually may pass $\circledcirc_{\mathcal{P}}$ over to the quotient algebra.

3.3.1 Convention. Let $(\mathscr{A}_j)_{j \in [k]}$ be a k-tuple of algebras for some $k \in \mathbb{N}$. Consider the following commutative diagram for all $r \in [k]$

$$\begin{array}{ccc} \bigoplus_{i=1}^k \mathscr{A}_i & \xhookrightarrow{\ \text{inc}\ } & \mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i) \\ & {\scriptstyle \bigoplus_{i=1}^k \Delta_{i,r}} \searrow & \Big\downarrow {\scriptstyle \mathcal{T}(\bigoplus_{i=1}^k \Delta_{i,r})} \\ & & \mathscr{A}_r \end{array} \tag{3.3.1}$$

wherein we put ("map-like Kronecker delta")

$$\forall\, r \in [k]\colon \Delta_{i,r} := \begin{cases} 0\colon \mathscr{A}_i \longrightarrow \mathscr{A}_r & \text{for } i \neq r \\ \mathsf{id}_{\mathscr{A}_r}\colon \mathscr{A}_r \longrightarrow \mathscr{A}_r & \text{for } i = r. \end{cases} \tag{3.3.2}$$

We shall use the following abbreviation

$$\forall\, r \in [k]\colon \mathrm{j}_i := \mathcal{T}\Big(\bigoplus_{i=1}^{k} \Delta_{i,r}\Big). \tag{3.3.3}$$

In Lemma 3.2.9 we have shown that $\circledcirc_{\mathcal{P}}$ is an associative binary operation. Hence, we can omit putting brackets for products with respect to $\circledcirc_{\mathcal{P}}$. Therefore, the following statement makes sense.

3.3.2 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let $(\mathscr{A}_i)_{i\in[k]}$ be a k-tuple of algebras and $(\varphi_i)_{i\in[k]} \in \prod_{i=1}^{k} \operatorname{Lin}(\mathscr{A}_i, \mathbb{C}))$ for some $k \in \mathbb{N} \setminus \{1\}$. Let $n \in \mathbb{N} \setminus \{1\}$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [k]^{\times n}$ and assume that the tuple ε has the property

$$\exists\, \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}\colon \varepsilon_{\ell+1} = \varepsilon_{\ell+2}. \tag{3.3.4}$$

Then, for any $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$ the equation

$$\begin{aligned} &\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)\overbrace{(a_1 \otimes \cdots \otimes a_{\ell+1} \otimes a_{\ell+2} \otimes \cdots \otimes a_n)}^{\in \mathrm{T}(\bigoplus_{i=1}^{k} \mathscr{A}_i)} \\ &\qquad = \big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)(a_1 \otimes \cdots \otimes (a_{\ell+1} \cdot a_{\ell+2}) \otimes \cdots \otimes a_n) \end{aligned} \tag{3.3.5}$$

holds.

Proof: We can consider two different cases. In both cases let $a_1 \otimes \cdots \otimes a_n \in \mathscr{A}_{\varepsilon_1} \otimes \cdots \otimes \mathscr{A}_{\varepsilon_n}$, where $(\varepsilon_i)_{i\in[n]} \in [k]^{\times n}$ satisfies equation (3.3.4). In the first case it is assumed, that

$$\exists\, c \in [k],\ \forall\, i \in [n]\colon \varepsilon_i = c \tag{I}$$

and in the second case that

$$\exists\, j \in [n] \setminus \{\ell+1, \ell+2\}\colon\ \varepsilon_j \neq \varepsilon_{\ell+1}. \tag{II}$$

We begin with the proof for the first case and calculate

$$\begin{aligned} &\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)(a_1 \otimes \cdots \otimes a_n) \\ &= \Big(\exp_{\mathcal{P}}\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)\Big)(a_1 \otimes \cdots \otimes a_n) \quad [\![\text{ eq. (3.2.8) }]\!] \\ &= \sum_{\pi\in\mathcal{P}_n} \prod_{b\in\pi} \Big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b) \quad [\![\text{ property of } \exp_{\mathcal{P}} \text{ in eq. (3.2.4) }]\!] \end{aligned}$$

$$= \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \big(\log_{\mathcal{P}}(\varphi_c \circ \mathrm{j}_c)\big)(a_b) \quad \left[\!\!\left[\begin{array}{l} \text{case defined in (I)} \\ \implies \forall j \in [n], \forall i \in [k] \setminus \{c\} \colon a_j \in \ker \mathrm{j}_i \\ \text{apply Lemma 3.2.12 to } \varphi_i \text{ instead of } \varphi \\ \text{and } \mathrm{j}_i \text{ instead of } f \text{, use Convention 3.3.1} \end{array} \right]\!\!\right]$$

$$= \Big(\exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi_c \circ \mathrm{j}_c)\big)\Big)(a_1 \otimes \cdots \otimes a_n) \quad [\![\text{ property of } \exp_{\mathcal{P}} \text{ in eq. (3.2.4) }]\!]$$

$$= (\varphi_c \circ \mathrm{j}_c)(a_1 \otimes \cdots \otimes a_n) \quad [\![\text{ by Lemma 3.2.7 } \exp_{\mathcal{P}} \text{ is inverse to } \log_{\mathcal{P}}]\!]$$

$$= \varphi_c\big(\mathrm{j}_c(a_1) \otimes \cdots \otimes \mathrm{j}_c(a_n)\big) \quad [\![\mathrm{j}_c \in \mathrm{Alg}(\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2), \mathscr{A}_1)]\!]$$

$$= \varphi_c(a_1 \cdot \cdots \cdot (a_{\ell+1} \cdot a_{\ell+2}) \cdot \cdots \cdot a_n)$$

$$= (\varphi_1 \circ \mathrm{j}_c)(a_1 \otimes \cdots \otimes (a_{\ell+1} \cdot a_{\ell+2}) \otimes \cdots \otimes a_n)$$

$$= \Big(\exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi_c \circ \mathrm{j}_c)\big)\Big)(a_1 \otimes \cdots \otimes (a_{\ell+1} \cdot a_{\ell+2}) \otimes \cdots \otimes a_n)$$

$$[\![\text{ property of } \exp_{\mathcal{P}} \cdot \text{ from eq. (3.2.4) }]\!]$$

$$= \ldots = \Big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\Big)(a_1 \otimes \cdots \otimes (a_{\ell+1} \cdot a_{\ell+2}) \otimes \cdots \otimes a_n)$$

$$\left[\!\!\left[\begin{array}{l} \text{same steps apply as in the beginning of the calculation,} \\ \text{since } \forall i \in [n], \forall j \in [k] \setminus \{c\} \colon a_i \in \ker \mathrm{j}_j \end{array} \right]\!\!\right].$$

Before proving the second case, we set

$$\begin{aligned} \forall i \in [n] \colon x_i &\coloneqq a_i \\ \forall i \in \{1, \ldots, \ell\} \subseteq \mathbb{N} \colon y_i &\coloneqq x_i \\ y_{\ell+1} &\coloneqq a_{\ell+1} \cdot a_{\ell+2} \\ \forall i \in \{\ell+2, \ldots, n-1\} \subseteq \mathbb{N} \colon y_i &\coloneqq a_{i+1}. \end{aligned} \tag{III}$$

For the second case we verify the assertion by direct calculation for some $n \in \mathbb{N} \setminus \{1\}$.

$$\begin{aligned} &\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)(a_1 \otimes \cdots \otimes a_n) \\ &\quad = \big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)(x_1 \otimes \cdots \otimes x_n) \\ &\quad = \Big(\exp_{\mathcal{P}}\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)\Big)(x_1 \otimes \cdots \otimes x_n) \quad [\![\text{ eq. (3.2.8) }]\!] \\ &\quad = \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) \quad [\![\text{ property of } \exp_{\mathcal{P}} \cdot \text{ from eq. (3.2.4) }]\!] \\ &\quad = \sum_{k=1}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \vee (\ell+2)}}} \prod_{b \in \pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) + \sum_{k=1}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1) \wedge (\ell+2)}}} \prod_{b \in \pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) \end{aligned}$$

$$\left[\!\!\left[\bigcup_{k=1}^{n} (\mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)} \mathbin{\dot\cup} \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}) = \mathcal{P}_n \right]\!\!\right]$$

$$= \sum_{\substack{\{b_1,b_2\}\in \\ \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)}}} \Big(\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_{b_1}) \Big) \Big(\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_{b_2}) \Big)$$

$$+ \sum_{k=3}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) + \sum_{k=1}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$$[\![\, \mathcal{P}_{n,1}^{(\ell+1)\vee(\ell+2)} = \emptyset \,]\!]$$

$$= 0 + \sum_{k=3}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$$+ \sum_{k=1}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$$\left[\!\!\left[\begin{array}{l} \text{according to Conv. 3.3.1 we have } \mathrm{j}_i = \mathcal{T}(\bigoplus_{j=1}^{k} \Delta_{j,i}) \\ \text{and } \forall\, i \in [k] \setminus \{\varepsilon_{\ell+1}\}\colon a_{\ell+1}, a_{\ell+2} \in \ker(\mathrm{j}_i) \\ \text{application of Lemma 3.2.12 to } \sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i) \text{ yields} \\ \text{case defined in eq. (II)} \\ \implies \forall \pi \in \mathcal{P}_{n,2}^{(\ell+1)\vee(\ell+2)},\ \forall b \in \pi\colon x_b \in \ker\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big) \end{array} \right]\!\!\right]$$

$$= \sum_{k=3}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) + \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_1 \otimes \cdots \otimes x_n)$$

$$+ \sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$$= \sum_{k=3}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) + 0 + \sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$$[\![\, \mathcal{P}_{n,n}^{(\ell+1)\wedge(\ell+2)} = \emptyset \,]\!]$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) + \sum_{k=3}^{n} \sum_{\substack{\pi\in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Big(\prod_{\substack{b\in\pi \\ b\neq b'}} \Big(\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) \Big) \cdot \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_{b'}) \Big)$$

$$+ \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$\llbracket\, \forall \pi \in \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\ \exists b' \in \pi\colon \ell+1,\ \ell+2 \in \mathsf{set}\, b' \,\rrbracket$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Big(\prod_{\substack{b\in\pi \\ b\neq b'}} \Big(\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) \Big) \cdot \log_{\mathcal{P}}(\varphi_{\varepsilon_{\ell+1}} \circ \mathrm{j}_{\varepsilon_{\ell+1}})(x_{b'}) \Big)$$

$$+ \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$\llbracket\, \forall i \in [k] \setminus \{\varepsilon_{\ell+1}\}\colon a_{\ell+1}, a_{\ell+2} \in \ker(\mathrm{j}_i)$, then apply Lemma 3.2.12 $\rrbracket$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\prod_{\substack{b\in\pi \\ b\neq b'}} \Big(\big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b) \Big)$$

$$\cdot \Big(\big(\log_{\mathcal{P}}(\varphi_{\varepsilon_{\ell+1}} \circ \mathrm{j}_{\varepsilon_{\ell+1}})\big)(y_{\mathsf{delete}_{\ell+2,\pi}(b')})$$

$$- \sum_{\substack{\{\sigma_1,\sigma_2\} \in \\ \mathcal{P}_{|b'|,2}^{\mathsf{pos}_{b'}(\ell+1)\vee \mathsf{pos}_{b'}(\ell+2)}}} \big(\log_{\mathcal{P}}(\varphi_{\varepsilon_{\ell+1}} \circ \mathrm{j}_{\varepsilon_{\ell+1}})\big)\big((x_{b'})_{\sigma_1}\big) \big(\log_{\mathcal{P}}(\varphi_{\varepsilon_{\ell+1}} \circ \mathrm{j}_{\varepsilon_{\ell+1}})\big)\big((x_{b'})_{\sigma_2}\big) \Big) \Bigg)$$

$$+ \sum_{k=3}^{n} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}} \prod_{b\in\pi} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(x_b)$$

$\llbracket$ prequisites of Lemma 3.2.15 are satisfied, then apply eq. (3.2.15) to $\varphi_{\varepsilon_{\ell+1}} \circ \mathrm{j}_{\varepsilon_{\ell+1}}$ instead of φ and $(x_i)_{i\in \mathsf{set}(b')}$ instead of $(x_i)_{i\in[n]}$ $\rrbracket$

$$= \sum_{k=2}^{n-1} \sum_{\substack{\pi \in \\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}} \Bigg(\Big(\prod_{\substack{b\in\pi \\ b\neq b'}} \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(\underbrace{x_b}_{=y_b}) \Big) \cdot \big(\sum_{i=1}^{k} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(y_{\mathsf{delete}_{\ell+2,\pi}(b')}) \Bigg)$$

$$-\sum_{k=2}^{n-1}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}}\Bigg(\prod_{\substack{b\in\pi\\ b\neq b'}}\Big(\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(x_b)$$

$$\cdot\sum_{\substack{\{\sigma_1,\sigma_2\}\in\\ \mathcal{P}_{|b'|,2}^{\mathsf{pos}_{b'}(\ell+1)\vee\mathsf{pos}_{b'}(\ell+2)}}}\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)\big((x_{b'})_{\sigma_1}\big)\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)\big((x_{b'})_{\sigma_2}\big)\Big)\Bigg)$$

$$+\sum_{k=3}^{n}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}}\prod_{b\in\pi}\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(x_b)$$

⟦ $\forall i\in[k]\setminus\{\varepsilon_{\ell+1}\}\colon a_{\ell+1},a_{\ell+2},a_{\ell+1}\cdot a_{\ell+2}\in\ker(\mathrm{j}_i)$ then apply Lemma 3.2.12 ⟧

$$=\sum_{k=2}^{n-1}\sum_{\substack{\pi\in\mathcal{P}_{n-1,k}\\ \pi\neq\mathbb{1}_{n-1}}}\prod_{b\in\pi}\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(y_b)$$

$$-\sum_{k=2}^{n-1}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}}}\Bigg(\prod_{\substack{b\in\pi\\ b\neq b'}}\Big(\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(x_b)\Big)$$

$$\cdot\sum_{\substack{\{\sigma_1,\sigma_2\}\in\\ \mathcal{P}_{|b'|,2}^{\mathsf{pos}_{b'}(\ell+1)\vee\mathsf{pos}_{b'}(\ell+2)}}}\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)\big((x_{b'})_{\sigma_1}\big)\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)\big((x_{b'})_{\sigma_2}\big)\Bigg)$$

$$+\sum_{k=3}^{n}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}}\prod_{b\in\pi}\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(x_b)$$

⟦ def. of $\mathsf{double}_{n-1,\ell+1}\restriction\mathcal{P}_{n-1,k}\colon\mathcal{P}_{n-1,k}\longrightarrow\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}$ in eq. (3.1.39) $\mathsf{double}_{n-1,\ell+1}\restriction\mathcal{P}_{n-1,k}$ is bijective by Lemma 3.1.11 **(a)**, Thm. 3.2.13 ⟧

$$=\sum_{k=2}^{n-1}\sum_{\substack{\pi\in\mathcal{P}_{n-1,k}\\ \pi\neq\mathbb{1}_{n-1}}}\prod_{b\in\pi}\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(y_b)$$

$$-\sum_{k=2}^{n-1}\sum_{\substack{(\pi,\{\sigma_1,\sigma_2\})\\ \in\mathrm{sub}\left(\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\right)}}\Bigg(\prod_{\substack{b\in\pi\\ b\neq b'}}\Big(\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)(x_b)\Big)$$

$$\cdot\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)\big((x_{b'})_{\sigma_1}\big)\big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\big)\big((x_{b'})_{\sigma_2}\big)\Bigg)$$

$$+\sum_{k=3}^{n}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}}\prod_{b\in\pi}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(x_b)$$

⟦ def. of $\mathrm{sub}\big(\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\big)$ in eq. (3.1.31) ⟧

$$=\sum_{k=2}^{n-1}\sum_{\pi\in\mathcal{P}_{n-1,k}}\prod_{b\in\pi}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(y_b)$$

$$-\sum_{k=3}^{n}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}}\prod_{b\in\pi}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(x_b)+\sum_{k=3}^{n}\sum_{\substack{\pi\in\\ \mathcal{P}_{n,k}^{(\ell+1)\vee(\ell+2)}}}\prod_{b\in\pi}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(x_b)$$

⟦ $\mathsf{UMem}_{n,k+1}^{\ell+1}\restriction\mathcal{P}_{n,k+1}^{(\ell+1)\vee(\ell+2)}\colon\mathcal{P}_{n,k+1}^{(\ell+1)\vee(\ell+2)}\longrightarrow\mathrm{sub}\big(\mathcal{P}_{n,k}^{(\ell+1)\wedge(\ell+2)}\big)$ is bijective by Lemma 3.1.11 **(b)**, Thm. 3.2.13 ⟧

$$=\underbrace{\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(y_1\otimes\cdots\otimes y_{n-1})}_{=0}+\sum_{k=2}^{n-1}\sum_{\pi\in\mathcal{P}_{n-1,k}}\prod_{b\in\pi}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(y_b)$$

⟦ $\forall\, i\in[k]\setminus\{\varepsilon_{\ell+1}\}\colon y_{\ell+1}=a_{\ell+1}\cdot a_{\ell+2}\in\ker(\mathrm{j}_i)$, then apply Lemma 3.2.12 ⟧

$$=\underbrace{\sum_{k=1}^{n-1}\sum_{\pi\in\mathcal{P}_{n-1,k}}}_{=\sum_{\pi\in\mathcal{P}_{n-1}}}\prod_{b\in\pi}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)(y_b)$$

$$=\Big(\exp_{\mathcal{P}}\Big(\sum_{i=1}^{k}\log_{\mathcal{P}}(\varphi_i\circ \mathrm{j}_i)\Big)\Big)(y_1\otimes\cdots\otimes y_{n-1})$$ ⟦ property of $\exp_{\mathcal{P}}$ from eq. (3.2.4) ⟧

$$=\big((\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ \mathrm{j}_k)\big)(y_1\otimes\cdots\otimes y_{n-1})$$ ⟦ definition of $\circledcirc_{\mathcal{P}}$ in eq. (3.2.7) ⟧

$$=\big((\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ \mathrm{j}_k)\big)\big(a_1\otimes\cdots\otimes(a_{\ell+1}\cdot a_{\ell+2})\otimes\cdots\otimes\big)$$

⟦ def. of $(y_i)_{i\in[n]}$ in eq. (III) ⟧. □

3.3.3 Remark. Assume $(\mathscr{A}_i)_{i\in[k]}$ is an k-tuple of algebras for some $k\in\mathbb{N}\setminus\{1\}$, then we define

$$S_{1,\dots,k}:=\{\,a\otimes a'-a\cdot a'\mid i\in[k]\colon a,\,a'\in\mathscr{A}_i\,\}\subseteq \mathrm{T}\Big(\bigoplus_{i=1}^{k}\mathscr{A}_i\Big),\tag{3.3.6}$$

$$I_{1,\dots,k}:=\langle S_{1,\dots,k}\rangle\tag{3.3.7}$$

i. e., $I_{1,\dots,k}$ denotes the smallest two-sided ideal in $\mathrm{T}(\bigoplus_{i=1}^{k}\mathscr{A}_i)$ such that $I_{1,\dots,k}\supseteq S_{1,\dots,k}$. If we set

$$I_{S_{1,\dots,k}}:=\left\{\sum_{i=1}^{n}\Big(cs_i+xs_i+s_iy+\sum_{j=1}^{N_i}x_js_iy_j\Big)\,\middle|\,\begin{array}{l}n,\,N_i\in\mathbb{N},\ c\in\mathbb{C},\ s_i\in S_{1,\dots,k},\\ x,\,y,\,x_j,\,y_j\in\mathrm{T}(\bigoplus_{i=1}^{k}\mathscr{A}_i)\end{array}\right\},\tag{3.3.8}$$

then it is a standard task to show that $I_{1,\dots,2} = I_{S_{1,\dots,k}}$ (for instance look at [MV04, Prop. 23.3]).

3.3.4 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Let $(\mathscr{A}_i)_{i\in[k]}$ be a k-tuple of algebras for some $k \in \mathbb{N} \setminus \{1\}$ and $\varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ for each $i \in [k]$. Then $I_{S_{1,\dots,k}} \subseteq \ker((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)) \subseteq \mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i)$.

Proof: Due to equation (3.3.8) any element of $I_{S_{1,\dots,k}}$ is a linear combination of elements with certain elements from the set $S_{1,\dots,k}$ multiplied with arbitrary elements from $\mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i)$ and due to the fact that $(\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)$ is a linear map, it suffices to show

$$\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)\big(a_1 \otimes \cdots \otimes a_{\ell+1} \otimes a_{\ell+2} \otimes \cdots \otimes a_n \\ - a_1 \otimes \cdots \otimes (a_{\ell+1} \cdot a_{\ell+2}) \otimes \cdots \otimes a_n\big) = 0$$

for any $n \in \mathbb{N} \setminus \{1\}$, for any $\ell \in \{0, \dots, n-2\}$, for any $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [k]^{\times n}$ with $\varepsilon_{\ell+1} = \varepsilon_{\ell+2}$ and for any $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_i}$. But, the above equation is just a restatement of the assertion of Lemma 3.3.2. Therefore, equation (3.3.5) implies the assertion. □

Because of Lemma 3.3.4, we may define the following:

3.3.5 Definition (Universal product induced by a universal class of partitions). Let $\mathcal{P}$ be a universal class of partitions. Let $(\mathscr{A}_i)_{i\in[k]}$ be a k-tuple of algebras and $\varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$, $i \in [k]$ for some $k \in \mathbb{N} \setminus \{1\}$. From the universal property of the quotient space and by Lemma 3.3.4 there exists a unique linear map $\mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big) \in \mathrm{Lin}(\mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i)/I_{1,\dots,k}, \mathbb{C})$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i) & \xrightarrow{(\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ \mathrm{j}_k)} & \mathbb{C} \\ \downarrow \mathsf{pr} & \nearrow_{\mathsf{lift}\left((\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ \mathrm{j}_k)\right)} & \\ \mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i)\big/I_{1,\dots,k} & & \end{array} . \tag{3.3.9}$$

If we apply the above setting to the case $k = 2$, then we set

$$(\varphi_1 \circ \mathrm{j}_1) \mathbin{\widetilde{\circledcirc_{\mathcal{P}}}} (\varphi_2 \circ \mathrm{j}_2) := \mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2)\big). \tag{3.3.10}$$

Let us denote by $\mathbf{i}_j \colon \mathscr{A}_j \hookrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I_{1,2}$ the canonical injections. Then,

$$\mathbf{i}_1 \sqcup \mathbf{i}_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I_{1,2} \tag{3.3.11}$$

is the canonical isomorphism of algebras, depicted by the following diagram for $i \in [2]$

$$\begin{array}{ccc} & & \mathscr{A}_1 \sqcup \mathscr{A}_2 \\ & \nearrow^{\iota_i} & \downarrow \mathbf{i}_1\sqcup\mathbf{i}_2 \\ \mathscr{A}_i & \xrightarrow{\mathbf{i}_i} & \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)\big/I_{1,2} \quad \mathsf{id} \\ & \searrow_{\iota_i} & \downarrow \iota_1 \sqcup \iota_2 \\ & & \mathscr{A}_1 \sqcup \mathscr{A}_2 \end{array} . \tag{3.3.12}$$

By this we define

$$\varphi_1 \odot_{\mathcal{P}} \varphi_2 \colon \begin{cases} \mathcal{A}_1 \sqcup \mathcal{A}_2 \longrightarrow \mathbb{C} \\ a \longmapsto \Big(\big((\varphi_1 \circ \mathrm{j}_1) \widetilde{\circledcirc_{\mathcal{P}}} (\varphi_2 \circ \mathrm{j}_2) \big) \circ (\mathbf{i}_1 \sqcup \mathbf{i}_2) \Big)(a). \end{cases} \tag{3.3.13}$$

Finally, we set

$$\odot_{\mathcal{P}} \colon \begin{cases} \mathrm{Lin}(\mathcal{A}_1, \mathbb{C}) \times \mathrm{Lin}(\mathcal{A}_2, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{A}_1 \sqcup \mathcal{A}_2, \mathbb{C}) \\ (\varphi_1, \varphi_2) \longmapsto \varphi_1 \odot_{\mathcal{P}} \varphi_2. \end{cases} \tag{3.3.14}$$

3.3.6 Remark. The construction of the free product of algebras from Theorem 2.1.2 can be generalized to a finite indexed family of algebras $(\mathcal{A}_i)_{i\in I}$ with $|I| < \infty$. This means for $I = [k]$ for some $k \in \mathbb{N}$ by the definition of the ideal $I_{1,\dots,k}$ from equation (3.3.7) we have

$$\mathrm{T}\big(\textstyle\bigoplus_{i=1}^k \mathcal{A}_i\big)\Big/ I_{1,\dots,k} \cong \displaystyle\coprod_{i\in[k]} \mathcal{A}_i. \tag{3.3.15}$$

In other words, the left hand side of the above equation satisfies the universal property of the coproduct for $(\mathcal{A}_i)_{i\in[k]}$ in the category Alg in the sense discussed in Remark 1.1.7.

By the above remark in mind, the assertion of the next lemma makes sense.

3.3.7 Lemma. Let $\mathcal{P}$ be a universal class of partitions. For any $k \in \mathbb{N} \setminus \{1\}$ and $(\mathcal{A}_i, \varphi_i)_{i\in[k]} \in (\mathsf{Obj}(\mathsf{AlgP}))^{\times k}$ we have

$$\begin{aligned} &\Big(\big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \odot_{\mathcal{P}} \varphi_3 \big) \cdots \Big) \odot_{\mathcal{P}} \varphi_k \\ &\qquad = \mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k) \big) \circ \mathsf{can}, \end{aligned} \tag{3.3.16}$$

where $\circledcirc_{\mathcal{P}}$ here denotes the binary operation on the dual space of $\mathrm{T}(\bigoplus_{i=1}^k \mathcal{A}_i)$ and can is the canonical isomorphism of algebras, i. e., $\mathsf{can} \colon (((\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3) \dots) \sqcup \mathcal{A}_k \longrightarrow \mathrm{T}(\bigoplus_{i=1}^k \mathcal{A}_i)/I_{1,\dots,k}$.

Proof: We prove the statement by induction over $k \in \mathbb{N} \setminus \{1\}$. The induction base $k = 2$ holds, since equation (3.3.13) gives us

$$\varphi_1 \odot_{\mathcal{P}} \varphi_2 = \big((\varphi_1 \circ \mathrm{j}_1) \widetilde{\circledcirc_{\mathcal{P}}} (\varphi_2 \circ \mathrm{j}_2) \big) \circ (\mathbf{i}_1 \sqcup \mathbf{i}_2),$$

where $\mathbf{i}_1 \sqcup \mathbf{i}_2 \colon \mathcal{A}_1 \sqcup \mathcal{A}_2 \longrightarrow \mathrm{T}(\mathcal{A}_1 \oplus \mathcal{A}_2)/I_{1,2}$ denotes the canonical isomorphism of algebras.

We now perform the induction step $k \to k+1$ and assume that $k \in \mathbb{N} \setminus \{1\}$, and that equation (3.3.16) holds for all $k' \leq k$. We set

$$\forall k' \in \mathbb{N} \setminus \{1\} \colon \mathcal{A}_{1\sqcup\dots\sqcup k'} \coloneqq \Big(\big((\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 \big) \cdots \Big) \sqcup \mathcal{A}_{k'}.$$

By $I_{1\sqcup\dots\sqcup k, k+1} \subseteq \mathrm{T}(\mathcal{A}_{1\sqcup\dots\sqcup k} \oplus \mathcal{A}_{k+1})$ we mean the two-sided ideal from Remark 3.3.3 in equation (3.3.7), where we use $\mathcal{A}_{1\sqcup\dots\sqcup k}$ instead of $\mathcal{A}_1$ and $\mathcal{A}_{k+1}$ instead of $\mathcal{A}_2$. Moreover, we recall the definition for all $k' \in \mathbb{N} \setminus \{1\}$ that $I_{1,\dots,k'} \subseteq \mathrm{T}(\bigoplus_{i\in[k']} \mathcal{A}_i)$ denotes the two-sided ideal generated by the set $\{\, a \otimes a' - a \cdot a' \mid i \in [k'] \colon a,\, a' \in \mathcal{A}_i \,\}$.

In the first step we claim that the following diagram is a commutative diagram of algebra homomorphisms

$$
\begin{array}{ccccc}
 & \overset{\mathrm{T}\left(\bigoplus_{i=1}^{k+1}\mathsf{inc}_{\mathcal{A}_i,\mathcal{A}_{1\sqcup\cdots\sqcup k}\oplus\mathcal{A}_{k+1}}\right)}{\nearrow} & \mathrm{T}(\mathcal{A}_{1\sqcup\cdots\sqcup k}\oplus\mathcal{A}_{k+1}) & \xrightarrow{\mathcal{T}(\mathsf{id}_{\mathcal{A}_{1\sqcup\cdots\sqcup k}}\oplus 0)} & \mathcal{A}_{1\sqcup\cdots\sqcup k} \\
\mathrm{T}(\bigoplus_{i=1}^{k+1}\mathcal{A}_i) & & & & \Big\downarrow \mathsf{can} \\
 & \underset{\mathrm{T}(\bigoplus_{i=1}^{k}\mathsf{id}_{\mathcal{A}_i}\oplus 0)}{\searrow} & \mathrm{T}(\bigoplus_{i=1}^{k}\mathcal{A}_i) & \xrightarrow[\mathsf{pr}]{} & \mathrm{T}(\bigoplus_{i=1}^{k}\mathcal{A}_i)\Big/I_{1,\dots,k}.
\end{array}
\qquad . \qquad \text{(I)}
$$

Because all maps of consideration are homomorphisms of algebras and since the tensor algebra $\mathrm{T}(\bigoplus_{i=1}^{3}\mathcal{A}_i)$ is generated by elements of $\bigoplus_{i=1}^{3}\mathcal{A}_i$, it suffices to prove the commutativity for an element of the form $\mathsf{inc}_{\mathcal{A}_i,\mathrm{T}(\bigoplus_{i=1}^{3}\mathcal{A}_i)}(a)$ for some $a\in\mathcal{A}_i$ and any $i\in[k+1]$. Let $i\in[k]$, then for the lower path in the diagram we have

$$
\begin{aligned}
&\big(\mathsf{pr}\circ\mathrm{T}(\bigoplus_{i=1}^{k}\mathsf{id}_{\mathcal{A}_i}\oplus 0)\big)\big(\mathsf{inc}_{\mathcal{A}_i,\mathrm{T}(\bigoplus_{i=1}^{k+1}\mathcal{A}_i)}(a)\big)\\
&\quad=\mathsf{pr}\circ\mathsf{inc}_{\mathcal{A}_i,\mathrm{T}(\mathcal{A}_1\oplus\mathcal{A}_2)}(a)\quad [\![\text{ UMP of tensor algebra }]\!]
\end{aligned}
$$

And for the upper path in the diagram we have

$$
\begin{aligned}
&\Big(\mathsf{can}\circ\mathcal{T}(\mathsf{id}_{\mathcal{A}_{1\sqcup\cdots\sqcup k}}\oplus 0)\circ\mathrm{T}\big(\bigoplus_{i=1}^{k+1}\mathsf{inc}_{\mathcal{A}_i,\mathcal{A}_{1\sqcup\cdots\sqcup k}\oplus\mathcal{A}_{k+1}}\big)\Big)\big(\mathsf{inc}_{\mathcal{A}_i,\mathrm{T}(\bigoplus_{i=1}^{k+1}\mathcal{A}_i)}(a)\big)\\
&=\Big(\mathsf{can}\circ\mathcal{T}(\mathsf{id}_{\mathcal{A}_{1\sqcup\cdots\sqcup k}}\oplus 0)\circ\mathsf{inc}_{\mathcal{A}_i,\mathrm{T}(\mathcal{A}_{1\sqcup\cdots\sqcup k}\oplus\mathcal{A}_{k+1})}\Big)(a)\\
&=\mathsf{can}\big(\iota_{\mathcal{A}_i,\mathcal{A}_{1\sqcup 2}}(a)\big)\\
&=\mathsf{pr}\circ\mathsf{inc}_{\mathcal{A}_i,\mathrm{T}(\mathcal{A}_1\oplus\mathcal{A}_2)}(a)\quad [\![\text{ UMP of free product of algebras }]\!].
\end{aligned}
$$

If $a\in\mathcal{A}_{k+1}$ both sides of the equation give zero. This proves the statement of equation (I).

We use the above commutativity of the diagram for the following calculation

$$
\begin{aligned}
&\mathsf{lift}\big((\varphi_1\circ\mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ\mathrm{j}_k)\big)\circ\mathsf{can}\circ\mathcal{T}(\mathsf{id}_{\mathcal{A}_{1\sqcup\cdots\sqcup k}}\oplus 0)\circ\mathrm{T}\big(\bigoplus_{i=1}^{k+1}\mathsf{inc}_{\mathcal{A}_i,\mathcal{A}_{1\sqcup\cdots\sqcup k}\oplus\mathcal{A}_{k+1}}\big)\\
&=\mathsf{lift}\big((\varphi_1\circ\mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ\mathrm{j}_k)\big)\circ\mathsf{pr}\circ\mathrm{T}(\bigoplus_{i=1}^{k}\mathsf{id}_{\mathcal{A}_i}\oplus 0)\quad [\![\text{ eq. (I) }]\!]\\
&=\big((\varphi_1\circ\mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ\mathrm{j}_k)\big)\circ\mathrm{T}(\bigoplus_{i=1}^{k}\mathsf{id}_{\mathcal{A}_i}\oplus 0)\quad [\![\text{ eq. (3.3.9) }]\!]\\
&=\big(\varphi_1\circ\mathcal{T}(\bigoplus_{j=1}^{k}\Delta_{j,1}\circ\mathrm{T}(\bigoplus_{i=1}^{k}\mathsf{id}_{\mathcal{A}_i}\oplus 0)\big)\\
&\qquad\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}\big(\varphi_k\circ\mathcal{T}(\bigoplus_{j=1}^{k}\Delta_{j,k}\circ\mathrm{T}(\bigoplus_{i=1}^{k}\mathsf{id}_{\mathcal{A}_i}\oplus 0)\big)
\end{aligned}
$$

$$\left[\!\!\left[\, \text{Conv. 3.3.1}, \bigoplus_{i=1}^{k} \mathsf{id}_{\mathscr{A}_i} \oplus 0 \in \mathrm{Lin}\big(\bigoplus_{i=1}^{k+1} \mathscr{A}_i, \bigoplus_{i=1}^{k} \mathscr{A}_i\big), \text{Lemma 3.2.11 } \mathbf{(c)} \,\right]\!\!\right]$$

$$= \Big(\varphi_1 \circ \mathcal{T}\big(\bigoplus_{j=1}^{k} \Delta_{j,1} \oplus 0\big)\Big) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \Big(\varphi_k \circ \mathcal{T}\big(\bigoplus_{j=1}^{k} \Delta_{j,k} \oplus 0\big)\Big). \tag{II}$$

As a last preparatory step, we claim that the following diagram is commutative for all $i \in [k+1]$

$$\begin{array}{ccccc}
\mathscr{A}_i & \hookrightarrow & \bigoplus_{i=1}^{k+1} \mathscr{A}_i & \hookrightarrow & \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathscr{A}_i) \\
\downarrow \widetilde{\mathsf{inc}}_i & & \downarrow \bigoplus_{i=1}^{k} \widetilde{\mathsf{inc}}_i & & \downarrow \mathrm{T}(\bigoplus_{i=1}^{k} \widetilde{\mathsf{inc}}_i) \\
\mathscr{A}_{1\sqcup\cdots\sqcup k} \oplus \mathscr{A}_{k+1} & \hookrightarrow & \mathscr{A}_{1\sqcup\cdots\sqcup k} \bigoplus \mathscr{A}_{k+1} & \hookrightarrow & \mathrm{T}(\mathscr{A}_{1\sqcup\cdots\sqcup k} \bigoplus \mathscr{A}_{k+1})
\end{array} . \tag{III}$$

The commutativity of the diagram is given by the universal mapping property of the tensor algebra. Now, we calculate for an arbitrary basis element of $\mathscr{A}_{1\sqcup\cdots\sqcup(k+1)}$ with $n \in \mathbb{N}$, $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k+1])$ and $\forall\, i \in [n]$: $a_i \in \mathscr{A}_{\varepsilon_i}$ such that a_i is a basis vector of $\mathscr{A}_{\varepsilon_i}$

$$\left(\Big(\big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \odot_{\mathcal{P}} \varphi_3\big) \cdots\Big) \odot_{\mathcal{P}} \varphi_{k+1}\right)\big(\iota_{\mathscr{A}_{\varepsilon_1},\mathscr{A}_{1\sqcup\cdots\sqcup(k+1)}}(a_1) \cdot \cdots \cdot \iota_{\mathscr{A}_{\varepsilon_n},\mathscr{A}_{1\sqcup\cdots\sqcup(k+1)}}(a_n)\big)$$

$$= (\varphi_{1\sqcup\cdots\sqcup k} \odot_{\mathcal{P}} \varphi_{k+1})\big(\iota_{\mathscr{A}_{\varepsilon_1},\mathscr{A}_{1\sqcup\cdots\sqcup k}\sqcup\mathscr{A}_{k+1}}(a_1) \cdot \cdots \cdot \iota_{\mathscr{A}_{\varepsilon_n},\mathscr{A}_{1\sqcup\cdots\sqcup k}\sqcup\mathscr{A}_{k+1}}(a_n)\big)$$

$$\left[\!\!\left[\, \varphi_{1\sqcup\cdots\sqcup k} := \Big(\big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \odot_{\mathcal{P}} \varphi_3\big) \cdots\Big) \odot_{\mathcal{P}} \varphi_k,\ \mathscr{A}_{1\sqcup\cdots\sqcup(k+1)} = \mathscr{A}_{1\sqcup\cdots\sqcup k} \sqcup \mathscr{A}_{k+1} \,\right]\!\!\right]$$

$$= \Big(\big(\varphi_{1\sqcup\cdots\sqcup k} \circ \mathcal{T}(\mathsf{id}_{\mathscr{A}_{1\sqcup\cdots\sqcup k}} \oplus 0)\big) \widetilde{\circledcirc_{\mathcal{P}}} \big(\varphi_{k+1} \circ \mathcal{T}(0 \oplus \mathsf{id}_{\mathscr{A}_{k+1}})\big)\Big)\big(\mathbf{i}_{\varepsilon_1}(a_1) \cdot \cdots \cdot \mathbf{i}_{\varepsilon_n}(a_n)\big)$$

$$\left[\!\!\left[\, \begin{array}{l} \forall\, j \in [k]\colon \mathbf{i}_j\colon \mathscr{A}_j \hookrightarrow \mathrm{T}(\mathscr{A}_{1\sqcup\cdots\sqcup k} \oplus \mathscr{A}_{k+1}) \twoheadrightarrow \mathrm{T}(\mathscr{A}_{1\sqcup\cdots\sqcup k} \oplus \mathscr{A}_{k+1})/I_{1\sqcup\cdots\sqcup k,k+1} \\ \text{induction base applied to } (\mathscr{A}_{1\sqcup\cdots\sqcup k}, \varphi_{1\sqcup\cdots\sqcup k}),\ (\mathscr{A}_{k+1}, \varphi_{k+1}) \in \mathsf{Obj(AlgP)} \end{array} \,\right]\!\!\right]$$

$$= \Big(\big(\varphi_{1\sqcup\cdots\sqcup k} \circ \mathcal{T}(\mathsf{id}_{\mathscr{A}_{1\sqcup\cdots\sqcup k}} \oplus 0)\big) \circledcirc_{\mathcal{P}} \big(\varphi_{k+1} \circ \mathcal{T}(0 \oplus \mathsf{id}_{\mathscr{A}_{k+1}})\big)\Big)$$
$$\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_{1\sqcup\cdots\sqcup k}\oplus\mathscr{A}_{k+1})}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_{1\sqcup\cdots\sqcup k}\oplus\mathscr{A}_{k+1})}(a_n)\big)$$

$$\left[\!\!\left[\, \text{eq. (3.3.9), projection is homomorphism of algebras} \,\right]\!\!\right]$$

$$= \bigg(\Big(\big(\varphi_{1\sqcup\cdots\sqcup k} \circ \mathcal{T}(\mathsf{id}_{\mathscr{A}_{1\sqcup\cdots\sqcup k}} \oplus 0)\big) \circledcirc_{\mathcal{P}} \big(\varphi_{k+1} \circ \mathcal{T}(0 \oplus \mathsf{id}_{\mathscr{A}_{k+1}})\big)\Big)$$
$$\circ\, \mathrm{T}\big(\bigoplus_{i=1}^{k} \mathsf{inc}_{\mathscr{A}_i,\mathscr{A}_{1\sqcup\cdots\sqcup k}\oplus\mathscr{A}_{k+1}}\big)\bigg)\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\bigoplus_{i=1}^{k+1} \mathscr{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\bigoplus_{i=1}^{k+1} \mathscr{A}_i)}(a_n)\big)$$

$$\left[\!\!\left[\, \text{eq. (III), T}(\,\cdot\,) \text{ is homomorphism of algebras} \,\right]\!\!\right]$$

$$= \bigg(\Big(\varphi_{1\sqcup\cdots\sqcup k} \circ \mathcal{T}(\mathsf{id}_{\mathscr{A}_{1\sqcup\cdots\sqcup k}} \oplus 0) \circ \mathrm{T}\big(\bigoplus_{i=1}^{k+1} \mathsf{inc}_{\mathscr{A}_i,\mathscr{A}_{1\sqcup\cdots\sqcup k}\oplus\mathscr{A}_{k+1}}\big)\Big)$$
$$\circledcirc_{\mathcal{P}} \Big(\varphi_{k+1} \circ \mathcal{T}(0 \oplus \mathsf{id}_{\mathscr{A}_{k+1}}) \circ \mathrm{T}\big(\bigoplus_{i=1}^{k+1} \mathsf{inc}_{\mathscr{A}_i,\mathscr{A}_{1\sqcup\cdots\sqcup k}\oplus\mathscr{A}_{k+1}}\big)\Big)\bigg)$$
$$\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\bigoplus_{i=1}^{k+1} \mathscr{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\bigoplus_{i=1}^{k+1} \mathscr{A}_i)}(a_n)\big)$$

⟦ Lem. 3.2.11 **(c)** ⟧

$$= \Big(\big(\varphi_{1\sqcup\cdots\sqcup k} \circ \mathcal{T}(\mathsf{id}_{\mathcal{A}_{1\sqcup\cdots\sqcup k}} \oplus 0) \circ \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathsf{inc}_{\mathcal{A}_i, \mathcal{A}_{1\sqcup\cdots\sqcup k} \oplus \mathcal{A}_{k+1}})\big)$$

$$\circledcirc_{\mathcal{P}} \big(\varphi_{k+1} \circ \mathcal{T}(\bigoplus_{i=1}^{k+1} \Delta_{i,k+1})\big)\Big)$$

$$\big(\mathsf{inc}_{\mathcal{A}_{\varepsilon_1}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathcal{A}_{\varepsilon_n}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_n)\big)$$

$$\left[\!\!\left[\mathcal{T}(0 \oplus \mathsf{id}_{\mathcal{A}_{k+1}}) \circ \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathsf{inc}_{\mathcal{A}_i, \mathcal{A}_{1\sqcup\cdots\sqcup k} \oplus \mathcal{A}_{k+1}}) = \mathcal{T}(\bigoplus_{i=1}^{k+1} \Delta_{i,k+1}) \right]\!\!\right]$$

$$= \Big(\Big(\mathsf{lift}\big((\varphi_1 \circ \mathsf{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathsf{j}_k)\big) \circ \mathsf{can}$$

$$\circ\, \mathcal{T}(\mathsf{id}_{\mathcal{A}_{1\sqcup\cdots\sqcup k}} \oplus 0) \circ \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathsf{inc}_{\mathcal{A}_i, \mathcal{A}_{1\sqcup\cdots\sqcup k} \oplus \mathcal{A}_{k+1}})\Big)$$

$$\circledcirc_{\mathcal{P}} \big(\varphi_{k+1} \circ \mathcal{T}(\bigoplus_{i=1}^{k+1} \Delta_{i,k+1})\big)\Big)$$

$$\big(\mathsf{inc}_{\mathcal{A}_{\varepsilon_1}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathcal{A}_{\varepsilon_n}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_n)\big)$$

⟦ induction hypothesis applied to $\varphi_{1\sqcup\cdots\sqcup k}$ ⟧

$$= \Big(\Big(\big(\varphi_1 \circ \mathcal{T}(\bigoplus_{j=1}^{k} \Delta_{j,1} \oplus 0)\big) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \big(\varphi_k \circ \mathcal{T}(\bigoplus_{j=1}^{k} \Delta_{j,k} \oplus 0)\big)\Big)$$

$$\circledcirc_{\mathcal{P}} \big(\varphi_{k+1} \circ \mathcal{T}(\bigoplus_{i=1}^{k+1} \Delta_{i,k+1})\big)\Big)$$

$$\big(\mathsf{inc}_{\mathcal{A}_{\varepsilon_1}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathcal{A}_{\varepsilon_n}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_n)\big) \quad [\![\text{ eq. (II) }]\!]$$

$$= \Big(\big(\varphi_1 \circ \mathcal{T}(\bigoplus_{j=1}^{k+1} \Delta_{j,1})\big) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \big(\varphi_k \circ \mathcal{T}(\bigoplus_{j=1}^{k+1} \Delta_{j,k+1})\big)\Big)$$

$$\big(\mathsf{inc}_{\mathcal{A}_{\varepsilon_1}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathcal{A}_{\varepsilon_n}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_n)\big)$$

⟦ $\circledcirc_{\mathcal{P}}$ is associative by Lem. 3.2.9 ⟧

$$= \Big(\mathsf{lift}\big((\varphi_1 \circ \mathsf{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_{k+1} \circ \mathsf{j}_{k+1})\big) \circ \mathsf{pr}\Big)$$

$$\big(\mathsf{inc}_{\mathcal{A}_{\varepsilon_1}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\mathcal{A}_{\varepsilon_n}, \mathrm{T}(\bigoplus_{i=1}^{k+1} \mathcal{A}_i)}(a_n)\big)$$

⟦ eq. (3.3.3), eq. (3.3.9) ⟧

$$= \Big(\mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_{k+1} \circ \mathrm{j}_{k+1})\big) \circ \mathsf{can}\Big)$$

$$\big(\iota_{\mathcal{A}_{\varepsilon_1}, \mathcal{A}_{1 \sqcup \cdots \sqcup (k+1)}}(a_1) \cdot \cdots \cdot \iota_{\mathcal{A}_{\varepsilon_n}, \mathcal{A}_{1 \sqcup \cdots \sqcup (k+1)}}(a_n)\big)$$

$[\![$ pr is homomorphism of algebras $]\!]$. □

We have a very similar result to Lemma 3.3.7 which turns out to be useful proving associativity of $\odot_{\mathcal{P}}$.

3.3.8 Lemma. Let $\mathcal{P}$ be a universal class of partitions. For any $k \in \mathbb{N} \setminus \{1\}$ and $(\mathcal{A}_i, \varphi_i)_{i \in [k]} \in (\mathsf{Obj}(\mathsf{AlgP}))^{\times k}$ we have

$$\varphi_1 \odot_{\mathcal{P}} \Big(\cdots \big(\varphi_{k-2} \odot_{\mathcal{P}} (\varphi_{k-1} \odot_{\mathcal{P}} \varphi_k)\big)\Big) \tag{3.3.17}$$
$$= \mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big) \circ \mathsf{can},$$

where $\circledcirc_{\mathcal{P}}$ here denotes the binary operation on the dual space of $\mathrm{T}(\bigoplus_{i=1}^{k} \mathcal{A}_i)$ and can is the canonical isomorphism of algebras, i. e., $\mathsf{can}\colon \mathcal{A}_1 \sqcup (\cdots (\mathcal{A}_{k-2} \sqcup (\mathcal{A}_{k-1} \sqcup \mathcal{A}_k))) \longrightarrow \mathrm{T}(\bigoplus_{i=1}^{k} \mathcal{A}_i)/I_{1,\dots,k}$.

Proof: The proof is analogously done like the proof of Lemma 3.3.7 by some "minor" notational modifications. We therefore omit the proof. □

3.3.9 Theorem. Let $\mathcal{P}$ be a universal class of partitions. The above defined map $\odot_{\mathcal{P}}\colon \mathrm{Lin}(\mathcal{A}_1, \mathbb{C}) \times \mathrm{Lin}(\mathcal{A}_2, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathcal{A}_1 \sqcup \mathcal{A}_2, \mathbb{C})$ fulfills the properties of a symmetric u.a.u.-product, which has the right ordered-monomials property, i. e.,

(a) $\odot_{\mathcal{P}}$ is unital: Let $\iota_i\colon \mathcal{A}_i \longrightarrow \mathcal{A}_1 \sqcup \mathcal{A}_2$ denote the canonical homomorphic insertions for $i \in \{1, 2\}$, then
$$\forall\, i \in \{1, 2\}\colon (\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ \iota_i = \varphi_i. \tag{3.3.18}$$

(b) $\odot_{\mathcal{P}}$ is associative:
$$\forall\, i \in \{1, 2, 3\}\ \forall\, \varphi_i \in \mathrm{Lin}(\mathcal{A}_i, \mathbb{C})\colon (\varphi_1 \odot_{\mathcal{P}} \varphi_2) \odot_{\mathcal{P}} \varphi_3 = \varphi_1 \odot_{\mathcal{P}} (\varphi_2 \odot_{\mathcal{P}} \varphi_3) \circ \mathsf{can}, \tag{3.3.19}$$
where $\mathsf{can}\colon (\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 \longrightarrow \mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3)$ is the canonical algebra homomorphism.

(c) $\odot_{\mathcal{P}}$ is universal: If $\kappa_i\colon \mathcal{B}_i \longrightarrow \mathcal{A}_i$ for $i \in \{1, 2\}$ are homomorphisms of algebras, then
$$(\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ (\kappa_1 \amalg \kappa_2) = (\varphi_1 \circ \kappa_1) \odot_{\mathcal{P}} (\varphi_2 \circ \kappa_2). \tag{3.3.20}$$

(d) $\odot_{\mathcal{P}}$ is symmetric: $\forall\, i \in \{1, 2\}\ \forall\, \varphi_i \in \mathcal{A}_i$:
$$\varphi_1 \odot_{\mathcal{P}} \varphi_2 = (\varphi_2 \odot_{\mathcal{P}} \varphi_1) \circ \mathsf{can}, \tag{3.3.21}$$
where $\mathsf{can}\colon \mathcal{A}_1 \sqcup \mathcal{A}_2 \longrightarrow \mathcal{A}_2 \sqcup \mathcal{A}_1$ is the canonical isomorphism of algebras.

(e) $\odot_{\mathcal{P}}$ has the right-ordered monomials property.

Proof: Ad **(a)**: Let us assume that $a \in \mathcal{A}_1$. We calculate

$$\big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ \iota_1\big)(a)$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}_1) \mathbin{\widetilde{\circledcirc_{\mathcal{P}}}} (\varphi_2 \circ \mathrm{j}_2)\big) \circ (\mathbf{i}_1 \sqcup \mathbf{i}_2)\Big)\big(\iota_1(a)\big) \quad [\![\text{ def. of } \odot \text{ in eq. (3.3.13) }]\!]$$

$$= \big((\varphi_1 \circ \mathrm{j}_1) \mathbin{\widetilde{\circledcirc_{\mathcal{P}}}} (\varphi_2 \circ \mathrm{j}_2)\big)\big((\mathbf{i}_1(a))\big) \quad [\![\text{ UMP of } \mathscr{A}_1 \sqcup \mathscr{A} \text{ in eq. (3.3.11) }]\!]$$

$$= \big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2)\big)(a) \quad \left[\!\!\left[\begin{array}{l} \mathbf{i}_1 \colon \mathscr{A}_1 \hookrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2) \twoheadrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I_{1,2}, \\ \text{definition of } \varphi_1 \mathbin{\widetilde{\circledcirc_{\mathcal{P}}}} \varphi_2 \text{ in eq. (3.3.9)} \end{array} \right]\!\!\right]$$

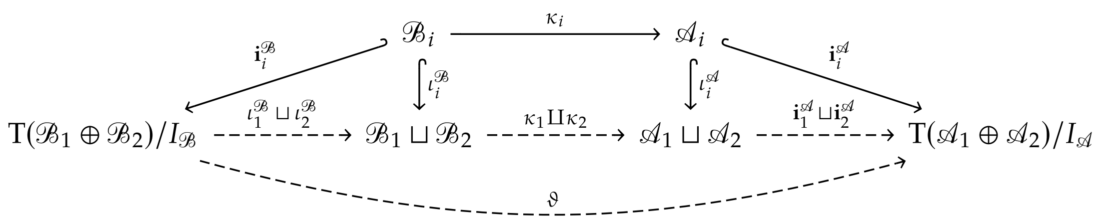
$$= \sum_{\pi \in \mathcal{P}_1} \prod_{b \in \pi} \big(\log_{\mathcal{P}}(\varphi_1 \circ \mathrm{j}_1) + \log_{\mathcal{P}}(\varphi_2 \circ \mathrm{j}_2)\big)(a_b) \quad [\![\text{ def. of } \circledcirc_{\mathcal{P}} \text{ in eq. (3.2.7) }]\!]$$

$$= \big(\log_{\mathcal{P}}(\varphi_1 \circ \mathrm{j}_1) + \log_{\mathcal{P}}(\varphi_2 \circ \mathrm{j}_2)\big)(a) \quad [\![\mathbb{1}_1 \in \mathcal{P}_1 \text{ by Def. 3.1.9 } \textbf{(a)}]\!]$$

$$= (\varphi_1 \circ \mathrm{j}_1)(a) + (\varphi_2 \circ \mathrm{j}_2)(a) \quad [\![\text{ property of } \log_{\mathcal{P}} \varphi \text{ in eq. (3.2.5) }]\!]$$

$$= \varphi_1(a) \quad [\![a \in \mathscr{A}_1 \text{ by assumption }]\!].$$

Since $a \in \mathscr{A}_1$ has been arbitrarily chosen, we have proven equation (3.3.18) for any element of $\mathscr{A}_1$. A similar proof holds for $(\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ \iota_2 = \varphi_2$.
Ad **(b)**: Associativity of $\odot_{\mathcal{P}}$ follows from associativity of $\circledcirc_{\mathcal{P}}$ in Lemma 3.2.9 and application of the assertions of Lemma 3.3.7 and Lemma 3.3.8.
Ad **(c)**: Let $\kappa_i \colon \mathscr{B}_i \longrightarrow \mathscr{A}_i$ be algebra homomorphisms for $i \in [2]$. By the universal mapping property of the free product of the algebras $\mathscr{B}_1$ and $\mathscr{B}_2$ we can see that the following diagram is commutative for each $i \in [2]$



(I)

Thus, we have

$$\vartheta \circ (\iota_1^{\mathscr{B}} \sqcup \iota_2^{\mathscr{B}})^{-1} = (\mathbf{i}_1^{\mathscr{A}} \sqcup \mathbf{i}_2^{\mathscr{A}}) \circ (\kappa_1 \amalg \kappa_2). \tag{II}$$

We calculate for any $n \in \mathbb{N}$, $(\varepsilon_i)_{i \in [n]} \in \{1,2\}^{\times n}$ and any $(b_i)_{i \in [n]} \in \prod_{i=1}^n \mathscr{B}_{\varepsilon_i}$

$$\big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ (\kappa_1 \amalg \kappa_2)\big)\big(\iota_{\varepsilon_1}^{\mathscr{B}}(b_1) \cdot \dots \cdot \iota_{\varepsilon_n}^{\mathscr{B}}(b_n)\big)$$

$$= \Big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ \big((\mathbf{i}_1^{\mathscr{A}} \sqcup \mathbf{i}_2^{\mathscr{A}})^{-1} \circ \vartheta \circ (\iota_1^{\mathscr{B}} \sqcup \iota_2^{\mathscr{B}})^{-1}\big)\Big)\big(\iota_{\varepsilon_1}^{\mathscr{B}}(b_1) \cdot \dots \cdot \iota_{\varepsilon_n}^{\mathscr{B}}(b_n)\big)$$

$$[\![\text{ eq. (II) \& } \mathbf{i}_1^{\mathscr{A}} \sqcup \mathbf{i}_2^{\mathscr{A}}, \iota_1^{\mathscr{B}} \sqcup \iota_2^{\mathscr{B}} \text{ are isomorphisms }]\!]$$

$$= \Bigg(\Big(\big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}}) \mathbin{\widetilde{\circledcirc_{\mathcal{P}}}} (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big) \circ (\mathbf{i}_1^{\mathscr{A}} \sqcup \mathbf{i}_2^{\mathscr{A}})\Big)$$

$$\circ \big((\mathbf{i}_1^{\mathscr{A}} \sqcup \mathbf{i}_2^{\mathscr{A}})^{-1} \circ \vartheta \circ (\iota_1^{\mathscr{B}} \sqcup \iota_2^{\mathscr{B}})^{-1}\big)\Bigg)\big(\iota_{\varepsilon_1}^{\mathscr{B}}(b_1) \cdot \dots \cdot \iota_{\varepsilon_n}^{\mathscr{B}}(b_n)\big)$$

$$[\![\text{ def. of } \varphi_1 \odot \varphi_2 \text{ in eq. (3.3.13) }]\!]$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}}) \mathbin{\widetilde{\circledcirc_{\mathcal{P}}}} (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big) \circ \vartheta\Big)\big(\mathbf{i}_{\varepsilon_1}^{\mathscr{B}}(b_1) \cdot \dots \cdot \mathbf{i}_{\varepsilon_n}^{\mathscr{B}}(b_n)\big)$$

$$\llbracket \text{ UMP of free product of algebras in left triangle diagram in eq. (I) } \rrbracket$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}})\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big)\Big)\Big(\vartheta\big(\mathbf{i}_{\varepsilon_1}^{\mathscr{B}}(b_1)\big) \cdot \cdots \cdot \vartheta\big(\mathbf{i}_{\varepsilon_n}^{\mathscr{B}}(b_n)\big)\Big)$$

$$\llbracket\ \vartheta \text{ is homomorphism of algebras } \rrbracket$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}})\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big)\Big)\Big(\mathbf{i}_{\varepsilon_1}^{\mathscr{A}}\big(\kappa_{\varepsilon_1}(b_1)\big) \cdot \cdots \cdot \mathbf{i}_{\varepsilon_n}^{\mathscr{A}}\big(\kappa_{\varepsilon_n}(b_n)\big)\Big)$$

$$\llbracket \text{ outer diagram in eq. (I) is commutative } \rrbracket$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}})\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big) \circ \mathsf{pr}_{I_{\mathscr{A}}}\Big)$$

$$\Big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1}, \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)}\big(\kappa_{\varepsilon_1}(b_1)\big) \cdot \cdots \cdot \mathsf{inc}_{\mathscr{A}_{\varepsilon_n}, \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)}\big(\kappa_{\varepsilon_n}(b_n)\big)\Big)$$

$$\llbracket\ \mathbf{i}_{\varepsilon_i}^{\mathscr{A}} = \mathsf{pr}_{I_{\mathscr{A}}} \circ \mathsf{inc}_{\mathscr{A}_{\varepsilon_i}, \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)} \ \&\ \mathsf{pr}_{I_{\mathscr{A}}} \text{ is homomorphism of algebras } \rrbracket$$

$$= \big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}}) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big)\Big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1}, \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)}\big(\kappa_{\varepsilon_1}(b_1)\big) \cdot \cdots \cdot \mathsf{inc}_{\mathscr{A}_{\varepsilon_1}, \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)}\big(\kappa_{\varepsilon_n}(b_n)\big)\Big)$$

$$\llbracket \text{ def. } \cdot \widetilde{\circledcirc_{\mathcal{P}}} \cdot \text{ in eq. (3.3.9) } \rrbracket$$

$$= \big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}}) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big)\big(\underbrace{\kappa_{\varepsilon_1}(b_1) \otimes \cdots \otimes \kappa_{\varepsilon_n}(b_n)}_{\in \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)}\big)$$

$$\llbracket \otimes \text{ is multiplication on } \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2) \text{ for pure tensors } \rrbracket$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}}) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}})\big) \circ \mathrm{T}(\kappa_1 \oplus \kappa_2)\Big)(b_1 \otimes \cdots \otimes b_n)$$

$$= \Big(\big(\varphi_1 \circ \mathrm{j}_1^{\mathscr{A}} \circ \mathrm{T}(\kappa_1 \oplus \kappa_2)\big) \circledcirc_{\mathcal{P}} \big(\varphi_2 \circ \mathrm{j}_2^{\mathscr{A}} \circ \mathrm{T}(\kappa_1 \oplus \kappa_2)\big)\Big)(b_1 \otimes \cdots \otimes b_n)$$

$$\llbracket\ \kappa_1 \oplus \kappa_2 \in \mathrm{Lin}(\mathscr{B}_1 \oplus \mathscr{B}_2, \mathscr{A}_1 \oplus \mathscr{A}_2), \text{ then apply Lemma 3.2.11 } \textbf{(c)} \rrbracket$$

$$= \big((\varphi_1 \circ \kappa_1 \circ \mathrm{j}_1^{\mathscr{B}}) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \kappa_2 \circ \mathrm{j}_1^{\mathscr{B}})\big)(b_1 \otimes \cdots \otimes b_n)$$

$$\left\llbracket \begin{array}{l} \text{it suffices to show } (\mathrm{j}_i^{\mathscr{A}} \circ \mathrm{T}(\kappa_1 \oplus \kappa_2))(b) = (\kappa_i \circ \mathrm{j}_i^{\mathscr{B}})(b) \text{ for } b \in \mathscr{B}_1 \oplus \mathscr{B}_2, \\ \text{because all maps are morphisms of algebras and} \\ \mathrm{T}(\mathscr{B}_1 \oplus \mathscr{B}_2) \text{ is generated by } \mathscr{B}_1 \oplus \mathscr{B}_2 \\ \text{moreover use Conv. 3.3.1} \end{array} \right\rrbracket$$

$$= \Big(\big((\varphi_1 \circ \kappa_1 \circ \mathrm{j}_1^{\mathscr{B}})\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_2 \circ \kappa_2 \circ \mathrm{j}_2^{\mathscr{B}})\big) \circ (\mathbf{i}_1^{\mathscr{B}} \sqcup \mathbf{i}_2^{\mathscr{B}})\Big)\big(\iota_{\varepsilon_1}^{\mathscr{B}}(b_1) \cdot \cdots \cdot \iota_{\varepsilon_n}^{\mathscr{B}}(b_n)\big)$$

$$\llbracket \text{ property of lifted map in eq. (3.3.11) } \rrbracket$$

$$= \big((\varphi_1 \circ \kappa_1) \odot (\varphi_2 \circ \kappa_2)\big)\big(\iota_{\varepsilon_1}^{\mathscr{B}}(b_1) \cdot \cdots \cdot \iota_{\varepsilon_n}^{\mathscr{B}}(b_n)\big) \quad \llbracket \text{ def. of } \varphi_1 \odot_{\mathcal{P}} \varphi_2 \text{ in eq. (3.3.13) } \rrbracket.$$

Since the algebra $\mathscr{B}_1 \sqcup \mathscr{B}_2$ is generated by elements from $\iota_1^{\mathscr{B}}(\mathscr{B}_1) \cup \iota_2^{\mathscr{B}}(\mathscr{B}_2)$ and we apply linear maps to such elements, the assertion follows from the above calculation.
AD **(d)**: We first want to show how $\mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I_{1,2}$ and $\mathrm{T}(\mathscr{A}_2 \oplus \mathscr{A}_1)/I_{2,1}$ are isomorphic as algebras. Therefore, let $f\colon \mathscr{A}_1 \oplus \mathscr{A}_2 \longrightarrow \mathscr{A}_2 \oplus \mathscr{A}_1$ be the map which swaps elements. We extend this as a

homomorphism of algebras by $\mathrm{T}(f)\colon \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2) \longrightarrow \mathrm{T}(\mathscr{A}_2 \oplus \mathscr{A}_1)$ and can furthermore lift this as a homomorphic quotient map $\widetilde{\mathrm{T}(f)}\colon \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I_{1,2} \longrightarrow \mathrm{T}(\mathscr{A}_2 \oplus \mathscr{A}_1)/I_{2,1}$, since $\mathrm{T}(f)$ respects the ideal $I_{1,2}$, which is easily shown. We can apply the same procedure to $g\colon \mathscr{A}_2 \oplus \mathscr{A}_1 \longrightarrow \mathscr{A}_1 \oplus \mathscr{A}_2$ and end up with $\widetilde{\mathrm{T}(g)}\colon \mathrm{T}(\mathscr{A}_2 \oplus \mathscr{A}_1)/I_{2,1} \longrightarrow \mathrm{T}(\mathscr{A}_1 \oplus \mathscr{A}_2)/I_{1,2}$. It can be shown that $\widetilde{\mathrm{T}(f)}$ and $\widetilde{\mathrm{T}(g)}$ are inverse to each other. For the following calculation we want to set some notations. Let $\iota_i\colon \mathscr{A}_1 \hookrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ and $\iota'_i\colon \mathscr{A}_i \hookrightarrow \mathscr{A}_2 \sqcup \mathscr{A}_1$ denote the canonical algebra homomorphisms. Furthermore, any map having a prime "$'$" attached to it, shall be understood in the setting associated to $\mathscr{A}_2 \sqcup \mathscr{A}_1$. Now let $(\varepsilon_i)_{i\in[n]} \in \mathbb{A}([2])$ and let each $a_i \in \mathscr{A}_{\varepsilon_i}$ be a basis vector. Then we calculate for a basis vector of $\mathscr{A}_2 \sqcup \mathscr{A}_1$

$$\big((\varphi_2 \odot_{\mathcal{P}} \varphi_1) \circ \mathsf{can}\big)\big(\iota_{\varepsilon_1}(a_1) \cdot \dots \cdot \iota_{\varepsilon_n}(a_n)\big)$$

$$= (\varphi_2 \odot_{\mathcal{P}} \varphi_1)\big(\iota'_{\varepsilon_1}(a_1) \cdot \dots \cdot \iota'_{\varepsilon_n}(a_n)\big) \quad [\![\text{ def. of } \mathsf{can}]\!]$$

$$= \Big(\big((\varphi_2 \circ \mathrm{j}'_1)\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_1 \circ \mathrm{j}'_2)\big) \circ (\mathbf{i}'_1 \sqcup \mathbf{i}'_2)\Big)\big(\iota'_{\varepsilon_1}(a_1) \cdot \dots \cdot \iota'_{\varepsilon_n}(a_n)\big)$$

$$[\![\text{ def. of } \odot \text{ in eq. (3.3.13) with } \mathbf{i}'_j\colon \mathscr{A}_j \hookrightarrow \mathrm{T}(\mathscr{A}_2 \oplus \mathscr{A}_1) \twoheadrightarrow \mathrm{T}(\mathscr{A}_2 \oplus \mathscr{A}_1)/I_{2,1}]\!]$$

$$= \big((\varphi_2 \circ \mathrm{j}'_1)\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_1 \circ \mathrm{j}'_2)\big)\big((\mathbf{i}'_{\varepsilon_1}(a_1) \cdots \mathbf{i}'_{\varepsilon_n}(a_n))\big) \quad [\![\text{ UMP of } \mathscr{A}_1 \sqcup \mathscr{A} \text{ in eq. (3.3.11) }]\!]$$

$$= \Big(\big((\varphi_2 \circ \mathrm{j}'_1)\, \widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_1 \circ \mathrm{j}'_2)\big) \circ \big(\widetilde{\mathrm{T}(f)} \circ \widetilde{\mathrm{T}(g)}\big)\Big)\big((\mathbf{i}'_{\varepsilon_1}(a_1) \cdots \mathbf{i}'_{\varepsilon_n}(a_n))\big)$$

$$[\![\ \widetilde{\mathrm{T}(f)} \circ \widetilde{\mathrm{T}(g)} = \mathsf{id}\ ]\!]$$

$$= \Big(\big((\varphi_2 \circ \mathrm{j}'_1) \circledcirc_{\mathcal{P}} (\varphi_1 \circ \mathrm{j}'_2)\big) \circ \big(\mathrm{T}(f) \circ \mathrm{T}(g)\big)\Big)\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_2\oplus\mathscr{A}_1)}(a_1) \otimes \dots \otimes \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_2\oplus\mathscr{A}_1)}(a_n)\big)$$

$$[\![\text{ def. of } \varphi_1 \widetilde{\circledcirc_{\mathcal{P}}} \varphi_2 \text{ in eq. (3.3.9), } \widetilde{\mathrm{T}(f)},\ \widetilde{\mathrm{T}(g)} \text{ are quotient maps }]\!]$$

$$= \Big(\big((\varphi_2 \circ \mathrm{j}'_1) \circledcirc_{\mathcal{P}} (\varphi_1 \circ \mathrm{j}'_2)\big) \circ \mathrm{T}(f)\Big)\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_1) \otimes \dots \otimes \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_n)\big)$$

$$[\![\text{ UMP of tensor algebra \& def. of } \mathrm{T}(g)]\!]$$

$$= \Big(\big(\varphi_2 \circ \mathrm{j}'_1 \circ \mathrm{T}(f)\big) \circledcirc_{\mathcal{P}} \big(\varphi_1 \circ \mathrm{j}'_2 \circ \mathrm{T}(f)\big)\Big)\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_1) \otimes \dots \otimes \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_n)\big)$$

$$[\![\text{ Lem. 3.2.11 } \textbf{(c)}]\!]$$

$$= \Big(\big(\varphi_2 \circ \mathrm{j}_2\big) \circledcirc_{\mathcal{P}} \big(\varphi_1 \circ \mathrm{j}_1\big)\Big)\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_1) \otimes \dots \otimes \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_n)\big)$$

$$[\![\ \mathrm{j}'_1 \circ \mathrm{T}(f) = \mathrm{j}_2,\ \mathrm{j}'_2 \circ \mathrm{T}(f) = \mathrm{j}_1\ ]\!]$$

$$= \Big(\big(\varphi_1 \circ \mathrm{j}_1\big) \circledcirc_{\mathcal{P}} \big(\varphi_2 \circ \mathrm{j}_2\big)\Big)\big(\mathsf{inc}_{\mathscr{A}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_1) \otimes \dots \otimes \mathsf{inc}_{\mathscr{A}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)}(a_n)\big)$$

$$[\![\ \circledcirc_{\mathcal{P}} \text{ is commutative by Lem. 3.2.9 }]\!]$$

$$= (\varphi_2 \odot_{\mathcal{P}} \varphi_1)\big(\iota_{\varepsilon_1}(a_1) \cdot \dots \cdot \iota_{\varepsilon_n}(a_n)\big) \quad [\![\text{ analogous steps from start of calculation }]\!].$$

Ad **(e)**: We need to show that equation (2.5.1) is satisfied according to $\odot_{\mathcal{P}}$, i. e., the only nonzero universal coefficients of $\odot_{\mathcal{P}}$ are the right-ordered ones. We show this by induction over $k \in \mathbb{N}\backslash\{1\}$. For the induction base $k = 2$ we let $n \in \mathbb{N}$, $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([2])$, $(\mathscr{A}_i, \varphi_i)_{i\in[2]} \in (\mathsf{Obj}(\mathsf{AlgP}))^{\times 2}$ and

$(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_i}$. Then, we calculate

$$
\begin{aligned}
&(\varphi_1 \odot_{\mathcal{P}} \varphi_2)\big(\iota_{\varepsilon_1}(a_1)\cdot\dots\cdot\iota_{\varepsilon_n}(a_n)\big)\\
&= \Big(\big((\varphi_1\circ \mathrm{j}_1)\,\widetilde{\circledcirc_{\mathcal{P}}}\,(\varphi_2\circ \mathrm{j}_2)\big)\circ(\mathbf{i}_1\sqcup\mathbf{i}_2)\Big)\big(\iota_{\varepsilon_1}(a_1)\cdot\dots\cdot\iota_{\varepsilon_n}(a_n)\big) \quad [\![\text{ def. of } \odot \text{ in eq. (3.3.13) }]\!]\\
&= \big((\varphi_1\circ \mathrm{j}_1)\,\widetilde{\circledcirc_{\mathcal{P}}}\,(\varphi_2\circ \mathrm{j}_2)\big)\big(\mathbf{i}_{\varepsilon_1}(a_1)\cdot\dots\cdot\mathbf{i}_{\varepsilon_n}(a_n)\big) \quad [\![\text{ UMP of } \mathscr{A}_1\sqcup\mathscr{A} \text{ in eq. (3.3.11) }]\!]\\
&= \Big(\big((\varphi_1\circ \mathrm{j}_1)\,\widetilde{\circledcirc_{\mathcal{P}}}\,(\varphi_2\circ \mathrm{j}_2)\big)\circ \mathsf{pr}\Big)\big(\mathsf{inc}_{\varepsilon_1}(a_1)\cdot\dots\cdot\mathsf{inc}_{\varepsilon_n}(a_n)\big)\\
&\qquad \left[\!\!\left[\begin{array}{l}\mathbf{i}_i\colon \mathscr{A}_1 \xhookrightarrow{\mathsf{inc}_i} \mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2) \xrightarrow{\mathsf{pr}} \mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)/I_{1,2}\\ \text{projection } \mathsf{pr} \text{ from eq. (3.3.9) is homomorphism of algebras}\end{array}\right]\!\!\right]\\
&= \big((\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}(\varphi_2\circ \mathrm{j}_2)\big)\big(\mathsf{inc}_{\varepsilon_1}(a_1)\cdot\dots\cdot\mathsf{inc}_{\varepsilon_n}(a_n)\big) \quad [\![\text{ def. of } \circledcirc_{\mathcal{P}} \text{ in eq. (3.3.9) }]\!]\\
&= \big((\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}(\varphi_2\circ \mathrm{j}_2)\big)(a_1\otimes\dots\otimes a_n)\\
&= \exp_{\mathcal{P}}\Big(\log_{\mathcal{P}}\big(\varphi_1\circ \mathrm{j}_1\big)+\log_{\mathcal{P}}\big(\varphi_2\circ \mathrm{j}_2\big)\Big)(a_1\otimes\dots\otimes a_n) \quad [\![\text{ def. of } \circledcirc_{\mathcal{P}} \text{ in eq. (3.2.7) }]\!]\\
&= \sum_{\pi\in\mathcal{P}_n}\prod_{b\in\pi}\Big(\log_{\mathcal{P}}\big(\varphi_1\circ \mathrm{j}_1\big)(a_b)+\log_{\mathcal{P}}\big(\varphi_2\circ \mathrm{j}_2\big)(a_b)\Big) \quad [\![\text{ def. of } \exp_{\mathcal{P}} \text{ in eq. (3.2.4) }]\!]\\
&= \sum_{\pi\in\mathcal{P}_n}\prod_{b\in\pi}\left(\sum_{\sigma\in\mathsf{Part}_{|\mathsf{set}\,b|}}\gamma_\sigma\left(\Big(\prod_{b'\in\sigma}(\varphi_1\circ \mathrm{j}_1)(a_{b'})\Big)+\Big(\prod_{b'\in\sigma}(\varphi_2\circ \mathrm{j}_2)(a_{b'})\Big)\right)\right) \quad [\![\text{ Lemma 3.2.16 }]\!].
\end{aligned}
$$

If we use Convention 2.5.7, we can see that linear functionals $\varphi_i \circ \mathrm{j}_i$ are only applied to ordered monomials $a_{j_1}\otimes\dots\otimes a_{j_\ell} \in \mathrm{T}(\mathscr{A}_1\oplus\mathscr{A}_2)$. Since $\odot_{\mathcal{P}}$ is a universal product we may consult equation (2.3.12) and conclude that terms with universal coefficients which are not right-ordered do not exist. This shows equation (2.5.1) in the case $k = 2$.

For the induction step $k \to k+1$ we assume that equation (2.5.1) is true for some $k \in \mathbb{N}\setminus\{1\}$. Then, we have by associativity

$$(\varphi_1 \odot_{\mathcal{P}} \dots \odot_{\mathcal{P}} \varphi_k \odot_{\mathcal{P}} \varphi_{k+1}) = \Big((\varphi_1 \odot_{\mathcal{P}} \dots \odot_{\mathcal{P}} \varphi_k)\odot_{\mathcal{P}} \varphi_{k+1}\Big).$$

Then, we may apply the induction base to the right hand side of the above equation and from the induction hypothesis we obtain the statement of equation (2.5.1) in the case for $k+1$. □

3.3.10 Definition (Partition induced universal product). Let $\odot$ be a symmetric u.a.u.-product with right-ordered monomials property in the category AlgP. We say that $\odot$ is a *partition induced universal product* (for $\mathcal{P}$) if and only if there exists some universal class of partitions $\mathcal{P}$ such that $\odot = \odot_{\mathcal{P}}$.

3.3.11 Remark. One might ask if it is possible to define an analogous partition induced universal product which is not symmetric. We have tried to do this but we could not succeed. We want to outline possible problems. The idea is the same as in the nonsymmetric case. The leitmotiv is formula (2.4.21) for a positive u.a.u.-product and incorporates Lemma 2.5.12. To keep things simple we only discuss the single-faced case. A possible universal class of partitions $\mathcal{P}$ should now consist of *ordered partitions*, i. .e., partitions $\pi \in \mathcal{P}$ are now to be understood as tuples of blocks rather than just a set of blocks as in the symmetric case. Then, the definitions of $\exp_{\mathcal{P}}$ and $\log_{\mathcal{P}}$ from Remark 3.2.6 are formally the same but now the sum runs over a set of ordered set partitions and therefore we need an occurring factor $\frac{1}{|\pi|!}$, in pure analogy to equation (2.5.20) and equation (2.5.21). Since $\mathcal{P}$ is now a set of ordered set partitions, this will put us in the position to define a non-trivial *partition induced commutator bracket* $[\,\cdot\,,\,\cdot\,]_{\mathcal{P}}$ w. r. t. $\mathcal{P}$ by

$$[\psi_1, \psi_2]_{\mathcal{P}}(x_1 \otimes \cdots \otimes x_n) := \sum_{(b_1,b_2)\in\mathcal{P}_n} \big(\psi_1(x_{b_1})\psi_2(x_{b_2}) - \psi_1(x_{b_2})\psi_2(x_{b_1})\big). \tag{3.3.22}$$

This equation is purely inspired by equation (2.5.23). The first thing we tried is to find sufficient conditions for $\mathcal{P}$ such that equation (3.3.22) leads to a definition of a Lie bracket. This should put some constraints on $\mathcal{P}$.
Assume we have found sufficient conditions for $\mathcal{P}$ which ensure that $[\,\cdot\,,\,\cdot\,]_{\mathcal{P}}$ is a Lie bracket, then we could set

$$\circledcirc_{\mathcal{P}} : \begin{cases} \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \times \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \\ (\varphi, \psi) \longmapsto \exp_{\mathcal{P}}\Big(\mathsf{BCH}_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi), \log_{\mathcal{P}}(\psi)\big)\Big), \end{cases} \tag{3.3.23}$$

where $\mathsf{BCH}_{\mathcal{P}}(\,\cdot\,,\,\cdot\,)$ denotes the BCH-series w. r. t. to the Lie-bracket $[\,\cdot\,,\,\cdot\,]_{\mathcal{P}}$ in the sense of equation (1.2.21). Thus, we arrive at a similar stage as done in Definition 3.2.8 in the symmetric case. But now things get really worse. In order to define a universal product on the free product of algebras $\mathscr{A}_1 \sqcup \mathscr{A}_2$ we need an analogous result to Lemma 3.3.2. In the symmetric case we had the idea that a partition induced cumulant $\log_{\mathcal{P}}$ needs to satisfy Lemma 3.2.15 as a sufficient condition to prove Lemma 3.3.2. Lemma 3.3.2 is the key to see most of the defining axioms of a partition induced universal products. So, yes, cumulants seem to be a powerful tool. But we could not succeed doing so in the nonsymmetric case. We had some ideas what a partition induced cumulant needs to satisfy but we were not able to prove an analogous result to Lemma 3.3.2 in the nonsymmetric setting. The problem is the BCH-formula itself. It leads to nearly uncontrollable appearances of $a_{\ell+1}$ and $a_{\ell+2}$ separated by different terms (in the sense of Lemma 3.3.2). In the nonsymmetric case not only the logarithm $\log_{\mathcal{P}}$ is a source for "splitting" $a_{\ell+1}$ and $a_{\ell+2}$ but also the the Lie bracket $[\,\cdot\,,\,\cdot\,]_{\mathcal{P}}$. Terms with $a_{\ell+1}$ and $a_{\ell+2}$ split in equation (3.3.23) can cancel at different "stages" because partitions π are now tuples of blocks and we have to deal with weighting factors $\frac{1}{|\pi|!}$ which complicates the calculations enormously, at least for us.
We were not able to see any potential structure for a universal class of partitions consisting of ordered partitions. Maybe, we need to find a proper mathematical object, already known to the mathematical community, which includes the definition of our universal classes of partitions. We can only speculate about a possible embedding of $\mathcal{P}$ into a bigger frame of mathematical objects but a hands on definition for $\mathcal{P}$ which leads to a partition induced universal product in the sense of equation (3.3.23) and Lemma 3.3.2 in the nonsymmetric case was out of reach for us.

3.4 Partition induced universal product: multi-colored case

We want to extend the above developed technique for the construction of a partition induced universal product $\varphi_1 \odot_{\mathcal{P}} \varphi_2$ on the free product of algebras $\mathscr{A}_1 \sqcup \mathscr{A}_2$ to the more specific assumption that each algebra $\mathscr{A}_i$ is isomorphic to an m-fold free product of subalgebras $\mathscr{A}_i^{(j)} \subseteq \mathscr{A}_i$, i. e.,

$\mathcal{A}_i \cong \mathcal{A}_i^{(1)} \sqcup \cdots \sqcup \mathcal{A}_i^{(m)}$. Such a universal product needs to satisfy equation (2.1.19) for morphisms in the category $\mathsf{Alg_m}$ and we may expect more examples than in the single-faced case. Once again, we are going to mimic equation (2.4.22) for a positive and symmetric u.a.u.-product in the category $\mathsf{AlgP_m}$. Therefore, our guidance are Sections 3.1 – 3.3 which will be extended to the m-faced case in this section. Since each algebra $\mathcal{A}$ now can have m-faces, we need to pay respect to this circumstance when we want to establish a partition induced universal product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Thus, we need partitions which are capable of this information. We call such partitions m-colored partitions.

We have chosen to treat the single-colored case separately from the m-colored case because our intention was to highlight the differences between these cases. Mostly, these differences are only of technical nature and formulas get extended by a color index. We hope that this might contribute to clarity of our presented results. To make this work not too bloated, we omit or sometimes shorten most of the proofs in this section because they have their counterparts in the single-colored case in the Sections 3.1 – 3.3.

3.4.1 Definition (Block over $\mathbb{N} \times [m]$, m-colored partition, block neighboring elements). Let $m \in \mathbb{N}$.

(a) If $n \in \mathbb{N}$, then we say that an n-tuple $x = \big((x_{i,1}, x_{i,2})\big)_{i\in[n]} \in (\mathbb{N} \times [m])^{\times n}$ is a *block* (*over* $\mathbb{N} \times [m]$) if $\mathsf{type}(x)$ is a block over $\mathbb{N}$, where $\mathsf{type}(\,\cdot\,)$ has been defined in (2.3.18).

(b) Let $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{N} \times [m]$ be a set. Then, we denote by $\mathsf{block}\, X \in (\mathbb{N} \times [m])^{\times n}$ the associated block originated from the finite set X such that $\mathsf{type}\big(\mathsf{block}\, X\big)$ is a block in the sense of Definition 2.5.6 **(b)**. The block $\mathsf{block}\, X$ is unique and always exists by the well-ordering principle of $\mathbb{N}$. By the operation $\mathsf{set}(\,\cdot\,)$ we turn any given block $x = \big((x_{i,1}, x_{i,2})\big)_{i\in[n]} \in (\mathbb{N} \times [m])^{\times n}$ into a set by

$$\mathsf{set}\, x := \bigcup_{i=1}^{k} \{(x_{i,1}, x_{i,2})\}. \tag{3.4.1}$$

Let $\emptyset \neq X \subseteq \mathbb{N} \times [m]$ and $|X| < \infty$.

(c) An *(m-colored) partition* π of $X \subseteq \mathbb{N} \times [m]$ is a finite set of blocks $\pi = \{b_1, \ldots, b_k\}$ such that the following conditions are fulfilled

- $\forall\, i \in [k]\colon\ \mathsf{set}\, b_i \neq \emptyset$,
- $\forall\, i,\, j \in [k]\colon\, i \neq j \implies \mathsf{set}\big(\mathsf{type}(b_i)\big) \cap \mathsf{set}\big(\mathsf{type}(b_j)\big) = \emptyset$,
- $X = \bigcup_{i=1}^{k} \mathsf{set}\, b_i$.

(d) The set of m-colored partitions of $X \subseteq \mathbb{N} \times [m]$ is denoted by Part_X. If $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ and $X = \{(1, \varepsilon_i), \ldots, (n, \varepsilon_n)\}$, then we set $\mathsf{Part}_\varepsilon := \mathsf{Part}_X$. The set of all partitions in $\mathsf{Part}_\varepsilon$ for a given $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ with k-blocks for $k \in [n]$ is denoted by $\mathsf{Part}_{\varepsilon,k}$.

(e) If $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ for some $n \in \mathbb{N}$, then we denote by $\mathbb{1}_\varepsilon \in \mathsf{Part}_\varepsilon$ the unique partition which consists of only one block and call it the *unit partition* of $\mathsf{Part}_\varepsilon$.

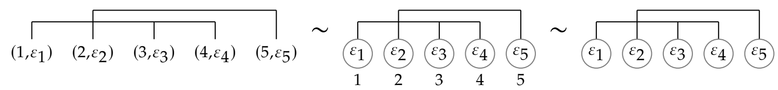
(f) Let $X \subseteq \mathbb{N} \times [m]$. For a given partition $\pi \in \mathsf{Part}_X$ we say two elements $x_1,\, x_2 \in X$ are *block neighboring* (w. r. t. $\pi \in \mathsf{Part}_X$) if and only if

- $x_1 \neq x_2$

- $\mathsf{col}(x_1) = \mathsf{col}(x_2)$
- there exists a block $b \in \pi$ such that $\mathsf{type}(x_1), \mathsf{type}(x_2) \in \mathsf{set}(b)$,
- there does not exist any element $x_3 \in X$ such that $\mathsf{type}(x_1) < \mathsf{type}(x_3) < \mathsf{type}(x_2)$ or $\mathsf{type}(x_2) < \mathsf{type}(x_3) < \mathsf{type}(x_1)$

Sometimes, we are sloppy and just speak of partitions although we mean m-colored partitions and not just partitions in the sense of Definition 3.1.1. It will be clear from the context for which value of $m \in \mathbb{N}$ certain partitions are m-colored. We will identify 1-colored or single-colored partitions with partitions in the sense of Definition 3.1.1.

3.4.2 Convention. Let $m, n \in \mathbb{N}$ and $\varepsilon \in [m]^{\times n}$. We introduce a graphical notation for a given partition $\pi \in \mathsf{Part}_\varepsilon$. We call elements of a block $b \in \pi$ a *leg* because we represent each element in $\mathsf{set}\, b$ by a vertical bar with a bracket attached at the bottom of the vertical bar. If the block $b = (b_1, \ldots, b_r)$ has $r \in \mathbb{N}$ elements, then we write for the i-th leg of the block b into this bracket $\mathsf{type}(b_i), \mathsf{col}(b_i)$. We connect vertical bars of the same block by a horizontal line at the top of the legs. If these connecting horizontal lines overlap, then we draw them at different height. The relative height between connecting horizontal lines does not matter and can be drawn arbitrarily.
Later, when we specialize to the case $m = 2$ the attached bracket at the bottom of a leg becomes a circle and in this circle will stand only the value of $\mathsf{col}(b_i)$. Thus, we have partitions where each leg has an attached color label at the bottom of the leg and the diagram may or may not have an attached bottom row which indicates $\mathsf{type}(\pi)$. Most of the times we will omit the bottom row which will be clear from the context. As an example we consider the partition $\pi = \{((1, \varepsilon_1), (3, \varepsilon_3), (4, \varepsilon_4)), ((2, \varepsilon_2), (5, \varepsilon_5))\}$ and its three equivalent graphical incarnations



(3.4.2)

Once again, we need to be able to annihilate and create legs with a certain color in an m-colored partition. This will be technically ensured by the following definition.

3.4.3 Definition (delete and double maps). Let $m \in \mathbb{N}$. This is an extension of Definition 3.1.3.

(a) Let $n \in \mathbb{N} \setminus \{1\}$ and $\pi = \{b_1, \ldots, b_k\} \in \mathsf{Part}_\varepsilon$ be a partition for some $k \in \mathbb{N}$. Each block is given by $b_i = ((b_{i,1,1}, b_{i,1,2}), \ldots, (b_{i,j_i,1}, b_{i,j_i,2})) \in (\mathbb{N} \times [m])^{\times j_i}$ for all $i \in \{1, \ldots, k\}$, where $(j_1, \ldots, j_k) \in \mathbb{N}^{\times k}$. Let $\ell \in \{0, \ldots, n-1\}$. Let π be some partition of k-blocks such that there exists $r \in [k]$ and $\ell + 1, \ell + 2 \in \mathsf{set}(\mathsf{type}(b_r))$ and $\varepsilon_{\ell+1} = \varepsilon_{\ell+2}$. If we set $\lambda := \mathsf{pos}_{\mathsf{type}(b_r)}(\ell + 1)$, then in particular $\lambda + 1 = \mathsf{pos}_{\mathsf{type}(b_r)}(\ell + 2)$. Then, we define for any such partition $\pi \in \mathsf{Part}_\varepsilon$ by the preceding notations and any block $b_i \in \pi$

$$\mathsf{delete}_{\ell+2,\pi}(b_i) = \mathsf{delete}_{\ell+2,\pi}\Big(\big((b_{i,1,1}, b_{i,1,2})\ldots, (b_{i,j_i,1}, b_{i,j_i,2})\big)\Big)$$

$$:= \begin{cases} \Big(\big(\mathsf{down}_{\ell+2}(b_{i,1,1}), b_{i,1,2}\big), \ldots, \big(\mathsf{down}_{\ell+2}(b_{i,j_i,1}), b_{i,j_i,2}\big)\Big) \\ \quad \in (\mathbb{N}\times\mathbb{N})^{\times j_i} \text{ for } i \neq r \\ \Big((b_{r,1,1}, b_{r,1,2}), \ldots, (b_{r,\lambda,1}, \varepsilon_{\ell+1}), (b_{r,\lambda+2,1}-1, b_{r,\lambda+2,2}), \ldots, (b_{r,j_r,1}-1, b_{r,j_r,2})\Big) \\ \quad \in (\mathbb{N}\times\mathbb{N})^{\times(j_i-1)} \text{ for } i = r. \end{cases} \tag{3.4.3}$$

By definition for each choice $i \in [k]$ the expression $\mathsf{delete}_{\ell+2,\pi}(b_i)$ is a block over $\mathbb{N} \times [m]$. Define for some $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ with the property there exists some $\ell \in \{0, \ldots, n-2\} \subseteq \mathbb{N}$ such that $\varepsilon_{\ell+1} = \varepsilon_{\ell+2}$

$$\begin{aligned} \mathsf{Part}_\varepsilon^{(\ell+1)\wedge(\ell+2)} &:= \{\, \pi \in \mathsf{Part}_\varepsilon \mid \exists b \in \pi \colon \ell+1,\ \ell+2 \in \mathsf{set}\big(\mathsf{type}(b)\big) \,\} \\ &\subseteq \mathsf{Part}_\varepsilon, \end{aligned} \tag{3.4.4}$$

then the following assignment is well-defined

$$\mathsf{delete}_{\varepsilon,\ell+2} \colon \begin{cases} \mathsf{Part}_\varepsilon^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathsf{Part}_{\tilde\varepsilon} \\ \pi = \{b_1, \ldots, b_k\} \\ \quad \longmapsto \Big\{\mathsf{delete}_{\ell+2,\pi}(b_1), \ldots, \mathsf{delete}_{\ell+2,\pi}(b_k)\Big\}, \end{cases} \tag{3.4.5}$$

where $\tilde\varepsilon \in [m]^{\times(n-1)}$ is determined by ε such that

$$\forall i \in [\ell+1] \colon \tilde\varepsilon_i = \varepsilon_i, \tag{3.4.6a}$$

$$\forall i \in [n-1] \setminus [\ell+1] \colon \tilde\varepsilon_i = \varepsilon_{i+1}. \tag{3.4.6b}$$

(b) Let $n \in \mathbb{N}$ and $\pi = \{b_1, \ldots, b_k\} \in \mathsf{Part}_{\tilde\varepsilon}$ for some $k \in [n]$ and $\tilde\varepsilon = (\tilde\varepsilon_i)_{i\in[n]} \in [m]^{\times(n)}$. Each block b_i is a tuple, i. e., $b_i = \big((b_{i,1,1}, b_{i,1,2}), \ldots, (b_{i,j_i,1}, b_{i,j_i,2})\big) \in (\mathbb{N}\times\mathbb{N})^{\times j_i}$ for $i \in \{1, \ldots, k\}$ and $(j_1, \ldots, j_k) \in \mathbb{N}^{\times k}$. Moreover, assume $\ell \in \{0, \ldots, n-1\}$. Then, there exists $r \in [k]$ such that $\ell + 1 \in \mathsf{set}\big(\mathsf{type}(b_r)\big)$. We set $\lambda := \mathsf{pos}_{\mathsf{type}(b_r)}(\ell+1)$. Then, we define for any partition $\pi \in \mathsf{Part}_{\tilde\varepsilon}$ and any block $b_i \in \pi$

$$\mathsf{double}_{\ell+1,\pi}(b_i) = \mathsf{double}_{\ell+1,\pi}\big((b_{i,1,1}, b_{i,1,2}), \ldots, (b_{i,j_i,1}, b_{i,j_i,2})\big)$$

$$:= \begin{cases} \Big(\big(\mathsf{up}_{\ell+1}(b_{i,1,1}), b_{i,1,2}\big)\ldots, \big(\mathsf{up}_{\ell+1}(b_{i,j_i,1}), b_{i,j_i,2}\big)\Big) \\ \quad \in (\mathbb{N}\times\mathbb{N})^{\times j_i} \text{ for } i \neq r \\ \Big((b_{r,1,1}, b_{r,1,2}), \ldots, (b_{r,\lambda,1}, \tilde\varepsilon_{\ell+1}), (b_{r,\lambda}+1, \tilde\varepsilon_{\ell+1}), \\ \quad (b_{r,\lambda+1,1}+1, b_{r,\lambda+1,2}), \ldots (b_{r,j_r,1}+1, b_{r,j_r,2})\Big) \\ \quad \in (\mathbb{N}\times\mathbb{N})^{\times(j_r+1)} \text{ for } i = r. \end{cases} \tag{3.4.7}$$

By definition for each choice $i \in [k]$ the expression $\mathsf{double}_{\ell+1,\pi}(b_i)$ is a block over $\mathbb{N} \times [m]$. Therefore, the following assignment is well-defined

$$\mathsf{double}_{\tilde{\varepsilon},\ell+1}\colon \begin{cases} \mathsf{Part}_{\tilde{\varepsilon}} \longrightarrow \mathsf{Part}_{\varepsilon} \\ \pi = \{b_1,\ldots,b_k\} \\ \quad \longmapsto \Big\{\mathsf{double}_{\ell+1,\pi}(b_1),\ldots,\mathsf{double}_{\ell+1,\pi}(b_k)\Big\}, \end{cases} \tag{3.4.8}$$

where $\varepsilon = (\varepsilon_i)_{i[n]} \in [m]^{\times n+1}$ is determined by $\tilde{\varepsilon}$ such that

$$\forall\, i \in [\ell+1]\colon\ \varepsilon_i = \tilde{\varepsilon}_i, \tag{3.4.9a}$$

$$\varepsilon_{\ell+2} = \tilde{\varepsilon}_{\ell+1}, \tag{3.4.9b}$$

$$\forall\, i \in [n+1] \setminus [\ell+2]\colon\ \varepsilon_i = \tilde{\varepsilon}_{i-1}. \tag{3.4.9c}$$

3.4.4 Lemma (delete and double are inverse).

(a) $\forall\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ \forall\, \ell \in \{0,\ldots,n-1\} \subseteq \mathbb{N}_0,\ \forall\, \tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i\in[n]} \in [m]^{\times n}\colon$ $\mathsf{double}_{\tilde{\varepsilon},\tilde{\varepsilon}_{\ell+1}}(\mathsf{Part}_{\tilde{\varepsilon}}) \subseteq \mathsf{Part}_{\varepsilon}^{(\ell+1)\wedge(\ell+2)}$.

(b) The maps

$$\mathsf{delete}_{\varepsilon,\ell+2}\colon\ \mathsf{Part}_{\varepsilon}^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathsf{Part}_{\tilde{\varepsilon}}, \tag{3.4.10}$$

$$\mathsf{double}_{\tilde{\varepsilon},\ell+1}\colon\ \mathsf{Part}_{\tilde{\varepsilon}} \longrightarrow \mathsf{Part}_{\varepsilon}^{(\ell+1)\wedge(\ell+2)} \tag{3.4.11}$$

are well-defined and inverse to each other, i. e.,

$\forall\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ \forall\, \ell \in \{0,\ldots,n-1\}, \subseteq \mathbb{N}_0,\ \forall\, \tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i\in[n]} \in [m]^{\times n}\colon$

$$\mathsf{delete}_{\varepsilon,\ell+2} \circ \mathsf{double}_{\tilde{\varepsilon},\ell+1} = \mathsf{id}_{\mathsf{Part}_{\tilde{\varepsilon}}} \tag{3.4.12}$$

and

$\forall\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N} \setminus \{1\},\ \forall\, \ell \in \{0,\ldots,n-1\} \subseteq \mathbb{N}_0,\ \forall\, \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}\colon$

$$\mathsf{double}_{\tilde{\varepsilon},\tilde{\varepsilon}_{\ell+1}} \circ \mathsf{delete}_{\varepsilon,\ell+2} = \mathsf{id}_{\mathsf{Part}_{\varepsilon}^{(\ell+1)\wedge(\ell+2)}}. \tag{3.4.13}$$

(c) $\forall\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N} \setminus \{1\},\ \forall\, \ell \in \{0,\ldots,n-2\} \subseteq \mathbb{N}_0,\ \forall\, \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall\, \pi \in \mathsf{Part}_{\varepsilon}^{(\ell+1)\wedge(\ell+2)}\colon |\pi| = |\mathsf{delete}_{\varepsilon,\ell+2}(\pi)|$.

(d) $\forall\, n \in \mathbb{N},\ \forall\, \ell \in \{0,\ldots,n-1\} \subseteq \mathbb{N}_0,\ \forall\, \tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i\in[n]} \in [m]^{\times n},\ \forall\, \pi \in \mathsf{Part}_{n-1}\colon |\pi| = |\mathsf{double}_{\tilde{\varepsilon},\ell+1}(\pi)|$.

PROOF: The proof is a straightforward application of the definitions and is omitted. □

An example for the above defined maps is postponed to Example 3.4.10.

3.4.5 Definition (Induced partition). Let $B = \big((\beta_{1,1},\beta_{1,2}),\ldots,(\beta_{n,1},\beta_{n,2})\big)$ be a block over $\mathbb{N} \times [m]$ for $n,\ m \in \mathbb{N}$. Moreover, let $\delta = \{\delta_1,\ldots,\delta_k\} \in \mathsf{Part}_{\varepsilon}$ be a partition with k blocks for $k \in [n]$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$, then δ induces a partition $\{B_1,\ldots,B_k\}$ of the set $\mathsf{set}\,B$ with k blocks, where the i-th block for each $i \in [k]$ is given by

$$B_i := \mathsf{block}\Big(\big\{ (\beta_{j,1}, \beta_{j,2}) \bigm| j \in \mathsf{set}\big(\mathsf{type}(\delta_i)\big) \big\}\Big)$$
$$\equiv \big((\beta_{j,1}, \beta_{j,2})\big)_{j\in\mathsf{set}(\mathsf{type}(\delta_i))} \quad [\![\text{ Conv. 2.5.5 } \textbf{(a)}]\!] \tag{3.4.14}$$

We denote this partition by $B \Vdash \delta$ and say that $B \Vdash \delta$ is *the partition of B induced by δ*.

3.4.6 Definition (split and UMem maps).

(a) Let $n \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{N}$, $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ and $\pi \in \mathsf{Part}_\varepsilon$ and $b, b' \in \pi$ some blocks in π. If we put

$$n' := |\mathsf{set}(b)| + |\mathsf{set}(b')| \in \mathbb{N}, \tag{3.4.15a}$$

$$\tilde{b} := \mathsf{block}\Big(\mathsf{set}(b) \cup \mathsf{set}(b')\Big) \in (\mathbb{N} \times \mathbb{N})^{n'}, \tag{3.4.15b}$$

$$\tilde{\varepsilon} := \mathsf{col}(\tilde{b}), \tag{3.4.15c}$$

then we define for the above two blocks $b = \big((b_{1,1}, b_{1,2}), \dots, (b_{r,1}, b_{r,2}))\big)$ and $b' = \big((b'_{1,1}, b_{1,2}), \dots, (b'_{r',1}, b_{r',2}))\big)$

$$\mathsf{release}_{\varepsilon,b,b'}(\pi) := \Bigg\{\Big(\big(\mathsf{pos}_{\tilde{b}}(b_{1,1}), b_{1,2}\big), \dots, \big(\mathsf{pos}_{\tilde{b}}(b_{r,1}), b_{r,2}\big)\Big),$$
$$\Big(\big(\mathsf{pos}_{\tilde{b}}(b'_{1,1}), b'_{1,2}\big), \dots, \big(\mathsf{pos}_{\tilde{b}}(b'_{r',1}), b'_{r',2}\big)\Big)\Bigg\} \in \mathsf{Part}_{\tilde{\varepsilon}}\,. \tag{3.4.16}$$

We want to define the following subsets of $\mathsf{Part}_\varepsilon$. Let $n \in \mathbb{N} \setminus \{1\}$, $k \in [n]$ and $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$, $m \in \mathbb{N}$, $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ with $\varepsilon_{\ell+1} = \varepsilon_{\ell+2}$. Then,

$$\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}$$
$$:= \{ \pi \in \mathsf{Part}_\varepsilon \mid |\pi| = k,\ \exists b \in \pi\colon \ell+1,\ \ell+2 \in \mathsf{set}(\mathsf{type}(b)) \}, \tag{3.4.17a}$$

$$\mathsf{Part}^{(\ell+1)\vee(\ell+2)}_{\varepsilon,k} := \left\{ \pi \in \mathsf{Part}_\varepsilon \;\middle|\; \begin{array}{l} |\pi| = k,\ \exists b, b' \in \pi\colon b \neq b', \\ \ell+1 \in \mathsf{set}(\mathsf{type}(b)),\ \ell+2 \in \mathsf{set}(\mathsf{type}(b')) \end{array} \right\}, \tag{3.4.17b}$$

$$\mathsf{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big)$$
$$:= \left\{ (\pi, \sigma) \;\middle|\; \begin{array}{l} \pi \in \mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k},\ \exists \hat{b} \in \pi\colon \ell+1, \ell+2 \in \mathsf{set}(\mathsf{type}(\hat{b})) \\ \sigma \in \mathsf{Part}_{\mathsf{col}(\hat{b})},\ |\sigma| = 2 \\ \nexists \beta \in (\hat{b} \Vdash \sigma)\colon \ell+1,\ \ell+2 \in \mathsf{set}\big(\mathsf{type}(\beta)\big) \end{array} \right\}. \tag{3.4.17c}$$

We define for $n \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{N}$, $k \in \{2, \dots, n\} \subseteq \mathbb{N}_0$, $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}$ and

$\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$

$$\mathsf{UMem}^{\ell+1}_{\varepsilon,k}\colon \begin{cases} \mathsf{Part}^{(\ell+1)\vee(\ell+2)}_{\varepsilon,k} \longrightarrow \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big) \\ \pi = \{b_1,\dots,\underbrace{b}_{\sim \ell+1\in\mathsf{set}(\mathsf{type}(b))},\dots,\underbrace{b'}_{\sim \ell+2\in\mathsf{set}(\mathsf{type}(b'))},\dots,b_k\} \\ \longmapsto \Big(\{b_1,\dots,\mathsf{block}\big(\mathsf{set}(b)\cup\mathsf{set}(b')\big),\dots,b_k\}, \\ \qquad \mathsf{release}_{\varepsilon,b,b'}(\pi)\Big). \end{cases} \tag{3.4.18}$$

(b) Let $n \in \mathbb{N}\setminus\{1\}$, $k \in \{2,\dots,n\} \subseteq \mathbb{N}$, $\ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}_0$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$

$$\mathsf{split}^{\ell+1}_{\varepsilon,k-1}\colon \begin{cases} \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big) \longrightarrow \mathsf{Part}^{(\ell+1)\vee(\ell+2)}_{\varepsilon,k} \\ (\pi,\tilde{\pi}) = \Big(\{b_1,\dots,\underbrace{b_r}_{\sim \ell+1,\ell+2\in\mathsf{set}(\mathsf{type}(b_r))},\dots,b_{k-1}\},\{\beta_1,\beta_2\}\Big) \\ \longmapsto \Bigg(\bigcup_{i\in[k-1]\setminus\{r\}}\{b_i\}\Bigg)\cup\big(b_r \Vdash \{\beta_1,\beta_2\}\big). \end{cases} \tag{3.4.19}$$

3.4.7 Lemma (split **and** UMem **are inverse).** For $n \in \mathbb{N}\setminus\{1\}$, $m \in \mathbb{N}$, $k \in \{2,\dots,n\} \subseteq \mathbb{N}_0$ and $\ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ the maps

$$\mathsf{UMem}^{\ell+1}_{\varepsilon,k}\colon\ \mathsf{Part}^{(\ell+1)\vee(\ell+2)}_{\varepsilon,k} \longrightarrow \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big), \tag{3.4.20}$$

$$\mathsf{split}^{\ell+1}_{\varepsilon,k-1}\colon\ \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big) \longrightarrow \mathsf{Part}^{(\ell+1)\vee(\ell+2)}_{\varepsilon,k} \tag{3.4.21}$$

are well-defined and inverse to each other, i. e.,

$$\mathsf{split}^{\ell+1}_{\varepsilon,k-1}\circ\mathsf{UMem}^{\ell+1}_{\varepsilon,k} = \mathsf{id}_{\mathsf{Part}^{(\ell+1)\vee(\ell+2)}_{\varepsilon,k}} \tag{3.4.22}$$

and

$$\mathsf{UMem}^{\ell+1}_{\varepsilon,k}\circ\mathsf{split}^{\ell+1}_{\varepsilon,k-1} = \mathsf{id}_{\mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big)}. \tag{3.4.23}$$

PROOF: The proof is a straightforward calculation and is omitted for convenience of the reader. □

An example for the above defined maps is postponed to Example 3.4.10.

3.4.8 Convention. We set for $n \in \mathbb{N}\setminus\{1\}$, $m \in \mathbb{N}$, $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$, $\ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}_0$, $k \in [n]$ and any subset $\mathcal{P} \subseteq \mathsf{Part}_\varepsilon$

$$\mathcal{P}_\varepsilon := \mathsf{Part}_\varepsilon \cap \mathcal{P}, \tag{3.4.24}$$

$$\mathcal{P}_{\varepsilon,k} := \{\, \pi \in \mathcal{P}_\varepsilon \mid |\pi| = k \,\}. \tag{3.4.25}$$

Now assume $\varepsilon_{\ell+1} = \varepsilon_{\ell+2}$

$$\mathcal{P}^{(\ell+1)\wedge(\ell+2)}_\varepsilon := \mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_\varepsilon \cap \mathcal{P}, \tag{3.4.26}$$

$$\mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} := \mathsf{Part}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} \cap \mathcal{P}, \tag{3.4.27}$$

$$\mathcal{P}_{\varepsilon,k}^{(\ell+1)\vee(\ell+2)} := \mathsf{Part}_{\varepsilon,k}^{(\ell+1)\vee(\ell+2)} \cap \mathcal{P}, \tag{3.4.28}$$

$$\mathrm{sub}\big(\mathcal{P}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big) := \mathrm{sub}\big(\mathsf{Part}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \cap (\mathcal{P}\times\mathcal{P}). \tag{3.4.29}$$

3.4.9 Definition (m-colored universal class of partitions). Let $m \in \mathbb{N}$. Let $\mathcal{P} \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{\varepsilon=(\varepsilon_i)_{i\in[n]}\in[m]^{\times n}} \mathsf{Part}_\varepsilon$. We say that $\mathcal{P}$ is an *m-colored universal class of partitions* (abbreviated by *m-colored u.c.p.*) if and only if the following properties are satisfied:

(a) $\forall n \in \mathbb{N},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([m])\colon\ \mathbb{1}_\varepsilon \in \mathcal{P}_\varepsilon$.

(b) $\forall n \in \mathbb{N}\setminus\{1\},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall \ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}_0\colon$

$$\Big(\varepsilon_{\ell+1} = \varepsilon_{\ell+2},\ \pi \in \mathcal{P}_{\varepsilon}^{(\ell+1)\wedge(\ell+2)} \implies \mathsf{delete}_{\varepsilon,\ell+2}(\pi) \in \mathcal{P}_{\tilde{\varepsilon}}\Big). \tag{3.4.30}$$

(c) $\forall n \in \mathbb{N},\ \forall \tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i\in[n]} \in [m]^{\times n},\ \forall \ell \in \{0,\dots,n-1\} \subseteq \mathbb{N}_0,\colon$

$$\Big(\pi \in \mathcal{P}_{\tilde{\varepsilon}} \implies \mathsf{double}_{\tilde{\varepsilon},\ell+1}(\pi) \in \mathcal{P}_\varepsilon\Big). \tag{3.4.31}$$

(d) $\forall n \in \mathbb{N}\setminus\{1\},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall k \in [n],\ \forall \ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}_0,\colon$

$$\Big(\varepsilon_{\ell+1} = \varepsilon_{\ell+2},\ \pi \in \mathcal{P}_{\varepsilon,k}^{(\ell+1)\vee(\ell+2)} \implies \mathsf{UMem}_{\varepsilon,k}^{\ell+1}(\pi) \in \mathcal{P}_\varepsilon\Big). \tag{3.4.32}$$

(e) $\forall n \in \mathbb{N}\setminus\{1\},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall k \in [n],\ \forall \ell \in \{0,\dots,n-2\} \subseteq \mathbb{N}_0\colon$

$$\Big(\varepsilon_{\ell+1} = \varepsilon_{\ell+2},\ (\pi,\tilde{\pi}) \in \mathrm{sub}\big(\mathcal{P}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \implies \mathsf{split}_{\varepsilon,k-1}^{\ell+1}(\pi,\tilde{\pi}) \in \mathcal{P}_\varepsilon\Big). \tag{3.4.33}$$

(f) Define for $n,\ k \in \mathbb{N},\ \delta,\ \delta' \in [m]$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$

$$\mathsf{cCol}_{\varepsilon,(\delta,\delta')}\colon \begin{cases} \mathsf{Part}_\varepsilon \longrightarrow \mathsf{Part}_{(\delta,\varepsilon_2,\dots,\varepsilon_{n-1},\delta')} \\ \{b_1,\dots,b_k\} \longmapsto \Big\{\big((1,\delta),\dots\big), b_2,\dots,b_{k-1}, \big((\dots,(n,\delta'))\big)\Big\}, \end{cases} \tag{3.4.34}$$

wherein b_1 denotes the unique block in $\pi \in \mathsf{Part}_\varepsilon$, such that $1 \in \mathsf{set}(\mathsf{pr}_1(b_1))$ and b_k denotes the unique block in $\pi \in \mathsf{Part}_\varepsilon$, such that $n \in \mathsf{set}(\mathsf{pr}_1(b_k))$. Then, it needs to hold

$\forall n \in \mathbb{N}\setminus\{1\},\ \forall k \in \mathbb{N},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall \delta \in [m],\ \forall \delta' \in [m]\colon$

$$\Big(\pi \in \mathcal{P}_\varepsilon \implies \mathsf{cCol}_{\varepsilon,(\delta,\delta')}(\pi) \in \mathcal{P}_{(\delta,\varepsilon_2,\dots,\varepsilon_{n-1},\delta')}\Big). \tag{3.4.35}$$

(g) Define for $n,\ m \in \mathbb{N},\ \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \pi \in \mathcal{P}_\varepsilon$ and any block $b = \big((b_{i,1},b_{i,2})\big)_{i\in[r]} \in \pi$

$$\mathsf{mirrorB}(b) := \Big(\big((n+1) - \mathsf{pos}_{\mathbb{1}_n}(b_{(r+1)-i,1}), b_{(r+1)-i,2}\big)\Big)_{i\in[r]} \in \big([n]\times[m]\big)^{\times r}. \tag{3.4.36}$$

By this we define

$$\mathsf{mirror}_\varepsilon\colon \begin{cases} \mathsf{Part}_\varepsilon \longrightarrow \mathsf{Part}_{(\varepsilon_{(r+1)-i})_{i\in[r]}} \\ \{b_1,b_2,\dots,b_k\} \\ \quad\longmapsto \{\mathsf{mirrorB}(b_k), \mathsf{mirrorB}(b_{k-1}),\dots,\mathsf{mirrorB}(b_1)\}. \end{cases} \tag{3.4.37}$$

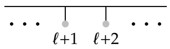
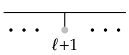
Then, it needs to hold

$\forall n \in \mathbb{N},\ \forall k \in \mathbb{N},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}\colon$

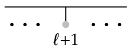
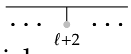
$$(\pi \in \mathcal{P}_\varepsilon \implies \mathsf{mirror}_\varepsilon(\pi) \in \mathcal{P}_{(\varepsilon_{(r+1)-i})_{i\in[r]}}). \tag{3.4.38}$$

In contrast to the single-colored case we have two more axioms and all the maps behave a little bit differently in the multi-colored case. In a more sloppy or condensed way we formulate once again the definitions for an m-colored universal class of partitions, where we try to avoid any technical notations and use some diagrams instead (Convention 3.4.2). Notice the gray color label which stands for some arbitrary color. Thus, we say that $\mathcal{P}$ as a subset of all m-colored partitions is an *m-colored universal class of partitions* if and only if

(1-block) it contains all alternating one-block partitions of m-colored partitions

(delete) $\pi \in \mathcal{P}$ with $\exists$ block $b \in \pi$ such that $b =$  $\implies \pi' \in \mathcal{P}$, where π' has the same blocks as π except $b =$ 

(double) reversal of **(delete)**

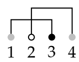
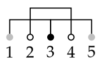
(UMem) $\pi \in \mathcal{P}$ with $\exists$ blocks $b_1, b_2 \in \pi$ such that $b_1 =$  and $b_2 =$  $\implies$ $(\pi', \sigma) \in \mathcal{P} \times \mathcal{P}$, where π' has the same blocks as π except b_1, b_2 which are replaced by their union and $\sigma \in \mathcal{P}$ is the two-block partition defined by b_1 and b_2.

(split) reversal of **(UMem)**

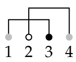
(cCol) $\pi \in \mathcal{P} \implies \pi' \in \mathcal{P}$, where π' has the same blocks as π except with changed color of the first or last leg

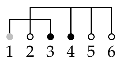
(mirror) $\pi \in \mathcal{P} \implies \pi' \in \mathcal{P}$, where π' is the "from-left-to-right" mirrored partition π

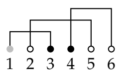
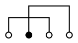
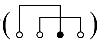
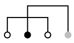
3.4.10 Example. Let $\mathcal{P}$ be a two-colored universal class of partition. We claim that

 $\in \mathcal{P} \implies$  $\in \mathcal{P} \wedge$  $\in \mathcal{P}$. (3.4.39)

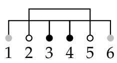
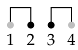
For the proof we calculate

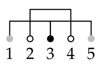
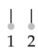
 $\in \mathcal{P}$

$\implies \pi_1 :=$  $\in \mathcal{P}$ ⟦two times application of cCol and double⟧

$\implies \pi_2 :=$  $\in \mathcal{P}$ ⟦ = mirror(), $\pi_2 = \text{split}^4_{\cdot,2}(\pi_1,$ $)$⟧

$\implies (\pi_3, \pi_4)$

$:=$ (, ) $\in \mathcal{P} \times \mathcal{P}$ ⟦$(\pi_3, \pi_4) = \text{UMem}^3_{\cdot,3}(\pi_2)$⟧

$\implies$ (, ) $\in \mathcal{P} \times \mathcal{P}$ ⟦multiple application of cCol and delete⟧.

3.4.11 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Then,

(a) $\forall n \in \mathbb{N}, \forall \varepsilon = (\varepsilon_i)_{i \in [n]} \in [m]^{\times n}: \mathbb{1}_\varepsilon \in \mathcal{P}_\varepsilon$

Let $n \in \mathbb{N} \setminus \{1\}$ and $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$ and $\varepsilon = (\varepsilon_i)_{i \in [n]} \in [m]^{\times n}$ with $\varepsilon_{\ell+1} = \varepsilon_{\ell+2}$ and let $\tilde{\varepsilon}$

be determined by ε according to equation (3.4.6), then

(b) $\mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} = \emptyset \iff \mathcal{P}_{\tilde{\varepsilon},k} = \emptyset$

(c) $\operatorname{sub}\big(\mathcal{P}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big) = \emptyset \iff \mathcal{P}_{\varepsilon,k}^{(\ell+1)\vee(\ell+2)} = \emptyset$

Proof: Ad **(a)**: By Definition 3.1.9 **(a)** we already know that for each alternating sequence $(\varepsilon_i)_{i\in[n]} \in \mathbb{A}([m])$ we have $\mathbb{1}_\varepsilon \in \mathcal{P}_\varepsilon$. It remains to show this assertion for non-alternating sequences $(\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$. This can be done by induction (over the number of application of double-operations). We omit this somehow technical, but not complicated, inductive proof. The proofs of **(b)** and **(c)** are analogous to the proofs of Lemma 3.1.10 **(b)** and **(c)**. We emphasize that the change color and mirror property of the m-colored universal class of partitions $\mathcal{P}$ are not needed for this proof. □

3.4.12 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $n \in \mathbb{N} \setminus \{1\}$, $\varepsilon = (\varepsilon)_i)_{i\in[m]} \in [m]^{\times n}$, $\ell \in \{0, \dots, n-2\} \subseteq \mathbb{N}_0$ and $k \in [n]$.

(a) If we set $\mathsf{delete}_{\varepsilon,\ell+2}(\emptyset) := \emptyset$ and $\mathsf{double}_{\tilde{\varepsilon},\ell+1}(\emptyset) := \emptyset$, then the following maps

$$\mathsf{delete}_{\varepsilon,\ell+2} \restriction \mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} \colon \mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} \longrightarrow \mathcal{P}_{\tilde{\varepsilon},k}, \tag{3.4.40}$$

$$\mathsf{double}_{\tilde{\varepsilon},\ell+1} \restriction \mathcal{P}_{\tilde{\varepsilon},k} \colon \mathcal{P}_{\tilde{\varepsilon},k} \longrightarrow \mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} \tag{3.4.41}$$

are well-defined and inverse to each other.

(b) If we set $\mathsf{UMem}_{\varepsilon,k}^{\ell+1}(\emptyset) := \emptyset$ and $\mathsf{split}_{\varepsilon,k-1}^{\ell+1}(\emptyset) := \emptyset$, then the following maps

$$\mathsf{UMem}_{\varepsilon,k}^{\ell+1} \restriction \mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} \colon \mathcal{P}_{\varepsilon,k}^{(\ell+1)\wedge(\ell+2)} \longrightarrow \operatorname{sub}\big(\mathcal{P}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big), \tag{3.4.42}$$

$$\mathsf{split}_{\varepsilon,k-1}^{\ell+1} \restriction \operatorname{sub}\big(\mathcal{P}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \colon \operatorname{sub}\big(\mathcal{P}_{\varepsilon,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \longrightarrow \mathcal{P}_{\varepsilon,k}^{(\ell+1)\vee(\ell+2)} \tag{3.4.43}$$

are well-defined and inverse to each other.

Proof: Ad **(a)**: The well-definedness for these maps follows from Definition 3.4.9 and Lemma 3.4.11 **(b)**. That the two maps are inverse to each other is an implication from Lemma 3.4.4 **(b)**. Ad **(b)**: The well-definedness for these maps follows from Definition 3.4.9 and Lemma 3.4.11 **(c)**. That the two maps are inverse to each other is an implication from Lemma 3.4.7. □

3.4.13 Lemma (cCol and mirror have left inverse). For $n, k \in \mathbb{N}$, $\delta, \delta' \in [m]$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ we have

$$\mathsf{cCol}_{(\delta,\varepsilon_2,\dots,\varepsilon_{n-1},\delta'),(\varepsilon_1,\varepsilon_n)} \circ \mathsf{cCol}_{\varepsilon,(\delta,\delta')} = \mathsf{id}_{\mathsf{Part}_\varepsilon} \tag{3.4.44}$$

$$\mathsf{mirror}_{(\varepsilon_{(r+1)-i})_{i\in[r]}} \circ \mathsf{mirror}_\varepsilon = \mathsf{id}_{\mathsf{Part}_\varepsilon} \tag{3.4.45}$$

Proof: The proof is a straightforward calculation and is omitted. □

3.4.14 Convention. We want to extend Convention 2.5.7 to the setting of blocks over $\mathbb{N}\times[m]$ for some $m \in \mathbb{N}$, in the following sense. Let V be a vector space, $n \in \mathbb{N}$ and $x := (x_i)_{i\in[n]} \in V^{\times n}$ be some n-tuple of elements in V. Then, we define for x and for any block $b = (b_i)_{i\in[\ell]} \in ([n]\times[m])^{\times\ell}$ with $\ell \in [n]$

$$x_b := x_{\mathsf{type}(b)} \equiv x_{\mathsf{type}(b_1)} \otimes \cdots \otimes x_{\mathsf{type}(b_n)} \in \mathrm{T}(V) \quad [\![\text{ Conv. 2.5.7 }]\!]. \tag{3.4.46}$$

3.4.15 Definition (Exponential and logarithm induced by $\mathcal{P}_\varepsilon$). Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces. Set $V := \bigoplus_{i=1}^{m} V_i$ and let $\varphi \in \operatorname{Lin}(\mathrm{T}(V), \mathbb{C})$.

(a) Let $n \in \mathbb{N}$. We define for any $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$

$$\exp_{\mathcal{P}_\varepsilon} \varphi \colon \begin{cases} V_{\varepsilon_1} \times \cdots \times V_{\varepsilon_n} \longrightarrow \mathbb{C} \\ (x_1, \ldots, x_n) \longmapsto \sum_{\pi \in \mathcal{P}_\varepsilon} \prod_{b \in \pi} \varphi(x_b). \end{cases} \tag{3.4.47}$$

(b) Let $n \in \mathbb{N}$. We recursively define a map $\log_{\mathcal{P}_\varepsilon} \varphi \colon \prod_{i=1}^{n} V_{\varepsilon_i} \longrightarrow \mathbb{C}$ by setting

$$\forall\, \varepsilon \in [m],\ \forall\, x \in V_\varepsilon, : (\log_{\mathcal{P}_\varepsilon} \varphi)(x) := \varphi(x), \tag{3.4.48a}$$

$$\forall\, \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall (x_i)_{i\in[n]} \in \prod_{i=1}^{n} V_{\varepsilon_i}:$$

$$(\log_{\mathcal{P}_\varepsilon} \varphi)(x_1, \ldots, x_n) := \tag{3.4.48b}$$

$$\varphi(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi \in \mathcal{P}_\varepsilon \\ \pi \neq \mathbb{1}_\varepsilon}} \prod_{b \in \pi} (\log_{\mathcal{P}_{(\varepsilon_\mathrm{i})_{\mathrm{i}\in\mathrm{set(col(b))}}}} \varphi), (x_b),$$

where we use Convention 2.5.5 **(a)** in equation (3.4.48b).

3.4.16 Lemma ($\exp_{\mathcal{P}_\varepsilon}$ and $\log_{\mathcal{P}_\varepsilon}$ are multilinear). Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces, Moreover, let $V := \bigoplus_{i=1}^{m} V_i$ and $\varphi \in \operatorname{Lin}(\mathrm{T}(V), \mathbb{C})$. Then,

(a) the map $\exp_{\mathcal{P}_\varepsilon} \varphi \colon \prod_{i\in[n]} V_{\varepsilon_i} \longrightarrow \mathbb{C}$ is multilinear for all $n \in \mathbb{N}$ and for all $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$,

(b) the map $\log_{\mathcal{P}_\varepsilon} \varphi \colon \prod_{i\in[n]} V_{\varepsilon_i} \longrightarrow \mathbb{C}$ is multilinear for all $n \in \mathbb{N}$ and for all $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$.

Proof: The proof of Lemma 3.2.4 can be adapted to the multi-colored case with analogous arguments. □

3.4.17 Lemma (Exponential and Logarithm induced by $\mathcal{P}$). Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces, Moreover, let $V := \bigoplus_{i=1}^{m} V_i$ and $\varphi \in \operatorname{Lin}(\mathrm{T}(V), \mathbb{C})$. Then,

(a) there exists a unique linear map $\exp_{\mathcal{P}} \varphi \colon \mathrm{T}(V) \longrightarrow \mathbb{C}$, such that for all $n \in \mathbb{N}$ and for each $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ the following diagram commutes

$$\begin{array}{ccc} \bigotimes_{i=1}^{n} V_{\varepsilon_i} & \xrightarrow{\mathcal{T}(\exp_{\mathcal{P}_\varepsilon} \varphi)} & \mathbb{C} \\ \Big\downarrow \text{inc} & \nearrow_{\exp_{\mathcal{P}} \varphi} & \\ \mathrm{T}(V) & & \end{array} , \tag{3.4.49}$$

(b) there exists a unique linear map $\log_{\mathcal{P}} \varphi\colon \mathrm{T}(V) \longrightarrow \mathbb{C}$, such that for all $n \in \mathbb{N}$ and for each $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ the following diagram commutes

$$\begin{array}{ccc} \bigotimes_{i=1}^n V_{\varepsilon_i} & \xrightarrow{\mathcal{T}(\log_{\mathcal{P}_\varepsilon} \varphi)} & \mathbb{C} \\ \Big\downarrow \text{inc} & \nearrow_{\log_{\mathcal{P}} \varphi} & \\ \mathrm{T}(V) & & \end{array} \quad . \tag{3.4.50}$$

Proof: We need several universal properties and the reasons are similar to the ones of the proof of Lemma 3.2.5 for the single-colored case. □

3.4.18 Remark. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. As a consequence of Lemma 3.4.17 and Definition 3.4.15, we obtain for any $m \in \mathbb{N}$, for any m-tuple of vector spaces $(V_i)_{i\in[m]}$ and for any $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$, where $V := \bigoplus_{i=1}^m V_i$, that the following maps are elements of $\mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$

$$\exp_{\mathcal{P}} \varphi\colon \begin{cases} \mathrm{T}(V) \cong \bigoplus\limits_{n\in\mathbb{N}} \bigoplus\limits_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[m]^{\times n}}} \bigotimes\limits_{i\in[n]} V_{\varepsilon_i} \longrightarrow \mathbb{C} \\ x_1 \otimes \cdots \otimes x_n \longmapsto \sum\limits_{\pi\in\mathcal{P}_\varepsilon} \prod\limits_{b\in\pi} \varphi(x_b), \end{cases} \tag{3.4.51}$$

$$\log_{\mathcal{P}} \varphi\colon \begin{cases} \mathrm{T}(V) \cong \bigoplus\limits_{n\in\mathbb{N}} \bigoplus\limits_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]}\\ \in[m]^{\times n}}} \bigotimes\limits_{i\in[n]} V_{\varepsilon_i} \longrightarrow \mathbb{C} \\ x_1 \otimes \cdots \otimes x_n \longmapsto \begin{cases} \varphi(x_1) & \text{for } n = 1 \\ \varphi(x_1 \otimes \cdots \otimes x_n) - \sum\limits_{\substack{\pi\in\mathcal{P}_\varepsilon\\ \pi\neq 1_\varepsilon}} \prod\limits_{b\in\pi} (\log_{\mathcal{P}} \varphi)(x_b) & \text{else.} \end{cases} \end{cases} \tag{3.4.52}$$

Here we have used an implication of equation (1.1.14) for the isomorphism of $\mathrm{T}(V)$.

3.4.19 Lemma ($\exp_{\mathcal{P}}$ and $\log_{\mathcal{P}}$ are inverse). Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces. Moreover, if $V := \bigoplus_{i=1}^m V_i$ and $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$, then

$$\exp_{\mathcal{P}}\big(\log_{\mathcal{P}} \varphi\big) = \varphi \quad \text{and} \quad \log_{\mathcal{P}}\big(\exp_{\mathcal{P}} \varphi\big) = \varphi. \tag{3.4.53}$$

Proof: The proof for the single-colored case (Lemma 3.2.7) can be directly extended to the purpose of the multi-colored case. □

3.4.20 Definition. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces. Set $V := \bigoplus_{i=1}^m V_i$ and let $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$. We denote by $\circledcirc_{\mathcal{P}}$ the following binary operation on $\mathrm{Lin}(\mathrm{T}(V), \mathbb{K})$

$$\circledcirc_{\mathcal{P}}\colon \begin{cases} \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \times \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathrm{T}(V), \mathbb{C}) \\ (\varphi, \psi) \longmapsto \exp_{\mathcal{P}}\big(\log_{\mathcal{P}}(\varphi) + \log_{\mathcal{P}}(\psi)\big). \end{cases} \tag{3.4.54}$$

We collect some properties of $\exp_{\mathcal{P}}$, $\log_{\mathcal{P}}$ and $\circledcirc_{\mathcal{P}}$, which will be of interest in the following sections.

3.4.21 Lemma ($\circledcirc_{\mathcal{P}}$ is associative and commutative). Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces. Set $V := \bigoplus_{i\in[m]} V_i$. Then, the binary mapping $\circledcirc_{\mathcal{P}}\colon \operatorname{Lin}(\mathrm{T}(V), \mathbb{C}) \times \operatorname{Lin}(\mathrm{T}(V), \mathbb{C}) \longrightarrow \operatorname{Lin}(\mathrm{T}(V), \mathbb{C})$ is associative and commutative.

Proof: The proof is formally the same as the proof in the single-colored case (Lemma 3.2.9). □

3.4.22 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$, $(V'_i)_{i\in[m]}$ be two m-tuples of vector spaces. We set $V := \bigoplus_{i=1}^m V_i$ and $V' := \bigoplus_{i=1}^m V'_i$. Assume $f \in \operatorname{Lin}(V, V')$ and $\forall i \in [m]\colon f(V_i) \subseteq V'_i$. Let $\varphi \in \operatorname{Lin}(\mathrm{T}(V'), \mathbb{C})$ and $(\varphi_i)_{i\in[k]} \in \big(\operatorname{Lin}(\mathrm{T}(V'), \mathbb{C})\big)^{\times k}$ for some $k \in \mathbb{N} \setminus \{1\}$, then

(a) $\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big) = (\log_{\mathcal{P}} \varphi) \circ \mathrm{T}(f)$,

(b) $\exp_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big) = (\exp_{\mathcal{P}} \varphi) \circ \mathrm{T}(f)$,

(c) $\big(\varphi_1 \circ \mathrm{T}(f)\big) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \big((\varphi_k \circ \mathrm{T}(f)\big) = (\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_k) \circ \mathrm{T}(f)$.

Proof: The tensor algebra $\mathrm{T}(V)$ is generated by pure tensors $x_1 \otimes \cdots \otimes x_n \in \bigotimes_{i\in[n]} V_{\varepsilon_i}$ for all $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ and for all $n \in \mathbb{N}$. Since all maps of consideration are linear maps, it suffices to show the stated equations on such generators of $\mathrm{T}(V)$.

Ad **(a)**: We show the statement by strong induction over $n \in \mathbb{N}$. The only difference to the single-colored case (Lemma 3.2.11) is that the linear map f is assumed to additionally fulfill $\forall i \in [m]\colon f(V_i) \subseteq V'_i$. Thus, for the induction base we can formally perform the same proof as in the single-colored case. Now, let $\mathbb{N} \ni n > 1$ and we assume that the expression of **(a)** is true for all pure tensors of length $\ell \in [n-1]$. Then, we calculate for any sequence $(x_i)_{i\in[n]} \in \prod_{i\in[n]} V_{\varepsilon_i}$ for any $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$

$$
\begin{aligned}
&\Big(\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big)\Big)(x_1 \otimes \cdots \otimes x_n) \\
&= \mathcal{T}\Big(\log_{\mathcal{P}_\varepsilon}\big(\varphi \circ \mathrm{T}(f)\big)\Big)(x_1 \otimes \cdots \otimes x_n) \quad [\![\text{ def. of } \log_{\mathcal{P}} \cdot \text{ in eq. (3.4.50) }]\!] \\
&= \big(\varphi \circ \mathrm{T}(f)\big)(x_1 \otimes \cdots \otimes x_n) - \sum_{\substack{\pi\in\mathcal{P}_\varepsilon \\ |\pi|\geq 2}} \prod_{b\in\pi} \Big(\log_{\mathcal{P}}\big(\varphi \circ \mathrm{T}(f)\big)\Big)(x_b) \\
&\qquad [\![\text{ def. of } \log_{\mathcal{P}_\varepsilon} \cdot \text{ in eq. (3.4.48b) on } \bigotimes_{i=1}^n V_{\varepsilon_i}]\!] \\
&= \varphi\big(\mathrm{T}(f)(x_1) \otimes \cdots \otimes \mathrm{T}(f)(x_n))\big) - \sum_{\substack{\pi\in\mathcal{P}_\varepsilon \\ |\pi|\geq 2}} \prod_{b\in\pi} (\log_{\mathcal{P}} \varphi)\big(\mathrm{T}(f)(x_b)\big) \\
&\qquad [\![\ \mathrm{T}(f) \in \operatorname{Alg}(\mathrm{T}(V), \mathrm{T}(V')), \text{ IH applied to } \ell = |b| \text{ and } (x_j)_{j\in\mathsf{set}(\mathsf{type}(b))} \in \prod_{j\in\mathsf{set}(\mathsf{type}(b))} V_{\varepsilon_j}]\!] \\
&= \varphi\big(f(x_1) \otimes \cdots \otimes f(x_n)\big) - \sum_{\substack{\pi\in\mathcal{P}_\varepsilon \\ |\pi|\geq 2}} \prod_{b\in\pi} (\log_{\mathcal{P}}(\varphi)\Big(\big(f(x)\big)_b\Big)
\end{aligned}
$$

$$[\![\mathrm{T}(f) \in \mathrm{Alg}(\mathrm{T}(V), \mathrm{T}(V')), \text{UMP of } \mathrm{T}(V)]\!]$$

$$= \varphi\big(y_1 \otimes \cdots \otimes y_n\big) - \sum_{\substack{\pi \in \mathcal{P}_\varepsilon \\ |\pi| \geq 2}} \prod_{b \in \pi} (\log_{\mathcal{P}} \varphi)(y_b)$$

$$[\![f \in \mathrm{Lin}(V, V') \implies \forall i \in [n]\colon f(x_i) =: y_i \in V']\!]$$

$$= (\log_{\mathcal{P}_\varepsilon} \varphi)(y_1 \otimes \cdots \otimes y_n)$$

$$\left[\!\!\left[\begin{array}{l} \forall i \in [n]\colon y_i \in f(V_i) \subseteq V_i' \implies y_1 \otimes \cdots \otimes y_n \in \bigotimes_{i=1}^n V'_{\varepsilon_i}, \\ \text{then apply def. of } \log_{\mathcal{P}_\varepsilon} \varphi \text{ in eq. (3.4.48b) on } \bigotimes_{i=1}^n V'_{\varepsilon_i} \end{array}\right]\!\!\right]$$

$$= \big((\log_{\mathcal{P}_\varepsilon} \varphi) \circ \mathrm{T}(f)\big)(x_1 \otimes \cdots \otimes x_n)$$

AD **(c)**: The proof is similar to the proof of Lemma 3.2.11 **(c)** and is therefore omitted. □

3.4.23 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces and $\mathscr{A}$ be an algebra. Furthermore, for $V := \bigoplus_{i=1}^m V_{\varepsilon_i}$ let $f \in \mathrm{Lin}(V, \mathscr{A})$ and $\varphi \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$. Then

$\forall n \in \mathbb{N},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall (x_i)_{i\in[n]} \in \prod_{i=1}^n V_{\varepsilon_i}\colon$

$$\Big(\big(\exists j \in [n]\colon x_j \in \ker f\big) \implies x_1 \otimes \cdots \otimes x_n \in \ker\big(\log_{\mathcal{P}}(\varphi \circ \mathcal{T}(f))\big)\Big) \tag{3.4.55}$$

PROOF: Formally, there is no difference to the single-colored case and we can extend the arguments used in the proof of Lemma 3.2.12 to there multi-colored case by suitable replacements of the occurring sets and maps. □

3.4.24 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]}$ be an m-tuple of vector spaces and set $V := \bigoplus_{i=1}^m V_i$. Assume we are given $n \in \mathbb{N} \setminus \{1\}$, $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$, $(x_i)_{i\in[n]} \in \prod_{i=1}^n V_{\varepsilon_i}$ and $(y_i)_{i\in[n-1]} \in V^{\times(n-1)}$ such that there exists $\ell \in \{0, 1, \ldots, n-2\} \subseteq \mathbb{N}_0$ with the property

$$\varepsilon_{\ell+1} = \varepsilon_{\ell+2} \tag{3.4.56}$$

and

$$\forall i \in \{0, 1, \ldots, \ell\}\colon x_i = y_i \in V_{\varepsilon_i}, \tag{3.4.57a}$$

$$y_{\ell+1} \in V_{\varepsilon_{\ell+1}} \tag{3.4.57b}$$

$$\forall i \in \{\ell+3, \ldots, n\}\colon x_i = y_{i-1} \in V_{\varepsilon_i}. \tag{3.4.57c}$$

Define the set

$$B_n :=:= \left\{ (i_j)_{j\in[k]} \in [n]^{\times k} \;\middle|\; \begin{array}{l} k \in [n],\ \ell+1,\ \ell+2 \in \{i_1, \ldots, i_m\}, \\ \forall j \in [k-1]\colon i_j < i_{j+1} \end{array} \right\} \tag{3.4.58}$$

Let $\varphi \in \mathrm{Lin}(\mathrm{T}(V), \mathbb{C})$ such that it satisfies

$\forall k \in [n]\ \forall (i_j)_{j\in[k]} \in B_n\colon$

$$\varphi(x_{i_1} \otimes \cdots \otimes x_{\ell+1} \otimes x_{\ell+2} \otimes \cdots x_{i_k}) = \varphi(y_{i_1} \otimes \cdots \otimes y_{\ell+1} \otimes \cdots \otimes y_{i_k - 1}).$$

Then, the following equation holds for the above choices

$$(\log_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) = (\log_{\mathcal{P}} \varphi)(y_1 \otimes \cdots \otimes y_{n-1}) - \sum_{\substack{\{\pi_1,\pi_2\}\in \\ \mathcal{P}_{\varepsilon,2}^{(\ell+1)\vee(\ell+2)}}} (\log_{\mathcal{P}} \varphi)(x_{\pi_1}) \cdot (\log_{\mathcal{P}} \varphi)(x_{\pi_2}). \tag{3.4.59}$$

Proof: The proof in the single-colored case (Lemma 3.2.15) can be directly adapted to the multi-colored case, since we have multi-colored counterparts for the delete and UMem maps, and each of these maps is bijective in the multi-colored case too. Check the sketch of the proof of Lemma 3.4.27 for a list of replacements for the transition from the single-colored to the multi-colored case. □

3.4.25 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $(V_i)_{i\in[m]})$ be an m-tuple of vector spaces. If $V := \bigoplus_{i=1}^m V_i$ and $\varphi \in \operatorname{Lin}(\mathrm{T}(V), \mathbb{C})$, then

$\forall n \in \mathbb{N},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n},\ \forall (x_i)_{i\in[n]} \in \prod_{j\in[n]} V_{\varepsilon_j},\ \forall \pi \in \mathsf{Part}_\varepsilon,\ \exists \gamma_\pi \in \mathbb{C}$:

$$(\log_{\mathcal{P}} \varphi)(x_1 \otimes \cdots \otimes x_n) = \sum_{\pi \in \mathsf{Part}_\varepsilon} \gamma_\pi \prod_{b\in\pi} \varphi(x_b) \tag{3.4.60}$$

holds.

Proof: The proof is done analogously to the proof of Lemma 3.2.16. We only have to replace Part_n by $\mathsf{Part}_\varepsilon$ and the calculations made in the proof of Lemma 3.2.16 formally remain the same. □

Now we are going to apply the statements from above to the situation of m-faced algebras.

3.4.26 Convention. Let $m \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$ $(\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$ and define

$$\mathcal{A}_{1\oplus\cdots\oplus k} := \bigoplus_{j=1}^{m} \mathcal{A}_{1\oplus\cdots\oplus k}^{(j)} := \bigoplus_{j=1}^{m} \Big(\bigoplus_{i=1}^{k} \mathcal{A}_i^{(j)} \Big). \tag{3.4.61}$$

If we set $\iota_r^{(j)} \colon \mathcal{A}_r^{(j)} \hookrightarrow \bigsqcup_{j=1}^m \mathcal{A}_r^{(j)}$ as the canonical insertion map, then we have the following commutative diagram for all $r \in [k]$

$$\begin{array}{ccc} \bigoplus_{j=1}^m \bigoplus_{i=1}^k \mathcal{A}_i^{(j)} & \xhookrightarrow{\ \mathsf{inc}\ } & \mathrm{T}\big(\bigoplus_{j=1}^m \bigoplus_{i=1}^k \mathcal{A}_i^{(j)}\big) \\ & {\scriptstyle \bigoplus_{j=1}^m \bigoplus_{i=1}^k \Delta_{i,r}^{(j)}} \searrow & \Big\downarrow {\scriptstyle \mathcal{T}\left(\bigoplus_{j=1}^m \bigoplus_{i=1}^k \Delta_{i,r}^{(j)}\right)} \\ & & \bigsqcup_{j\in[m]} \mathcal{A}_r^{(j)} \cong \mathcal{A}_r, \end{array} \tag{3.4.62}$$

wherein we put ("colored map-like Kronecker delta")

$$\forall j \in [m],\ \forall r \in [k] \colon \Delta_{i,r}^{(j)} := \begin{cases} 0 \colon \mathcal{A}_i^{(j)} \longrightarrow \mathcal{A}_r & \text{for } i \neq r \\ \iota_r^{(j)} \colon \mathcal{A}_r^{(j)} \hookrightarrow \mathcal{A}_r & \text{for } i = r. \end{cases} \tag{3.4.63}$$

We use the following abbreviation

$$\forall r \in [k] \colon \mathrm{j}_r := \mathcal{T}\Big(\bigoplus_{j=1}^{m} \bigoplus_{i=1}^{k} \Delta_{i,r}^{(j)} \Big). \tag{3.4.64}$$

With respect to the definitions made in equation (3.4.54), we have for $\varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ for $i \in \{1,2\}$ by Definition 3.4.20 and Convention 3.4.26 that

$$\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2)\big) \in \mathrm{Lin}\big(\mathrm{T}(\mathscr{A}_{1\oplus 2}), \mathbb{C}\big), \tag{3.4.65}$$

since $\mathscr{A}_{1\oplus 2} = \bigoplus_{i=1}^{m}(\mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)})$. This means we apply the definition of $\circledcirc_{\mathcal{P}}$ made in Definition 3.4.20 to the vector space $\mathscr{A}_{1\oplus 2}$ instead of V and to $\bigoplus_{i=1}^{m}(\mathscr{A}_1^{(i)} \oplus \mathscr{A}_2^{(i)})$ instead of $\bigoplus_{i=1}^{m} V_i$.

We make the following important statement, which will be the essential property to lift this binary operation between linear functionals to a map which possesses the properties of an universal product.

3.4.27 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $k \in \mathbb{N} \setminus \{1\}$, $(\mathscr{A}_i, (\mathscr{A}_i^j)_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$ and $(\varphi_i)_{i\in[k]} \in \prod_{i=1}^{k} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$). Let $n \in \mathbb{N} \setminus \{1\}$ and $(\varepsilon_i)_{i\in[n]} \in ([k] \times [m])^{\times n}$ and assume the tuple ε has the property that

$$\exists \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N} \colon \varepsilon_{\ell+1} = \varepsilon_{\ell+2}. \tag{3.4.66}$$

Then, for any $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\mathsf{type}(\varepsilon_i)}^{(\mathsf{col}(\varepsilon_i))}$ holds

$$\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)\overbrace{(a_1 \otimes \cdots \otimes a_{\ell+1} \otimes a_{\ell+2} \otimes \cdots \otimes a_n)}^{\in \mathrm{T}(\bigoplus_{j=1}^{m}\bigoplus_{i=1}^{k} \mathscr{A}_i^{(j)})}$$
$$= \big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)(a_1 \otimes \cdots \otimes (a_{\ell+1} \cdot a_{\ell+2}) \otimes \cdots \otimes a_n). \tag{3.4.67}$$

Proof: The proof is similar to the proof of Lemma 3.3.2 (single-colored case). Therefore, we do not do the proof in full detail, since the formal steps of reasoning from the proof of Lemma 3.3.2 are valid, if we use the following replacements (here we read the symbol "$\mapsto$" as "replaced by")

- $\varepsilon \in [k]^{\times n} \mapsto \varepsilon \in ([k] \times [m])^{\times n}$,
- $\mathcal{P}_n$ as u.c.p. defined in Definition 3.1.9 $\mapsto \mathcal{P}_{\mathsf{col}(\varepsilon)}$ as u.c.p. of m-colors defined in Definition 3.4.9,
- $\exp_{\mathcal{P}}$ and $\log_{\mathcal{P}}$ defined in Lemma 3.2.5 $\mapsto \exp_{\mathcal{P}}$ and $\log_{\mathcal{P}}$ defined in Lemma 3.4.17,
- j_i defined in equation (3.3.3) $\mapsto \mathrm{j}_i$ defined in equation (3.4.64),
- application of Lemma 3.2.15 $\mapsto$ application of Lemma 3.4.24.

Now, we can use the proof of Lemma 3.3.2 and apply the above list of replacements. □

3.4.28 Remark. Let $m \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$ and $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$, then we define

$$S_{1,\dots,k} := \{\, a \otimes a' - a \cdot a' \mid i \in [k],\, j \in [m] \colon a,\, a' \in \mathscr{A}_i^{(j)} \,\} \subseteq \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k}), \tag{3.4.68}$$

$$I_{1,\dots,k} := \langle S_{1,\dots,k} \rangle, \tag{3.4.69}$$

i. e., $I_{1,\dots,k}$ denotes the smallest two-sided ideal in $\mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})$ (here $\mathscr{A}_{1\oplus\cdots\oplus k}$ is defined in eq. (3.4.61)), such that $I_{1,\dots,k} \supseteq S_{1,\dots,k}$. If we set

$$I_{S_{1,\dots,k}} := \left\{ \sum_{i=1}^{n}\Big(cs_i + xs_i + s_i y + \sum_{j=1}^{N_i} x_j s_i y_j\Big) \;\middle|\; \begin{array}{l} n,\, N_i \in \mathbb{N},\; c \in \mathbb{C},\; s_i \in S_{1,\dots,k}, \\ x,\, y,\, x_j,\, y_j \in \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k}) \end{array} \right\}, \tag{3.4.70}$$

then it is a standard task to show that $I_{1,\dots,k} = I_{S_{1,\dots,k}}$.

3.4.29 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $k \in \mathbb{N} \setminus \{1\}$ $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$ and $(\varphi_i)_{i\in[k]} \in \prod_{i=1}^{k} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C}))$, then $I_{S_{1,\dots,k}} \subseteq \ker((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)) \subseteq \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})$.

Proof: The proof is formally the same as the proof of Lemma 3.3.4. We emphasize that in particular the statement of Lemma 3.4.27 is needed for this proof. □

3.4.30 Definition (Universal product induced by an m-colored universal class of partitions). Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $k \in \mathbb{N} \setminus \{1\}$ $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$ and $(\varphi_i)_{i\in[k]} \in \prod_{i=1}^{k} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C}))$. From the universal property of the quotient space and by Lemma 3.4.29 there exists a unique linear map $\mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big) \in \mathrm{Lin}(\mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})/I_{1,\dots,k}, \mathbb{C})$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k}) & \xrightarrow{(\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ \mathrm{j}_k)} & \mathbb{C} \\ \downarrow \mathsf{pr} & \nearrow \mathsf{lift}\big((\varphi_1\circ \mathrm{j}_1)\circledcirc_{\mathcal{P}}\cdots\circledcirc_{\mathcal{P}}(\varphi_k\circ \mathrm{j}_k)\big) & \\ \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})\big/ I_{1,\dots,k} & & \end{array} . \tag{3.4.71}$$

If we apply the above setting to the case $k = 2$, then we set

$$(\varphi_1 \circ \mathrm{j}_1) \,\widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_2 \circ \mathrm{j}_2) := \mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}_2)\big). \tag{3.4.72}$$

Let us denote by $\mathbf{i}_i^{(j)} \colon \mathscr{A}_i^{(j)} \hookrightarrow \mathrm{T}(\bigoplus_{j=1}^{m}(\mathscr{A}_1^{(j)} \oplus \mathscr{A}_2^{(j)})/I_{1,2}$ the canonical injections for each $i \in [2]$ and $j \in [m]$. Then,

$$\bigsqcup_{j\in[m]} (\mathbf{i}_1^{(j)} \sqcup \mathbf{i}_2^{(j)}) \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{T}(\mathscr{A}_{1\oplus 2})/I_{1,2} \tag{3.4.73}$$

is the canonical isomorphism of algebras, depicted by the following diagram for $i \in [2]$, $j \in [m]$

$$\begin{array}{ccccc} & & \bigsqcup_{j\in[m]}(\mathscr{A}_1^{(j)} \sqcup \mathscr{A}_2^{(j)}) & & \\ & \nearrow \iota_i^{(j)} & \downarrow \bigsqcup_{j\in[m]}(\mathbf{i}_1^{(j)} \sqcup \mathbf{i}_2^{(j)}) & & \\ \mathscr{A}_i^{(j)} & \xrightarrow{\mathbf{i}_i^{(j)}} & \mathrm{T}\big(\bigoplus_{j=1}^{m}(\mathscr{A}_1^{(j)} \oplus \mathscr{A}_2^{(j)})\big)\big/ I_{1,2} & & \mathsf{id} \\ & \searrow \iota_i^{(j)} & \downarrow \bigsqcup_{j\in[m]}(\iota_1^{(j)} \sqcup \iota_2^{(j)}) & & \\ & & \bigsqcup_{j\in[m]}(\mathscr{A}_1^{(j)} \sqcup \mathscr{A}_2^{(j)}) & & \end{array} . \tag{3.4.74}$$

Then, we define

$$\varphi_1 \odot_{\mathcal{P}} \varphi_2 \colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \overset{\mathsf{can}}{\cong} \bigsqcup_{i\in[m]} (\mathscr{A}_1^{(i)} \sqcup \mathscr{A}_2^{(i)}) \longrightarrow \mathbb{C} \\ \\ a \longmapsto \Big(\big((\varphi_1 \circ \mathrm{j}_1) \,\widetilde{\circledcirc_{\mathcal{P}}}\, (\varphi_2 \circ \mathrm{j}_2)\big) \circ \big(\bigsqcup_{j\in[m]} (\mathbf{i}_1^{(j)} \sqcup \mathbf{i}_2^{(j)})\big) \circ \mathsf{can}\Big)(a). \end{cases} \tag{3.4.75}$$

Finally, we set

$$\odot_{\mathcal{P}} \colon \begin{cases} \mathrm{Lin}(\mathscr{A}_1, \mathbb{C}) \times \mathrm{Lin}(\mathscr{A}_2, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathscr{A}_1 \sqcup \mathscr{A}_2, \mathbb{C}) \\ (\varphi_1, \varphi_2) \longmapsto \varphi_1 \odot_{\mathcal{P}} \varphi_2. \end{cases} \tag{3.4.76}$$

Similar to Remark 3.3.6 we have for $m \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$ $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$

$$\mathrm{T}\big(\mathscr{A}_{1\oplus\cdots\oplus k}\big)\Big/ I_{1,\ldots,k} \cong \bigsqcup_{j\in[m]} \Big(\bigsqcup_{i\in[k]} \mathscr{A}_i^{(j)} \Big), \tag{3.4.77}$$

$I_{1,\ldots,k}$ is the ideal from equation (3.4.69).

3.4.31 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $k \in \mathbb{N} \setminus \{1\}$ $(\mathscr{A}_i, (\mathscr{A}_i^{j})_{j\in[m]})_{i\in[k]} \in (\mathsf{Obj}(\mathsf{Alg_m}))^{\times k}$ and $(\varphi_i)_{i\in[k]} \in \prod_{i=1}^{k} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C}))$. Then

$$\begin{aligned} &\Big(\big((\varphi_1 \odot_{\mathcal{P}} \varphi_2) \odot_{\mathcal{P}} \varphi_3 \big) \cdots \Big) \odot_{\mathcal{P}} \varphi_k \\ &= \mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k) \big) \circ \mathsf{can}, \end{aligned} \tag{3.4.78}$$

where $\circledcirc_{\mathcal{P}}$ here denotes the binary operation on the dual space of $\mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})$ and can is the canonical isomorphism of algebras $\mathsf{can} \colon (((\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3) \ldots) \sqcup \mathscr{A}_k \longrightarrow \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})/I_{1,\ldots,k}$.

Proof: There is formally not much difference to the proof of Lemma 3.3.7, i. e., the case $m = 1$ but much "bigger clouds of indices". We will just outline the main differences which will result in highlighting, where the color index can appear. As in the case $m = 1$ for induction base there is nothing to show. Now, we want to consider the induction step $k \to k+1$. We need to be aware, where the color index might enter the stage. As for the case $m = 1$ we set

$$\forall k' \in \mathbb{N} \setminus \{1\} \colon \forall k' \in \mathbb{N} \setminus \{1\} \colon \mathscr{A}_{1\sqcup\cdots\sqcup k'} := \Big(\big((\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3 \big) \cdots \Big) \sqcup \mathscr{A}_{k'}.$$

We claim that the following diagram is commutative

$$\begin{array}{ccccc} & & \mathrm{T}\big(\bigoplus_{j=1}^{m} (\mathscr{A}_{1\sqcup\cdots\sqcup k}^{(j)} \oplus \mathscr{A}_{k+1}^{(j)}\big) & \xrightarrow{\mathcal{T}(\bigoplus_{j=1}^{m} (\mathsf{id}_{\mathscr{A}_{1\sqcup\cdots\sqcup k}^{(j)}} \oplus 0))} & \mathscr{A}_{1\sqcup\cdots\sqcup k} \\ & \nearrow^{\mathrm{T}\left(\bigoplus_{j=1}^{m} \bigoplus_{i\in[k+1]} \mathsf{inc}_{\mathscr{A}_i^{(j)}, \mathscr{A}_{1\sqcup\cdots\sqcup k}^{(j)} \oplus \mathscr{A}_{k+1}^{(j)}}\right)} & & & \Big\downarrow \mathsf{can} \\ \mathrm{T}\big(\bigoplus_{j=1}^{m} \bigoplus_{i\in[k+1]} \mathscr{A}_i^{(j)}\big) & & & & \\ & \searrow_{\mathrm{T}(\bigoplus_{j=1}^{m} (\bigoplus_{i=1}^{k} \mathsf{id}_{\mathscr{A}_i^{(j)}} \oplus 0))} & \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k}) & \xrightarrow[\mathsf{pr}]{} & \mathrm{T}(\mathscr{A}_{1\oplus\cdots\oplus k})\Big/ I_{1,\ldots,k} \end{array} .$$

The proof is formally the same as in the case $m = 1$. The main difference is that for the definition of the maps j_i we need to apply equation (3.4.64) instead of equation (3.3.3). Furthermore, the application of Lemma 3.2.11 **(c)** is replaced by the application of Lemma 3.4.22 **(c)**, which said that

$$\forall k \in \mathbb{N} \setminus \{1\} \colon \big(\varphi_1 \circ \mathrm{T}(f) \big) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \big((\varphi_k \circ \mathrm{T}(f) \big) = (\varphi_1 \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} \varphi_k) \circ \mathrm{T}(f)$$

for $f \in \mathrm{Lin}(V, V')$ and $\forall i \in [m] \colon f(V_i) \subseteq V_i'$. The last criterion displays a condition on the color index, we additionally have to check in the case $m > 1$. We claim that this prerequisite is fulfilled when we formally follow the steps of the proof of Lemma 3.3.7 since the map f in our situation is the direct sum of maps, i. e., $f = \bigoplus_{j=1}^{m} f^{(j)}$ with $f^{(j)} \colon V_j \longrightarrow V_j'$. We are not going to further elaborate on the proof since technically the same steps as in the case of Lemma 3.3.7 with the above modifications may be applied. □

3.4.32 Theorem. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$ and $\big(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i\big)_{i\in[2]} \in (\mathsf{Obj}(\mathsf{AlgP_m}))^{\times 2}$. Then, the above defined map $\odot_{\mathcal{P}}\colon \mathrm{Lin}(\mathscr{A}_1, \mathbb{C}) \times \mathrm{Lin}(\mathscr{A}_2, \mathbb{C}) \longrightarrow \mathrm{Lin}(\mathscr{A}_1 \sqcup \mathscr{A}_2, \mathbb{C})$ fulfills the properties of a symmetric m-faced u.a.u.-product with right-ordered monomials property, i. e.,

(a) $\odot_{\mathcal{P}}$ is unital: Let $\iota_i\colon \mathscr{A}_i \longrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ denote the canonical homomorphic insertions for $i \in \{1,2\}$, then

$$\forall i \in \{1,2\}\colon (\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ \iota_i = \varphi_i. \tag{3.4.79}$$

(b) $\odot_{\mathcal{P}}$ is associative:

$$\forall i \in \{1,2,3\}\ \forall \varphi_i \in \mathscr{A}_i\colon (\varphi_1 \odot_{\mathcal{P}} \varphi_2) \odot_{\mathcal{P}} \varphi_3 = \varphi_1 \odot_{\mathcal{P}} (\varphi_2 \odot_{\mathcal{P}} \varphi_3) \circ \mathsf{can}, \tag{3.4.80}$$

where $\mathsf{can}\colon (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3 \longrightarrow \mathscr{A}_1 \sqcup (\mathscr{A}_2 \sqcup \mathscr{A}_3)$ is the canonical algebra homomorphism.

(c) $\odot_{\mathcal{P}}$ is universal: If $\kappa_i\colon \mathscr{B}_i \longrightarrow \mathscr{A}_i$ for $i \in \{1,2\}$ are homomorphisms of algebras and

$$\forall i \in \{1,2\}, \forall j \in [m]\colon \kappa_i(\mathscr{B}_i^{(j)}) \subseteq \mathscr{A}_i^{(j)}, \tag{3.4.81}$$

then

$$(\varphi_1 \odot_{\mathcal{P}} \varphi_2) \circ (\kappa_1 \amalg \kappa_2) = (\varphi_1 \circ \kappa_1) \odot_{\mathcal{P}} (\varphi_2 \circ \kappa_2). \tag{3.4.82}$$

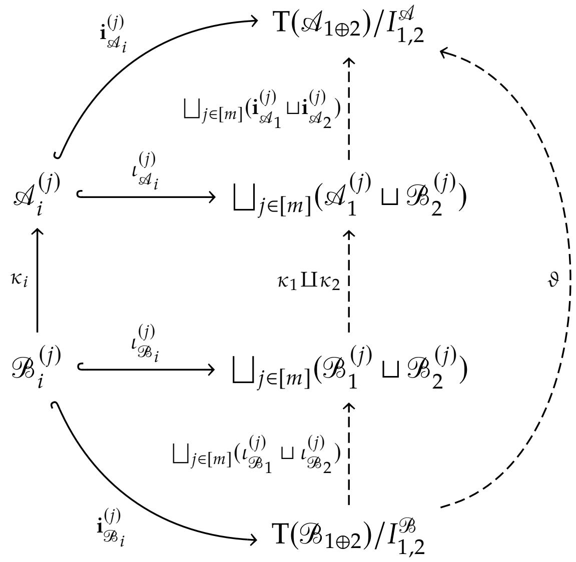
(d) $\odot_{\mathcal{P}}$ is symmetric: $\forall i \in \{1,2\}\ \forall \varphi_i \in \mathscr{A}_i$:

$$\varphi_1 \odot_{\mathcal{P}} \varphi_2 = (\varphi_2 \odot_{\mathcal{P}} \varphi_1) \circ \mathsf{can}, \tag{3.4.83}$$

where $\mathsf{can}\colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathscr{A}_2 \sqcup \mathscr{A}_1$ is the canonical isomorphism of algebras.

(e) $\odot_{\mathcal{P}}$ has the right-ordered monomials property.

Proof: Ad **(a)**: This proof is similar to the proof of Theorem 3.3.9 **(a)** and can be transferred to the multi-colored case.
Ad **(b)**: Associativity is shown formally in the way as in the case $m = 1$, i. e., in the proof of Theorem 3.3.9 **(b)**. Essential for this proof is the associativity of $\circledcirc_{\mathcal{P}}$, i. e., Lemma 3.4.21 and Lemma 3.4.31.
Ad **(c)**: Although the proof is similar to the proof of Theorem 3.3.9 **(c)**, we shall make the proof since morphisms in the m-faced setting need to respect the faces of the algebras and we want to see where this property enters the proof. Let $\kappa_i\colon \mathscr{B}_i \longrightarrow \mathscr{A}_i$ be an algebra homomorphism for $i \in [2]$. By the universal mapping property of the free product of the algebras $\mathscr{B}_1^{(j)}$ and $\mathscr{B}_2^{(j)}$ as in equation (3.4.74) we can see that the following diagram is commutative for each $i \in [2]$ and $j \in [m]$



(I)

Thus, we have

$$\vartheta\circ\Big(\underbrace{\bigsqcup_{j\in[m]}(\iota^{(j)}_{\mathscr{B}_1}\sqcup\iota^{(j)}_{\mathscr{B}_2})}_{=:\,\iota}\Big)^{-1}=\Big(\underbrace{\bigsqcup_{j\in[m]}(\mathbf{i}^{(j)}_{\mathscr{A}_1}\sqcup\mathbf{i}^{(j)}_{\mathscr{A}_2})}_{=:\mathbf{i}}\Big)\circ(\kappa_1\amalg\kappa_2). \tag{II}$$

We calculate for any $n\in\mathbb{N}$, $(\varepsilon_i)_{i\in[n]}\in\{1,2\}^{\times n}$, $(\delta_i)_{i\in[n]}\in[m]^{\times n}$ and any $(b^{(\delta_i)}_{\varepsilon_i})_{i\in[n]}\in\prod_{i=1}^n\mathscr{B}^{(\delta_i)}_{\varepsilon_i}$

$$\big((\varphi_1\odot_{\mathcal{P}}\varphi_2)\circ(\kappa_1\amalg\kappa_2)\big)\big(\iota^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1)\cdot\dots\cdot\iota^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big)$$

$$=\big((\varphi_1\odot_{\mathcal{P}}\varphi_2)\circ(\mathbf{i}^{-1}\circ\vartheta\circ\iota^{-1})\big)\big(\iota^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1)\cdot\dots\cdot\iota^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big)$$

⟦ eq. (II) & $\mathbf{i}$, ι are isomorphisms ⟧

$$=\Big(\Big(\big((\varphi_1\circ\mathrm{j}^{\mathscr{A}}_1)\mathbin{\widetilde{\circledcirc_{\mathcal{P}}}}(\varphi_2\circ\mathrm{j}^{\mathscr{A}}_2)\big)\circ\mathbf{i}\Big)\circ\big(\mathbf{i}^{-1}\circ\vartheta\circ\iota^{-1}\big)\Big)\big(\iota^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1)\cdot\dots\cdot\iota^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big)$$

⟦ def. of $\cdot\odot\cdot$ in eq. (3.4.75) ⟧

$$=\Big(\big((\varphi_1\circ\mathrm{j}^{\mathscr{A}}_1)\mathbin{\widetilde{\circledcirc_{\mathcal{P}}}}(\varphi_2\circ\mathrm{j}^{\mathscr{A}}_2)\big)\circ\vartheta\Big)\big(\mathbf{i}^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1)\cdot\dots\cdot\mathbf{i}^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big)$$

⟦ according to eq. (3.4.74) ι and $\mathbf{i}$ are inverse to each other ⟧

$$=\Big(\big((\varphi_1\circ\mathrm{j}^{\mathscr{A}}_1)\mathbin{\widetilde{\circledcirc_{\mathcal{P}}}}(\varphi_2\circ\mathrm{j}^{\mathscr{A}}_2)\big)\Big)\Big(\vartheta\big(\mathbf{i}^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1)\big)\cdot\dots\cdot\vartheta\big(\mathbf{i}^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big)\Big)$$

⟦ ϑ is homomorphism of algebras ⟧

$$=\Big(\big((\varphi_1\circ\mathrm{j}^{\mathscr{A}}_1)\mathbin{\widetilde{\circledcirc_{\mathcal{P}}}}(\varphi_2\circ\mathrm{j}^{\mathscr{A}}_2)\big)\Big)\Big(\mathbf{i}^{(\delta_1)}_{\mathscr{A}_{\varepsilon_1}}\big(\kappa_{\varepsilon_1}(b_1)\big)\cdot\dots\cdot\mathbf{i}^{(\delta_n)}_{\mathscr{A}_{\varepsilon_n}}\big(\kappa_{\varepsilon_n}(b_n)\big)\Big)$$

⟦ outer diagram in eq. (I) is commutative ⟧

$$=\Big(\big((\varphi_1\circ\mathrm{j}^{\mathscr{A}}_1)\mathbin{\widetilde{\circledcirc_{\mathcal{P}}}}(\varphi_2\circ\mathrm{j}^{\mathscr{A}}_2)\big)\circ\mathsf{pr}_{I^{\mathscr{A}}_{1,2}}\Big)\Big(\mathsf{inc}_{\mathscr{A}^{(\delta_1)}_{\varepsilon_1},\mathrm{T}(\mathscr{A}_{1,2})}\big(\kappa_{\varepsilon_1}(b_1)\big)\cdot\dots\cdot\mathsf{inc}_{\mathscr{A}^{(\delta_n)}_{\varepsilon_n},\mathrm{T}(\mathscr{A}_{1,2})}\big(\kappa_{\varepsilon_n}(b_n)\big)\Big)$$

$$[\![\ \mathbf{i}^{(j)}_{\mathscr{A}_i} = \mathsf{pr}_{I^{\mathscr{A}}_{1,2}} \circ \mathsf{inc}_{\mathscr{A}^{(j)}_i, \mathrm{T}(\mathscr{A}_{1,2})}, \ \mathsf{pr}_{I^{\mathscr{A}}_{1,2}} \text{ is homomorphism of algebras }]\!]$$

$$= \big((\varphi_1 \circ \mathrm{j}^{\mathscr{A}}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}^{\mathscr{A}}_2)\big)\Big(\mathsf{inc}_{\mathscr{A}^{(\delta_1)}_{\varepsilon_1}, \mathrm{T}(\mathscr{A}_{1,2})}\big(\kappa_{\varepsilon_1}(b_1)\big) \cdot \dots \cdot \mathsf{inc}_{\mathscr{A}^{(\delta_n)}_{\varepsilon_n}, \mathrm{T}(\mathscr{A}_{1,2})}\big(\kappa_{\varepsilon_n}(b_n)\big)\Big)$$

$$[\![\text{ def. } \cdot \widetilde{\circledcirc_{\mathcal{P}}} \cdot \text{ in eq. (3.4.71) }]\!]$$

$$= \big((\varphi_1 \circ \mathrm{j}^{\mathscr{A}}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}^{\mathscr{A}}_2)\big)\underbrace{\big(\kappa_{\varepsilon_1}(b_1) \otimes \dots \otimes \kappa_{\varepsilon_n}(b_n)\big)}_{\in \bigotimes_{j\in[n]} \mathscr{A}^{(\delta_j)}_{1,2} \subseteq \mathrm{T}(\mathscr{A}_{1,2})}$$

$$[\![\otimes \text{ is multiplication in } \mathrm{T}(\mathscr{A}_{1\oplus 2}) \text{ for pure tensors }]\!]$$

$$= \Big(\big((\varphi_1 \circ \mathrm{j}^{\mathscr{A}}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \mathrm{j}^{\mathscr{A}}_2)\big) \circ \mathrm{T}(\kappa_1 \oplus \kappa_2)\Big)(b_1 \otimes \dots \otimes b_n)$$

$$= \Big(\big(\varphi_1 \circ \mathrm{j}^{\mathscr{A}}_1 \circ \mathrm{T}(\kappa_1 \oplus \kappa_2)\big) \circledcirc_{\mathcal{P}} \big(\varphi_2 \circ \mathrm{j}^{\mathscr{A}}_2 \circ \mathrm{T}(\kappa_1 \oplus \kappa_2)\big)\Big)(b_1 \otimes \dots \otimes b_n)$$

$$\left[\!\!\left[\begin{array}{l} \kappa_1 \oplus \kappa_2 \in \mathrm{Lin}(\mathscr{B}_1 \oplus \mathscr{B}_2, \mathscr{A}_1 \oplus \mathscr{A}_2), \\ \text{by eq. (3.4.81) prequisites of Lem. 3.4.22 } \textbf{(c)} \text{ are fulfilled} \end{array}\right]\!\!\right]$$

$$= \big((\varphi_1 \circ \kappa_1 \circ \mathrm{j}^{\mathscr{B}}_1) \circledcirc_{\mathcal{P}} (\varphi_2 \circ \kappa_2 \circ \mathrm{j}^{\mathscr{B}}_1)\big)(b_1 \otimes \dots \otimes b_n)$$

$$\left[\!\!\left[\begin{array}{l} \text{it suffices to show } (\mathrm{j}^{\mathscr{A}}_i \circ \mathrm{T}(\kappa_1 \oplus \kappa_2))(b) = (\kappa_i \circ \mathrm{j}^{\mathscr{B}}_i)(b) \text{ for } b \in \mathscr{B}^{(j)}_1 \oplus \mathscr{B}^{(j)}_2, \\ \text{because all maps are morphisms of algebras and in particular linear,} \\ \mathrm{T}(\mathscr{B}_{1\oplus 2}) = \mathrm{T}(\bigoplus_{j=1}^m (\mathscr{B}^{(j)}_1 \oplus \mathscr{B}^{(j)}_2) \text{ is generated by } \mathscr{B}^{(j)}_1 \oplus \mathscr{B}^{(j)}_2 \text{ for all } j \in [m] \\ \text{moreover use Conv. 3.4.26 and assumption from eq. (3.4.81)} \end{array}\right]\!\!\right]$$

$$= \Big(\big((\varphi_1 \circ \kappa_1 \circ \mathrm{j}^{\mathscr{B}}_1) \widetilde{\circledcirc_{\mathcal{P}}} (\varphi_2 \circ \kappa_2 \circ \mathrm{j}^{\mathscr{B}}_2)\big) \circ \mathbf{i}\Big)\big(\iota^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1) \cdot \dots \cdot \iota^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big)$$

$$[\![\text{ property of lifted map in eq. (3.4.71) }]\!]$$

$$= \big((\varphi_1 \circ \kappa_1) \odot (\varphi_2 \circ \kappa_2)\big)\big(\iota^{(\delta_1)}_{\mathscr{B}_{\varepsilon_1}}(b_1) \cdot \dots \cdot \iota^{(\delta_n)}_{\mathscr{B}_{\varepsilon_n}}(b_n)\big) \quad [\![\text{ def. of } \varphi_1 \odot_{\mathcal{P}} \varphi_2 \text{ in eq. (3.4.75) }]\!]$$

Since the algebra $\mathscr{B}_1 \sqcup \mathscr{B}_2$ is generated by elements from $\bigcup_{i\in[2]} \bigcup_{j\in[m]} \iota^{(j)}_{\mathscr{B}_i}(\mathscr{B}^{(j)}_i)$ and we apply linear maps to such elements, the assertion follows from the above calculation.

Ad **(d)**: There is not much difference to the proof of Theorem 3.3.9 **(d)**. Thus, formally the same steps can be done. The only difference is to take care of the color index. Define $f^{(j)}\colon \mathscr{A}^{(j)}_1 \oplus \mathscr{A}^{(j)}_2 \longrightarrow \mathscr{A}^{(j)}_2 \oplus \mathscr{A}^{(j)}_1$ in the canonical way. Then $\mathrm{T}(\bigoplus_{j\in[m]} f^{(j)})\colon \mathrm{T}(\mathscr{A}_{1\oplus 2}) \longrightarrow \mathrm{T}(\mathscr{A}_{2\oplus 1})$ and $f^{(j)}(\mathscr{A}^{(j)}_1 \oplus \mathscr{A}^{(j)}_2) \subseteq \mathscr{A}^{(j)}_2 \oplus \mathscr{A}^{(j)}_1$ and therefore Lemma 3.4.22 **(c)** can be applied.

Ad **(e)**: Since the definition of the universal product $\odot_{\mathcal{P}}$ for an m-colored universal class of Partitions $\mathcal{P}$ formally looks the same as in the case for one color, we just refer to the proof of Theorem 3.3.9 **(e)** and notice that we have to replace in particular

- $\varepsilon \in [k]^{\times n}$ by $\varepsilon = (\varepsilon_{i,1}, \varepsilon_{i,2})_{i\in[n]} \in ([k] \times [m])^{\times n}$, this means $\iota_{\varepsilon_1}(a_1) \cdot \dots \cdot \iota_{\varepsilon_n}(a_n)$ is replaced by $\iota^{(\varepsilon_{2,1})}_{\varepsilon_{1,1}}(a_1) \cdot \dots \cdot \iota^{(\varepsilon_{n,2})}_{\varepsilon_{n,1}}(a_n)$. Here $\iota^{(i)}_j\colon \mathscr{A}^{(i)}_j \hookrightarrow \bigsqcup_{j=1}^2 \bigsqcup_{i=1}^m \mathscr{A}^{(i)}_j$ denotes the canonical homomorphic insertion map,
- the usage of Lemma 3.2.16 by Lemma 3.4.25.

□

3.4.33 Definition (Partition induced universal product). Let $m \in \mathbb{N}$. Let $\odot$ be a symmetric u.a.u.-product with right-ordered monomials property in the category $\mathsf{AlgP_m}$. We say that $\odot$ is a *partition induced universal product* (for $\mathcal{P}$) if and only if there exists some m-colored universal class of partitions $\mathcal{P}$ such that $\odot = \odot_{\mathcal{P}}$.

Chapter 4

On the classification of universal classes of partitions

In the preceding chapter we have seen that universal classes of partitions have the potential to define a partition induced universal product. This insight is the legitimization to look for concrete examples of universal classes of partitions to obtain possible new examples for symmetric u.a.u.-products. In the single- and two-colored case we were able to classify universal classes of partitions. This classification turns out to be fruitful once we have shown that any positive and symmetric u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$ induces an m-colored universal class of partitions. We do this in Chapter 5.

4.1 Classification of universal classes of partitions: single-colored case

4.1.1 Convention.

(a) Let X be a set, then we denote generic projection maps by

$$\forall i \in [2]\colon\ \mathsf{pr}_i\colon \begin{cases} X \times X \longrightarrow X \\ (x_1, x_2) \longmapsto x_i. \end{cases} \tag{4.1.1}$$

(b) We set for any $k \in \mathbb{N}$ and any universal class of partitions $\mathcal{P} \subseteq \mathsf{Part}$

$$\mathcal{P}_{\cdot\,,k} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_{n,k} \tag{4.1.2}$$

Thus, we have a distinction to the set of all partitions in $\mathcal{P}$ with n-legs denoted by $\mathcal{P}_n$.

(c) Let $\pi \in \mathsf{Part}$. Then, this partition has $n \in \mathbb{N}$ legs and $k \in \mathbb{N}$ blocks. Our maps delete, double, UMem and split carry this information about the number of legs and blocks in form of indices attached to the operator names. Sometimes we will just put a dot $\cdot$ instead of the concrete value for n and k, but the concrete values should be clear from the context whenever such a map is applied to a certain partition. As an example, let $\ell \in \{0, \ldots, n-2\}$, then we put

$$\mathsf{UMem}^{\ell+1}_{\cdot\,,\cdot}(\pi) = \mathsf{UMem}^{\ell+1}_{\cdot\,,k}(\pi) := \mathsf{UMem}^{\ell+1}_{n,k}(\pi) \tag{4.1.3}$$

We could have also introduced new maps without these indices attached to the operator names but we leave it like this to be more flexible whenever we want to highlight a dependence on the leg index or the block index. On the other hand, sometimes we will completely hide any dependence of indices of the above mentioned maps. Then, we understand such occurrences of the above mentioned maps without indices as a placeholder of the same concrete map with any valid index choice.

4.1.2 Definition. We want to recursively define a map $\mathsf{GenerateTwoBlocks}$ which assigns to each partition $\pi \in \mathsf{Part}$ with k blocks for some $k \in \mathbb{N} \setminus \{1\}$ a $(k-1)$-tuple of two-block partitions. Let $k = 2$ and $\pi \in \mathsf{Part}_{\cdot,k}$, then set

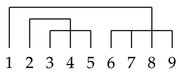
$$\mathsf{GenerateTwoBlocks}(\pi) = (\pi). \tag{4.1.4}$$

Now, assume $k \in \mathbb{N} \setminus \{2\}$ and $\mathsf{GenerateTwoBlocks}$ has been defined for each partition $\pi \in \mathsf{Part}_{\cdot,k-1}$. Let $\pi \in \mathsf{Part}_{\cdot,k}$. We can naturally order the blocks of π by their smallest legs in the blocks, i. e., for two blocks $b,\ b' \in \pi$ we can put $b < b'$ if and only if $\lambda < \lambda'$, where $\lambda = \min(\mathsf{set}\, b)$ and $\lambda' = \min(\mathsf{set}\, b')$. With respect to this order we have a first block $b_1 \in \pi$ and a second block $b_2 \in \pi$. In the first block b_1 there exists a leg $\ell + 1 \in \mathsf{set}\, b_1$, such that $\ell + 2 = \min(\mathsf{set}\, b_2)$. Then, the ordered tuple

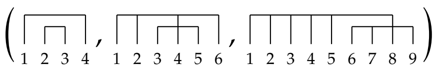
$$\Big((\mathsf{pr}_2 \circ \mathsf{UMem}^{\ell+1}_{\cdot,k})(\pi), \mathsf{GenerateTwoBlocks}\big((\mathsf{pr}_1 \circ \mathsf{UMem}^{\ell+1}_{\cdot,k})(\pi)\big) \Big) \tag{4.1.5}$$

defines a $(k-1)$-tuple of two-block partitions, where pr_1 denotes the projection to the first component of the pair and pr_2 denotes the projection to the second component of the pair.

4.1.3 Example. Let $\mathcal{P}$ be a universal class of partitions. Assume

$\pi =$  $\in \mathcal{P}_{9,4}$. (4.1.6)

Then, $\mathsf{GenerateTwoBlocks}(\pi)$ leads to

 $\in \mathcal{P}_{4,2} \times \mathcal{P}_{6,2} \times \mathcal{P}_{9,2}$. (4.1.7)

4.1.4 Lemma.

(a) For any $k \in \mathbb{N}$ the map $\mathsf{GenerateTwoBlocks}\colon \mathsf{Part}_{\cdot,k} \longrightarrow (\mathsf{Part}_{\cdot,2})^{\times(k-1)}$ is injective.

(b) Let $\mathcal{P} \subseteq \mathsf{Part}$ be a universal class of partitions, then $\mathsf{GenerateTwoBlocks}\restriction_{\mathcal{P}}\colon \mathcal{P} \longrightarrow (\mathcal{P}_{\cdot,2})^{\times(k-1)}$.

Proof: Ad **(a)**: We omit this direct but technical proof. We notice that we can "reconstruct" a partition $\pi \in \mathsf{Part}_{\cdot,k}$ from $\mathsf{GenerateTwoBlocks}(\pi)$, in other words give a left inverse to $\mathsf{GenerateTwoBlocks}$ by a successive application of the map split.

Ad **(b)**: This follows from the definition of the map $\mathsf{GenerateTwoBlocks}$ and the fact that $\mathcal{P}$ is a universal class of partitions and therefore needs to satisfy Definition 3.1.9 **(d)**. □

4.1.5 Proposition. Let $\mathcal{P}$ and $\mathcal{R}$ be two single-colored universal classes of partitions. Denote by $\mathcal{P}_{\cdot,2}$ the set of all two-block partitions of $\mathcal{P}$ respectively by $\mathcal{R}_{\cdot,2}$ the set of all two-block partitions of $\mathcal{R}$.

Tfae: **(a)** $\mathcal{P} = \mathcal{R}$,

(b) $\mathcal{P}_{\cdot,2} = \mathcal{R}_{\cdot,2}$, i. e., all the two-block partitions are the same.

Proof: **(a) $\Longrightarrow$ (b)** Since $\mathcal{P}_{\cdot,2} \subseteq \mathcal{P}$ and $\mathcal{R}_{\cdot,2} \subseteq \mathcal{R}$, the assertion follows directly from the assumption.
(b) $\Longrightarrow$ (a) We want to show that $\mathcal{R} \subseteq \mathcal{P}$. Therefore, let $\pi \in \mathcal{R}$ be any partition with k-blocks, i.e. $\pi \in \mathcal{R}_{\cdot,k}$ for some $k \in \mathbb{N} \setminus \{1,2\}$. Since the map GenerateTwoBlocks is injective by Lemma 4.1.4, we can uniquely associate a partition π to a $(k-1)$-tuple of two-block partitions from $\mathcal{R}$. Since $\mathcal{R}_{\cdot,2} = \mathcal{P}_{\cdot,2}$ and since the map GenerateTwoBlocks as an injection has a left inverse, we obtain that $\pi \in \mathcal{P}$. A similar proof holds for the equation $\mathcal{P} \subseteq \mathcal{R}$. □

Proposition 4.1.5 is our main tool to perform the classification in the single-colored case.

4.1.6 Definition (Crossing partition). For any partition $\pi \in \mathsf{Part}$ we set

$$\mathsf{Cross}(\pi) := \left\{ \big((p_1,p_2),(q_1,q_2)\big) \in \mathbb{N}^{\times 2} \times \mathbb{N}^{\times 2} \;\middle|\; \begin{array}{l} p_1 < q_1 < p_2 < q_2,\ \exists b,\, b' \in \pi\colon \\ b \neq b',\, \{p_1,p_2\} \subseteq \mathsf{set}\, b, \\ \{q_1,q_2\} \subseteq \mathsf{set}\, b' \end{array} \right\} \tag{4.1.8}$$

Elements of the set $\mathsf{Cross}(\pi)$ are called *crossings*. For a partition $\pi \in \mathsf{Part}$ we say it is a *partition with crossing* or just *crossing* if and only if $\mathsf{Cross}(\pi) \neq \emptyset$.

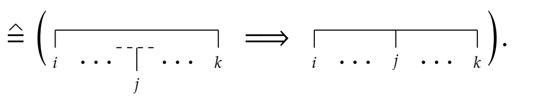
4.1.7 Definition.

(a) A partition $\pi \in \mathsf{Part}$ is said to be *one block* if and only if $|\pi| = 1$. The set of all partitions which are one block is denoted by $\mathsf{1B}$.

(b) A partition $\pi \in \mathsf{Part}_n$ is said to be *interval* for some $n \in \mathbb{N}$ if and only if

$\forall j \in [n],\ \forall i \in [j-1],\ \forall k \in [n] \setminus [j],\ \forall b \in \pi\colon$

$$\Big(i,\ k \in \mathsf{set}(b) \implies j \in \mathsf{set}(b)\Big) \tag{4.1.9a}$$



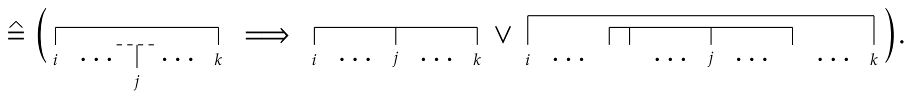
(4.1.9b)

The set of all interval partitions is denoted by I.

(c) A partition $\pi \in \mathsf{Part}_n$ is said to be *noncrossing* for some $n \in \mathbb{N}$ if and only if

$\forall j \in [n],\ \forall i \in [j-1],\ \forall k \in [n] \setminus [j],\ \forall b \in \pi\colon$

$$\Big(i,\ k \in \mathsf{set}\, b \implies \big(j \in \mathsf{set}\, b\big) \vee \big(\exists b' \in \pi \setminus \{b\}\colon j \in \mathsf{set}(b'),\ \mathsf{set}(b') \subseteq [i+1, k-1] \subseteq \mathbb{N}\big)\Big) \tag{4.1.10}$$



(4.1.11)

The set of all partitions which are noncrossing for any $n \in \mathbb{N}$ is denoted by NC.

4.1.8 Remark. It can be shown that our definition of interval and noncrossing partitions is equivalent to the one used in [Spe97].

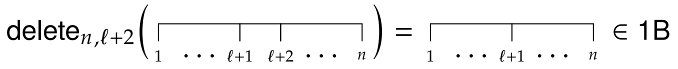
4.1.9 Lemma.

(a) The set of all one block partitions 1B is a universal class of partitions.

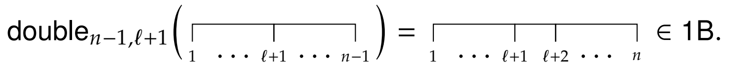
(b) The set of all interval partitions I is a universal class of partitions.

(c) The set of all noncrossing partitions NC is a universal class of partitions.

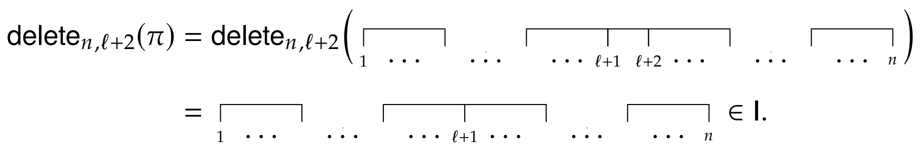
Proof: For the assertions we need to check the properties **(a)** till **(e)** of Definition 3.1.9.
Ad **(a)**: We show that 1B fulfills these axioms. Since 1B is the class of all one block partitions, this implies $\mathbb{1}_1 \in 1\mathsf{B}$. Moreover, the effect of the maps delete and double is diagrammatically represented by



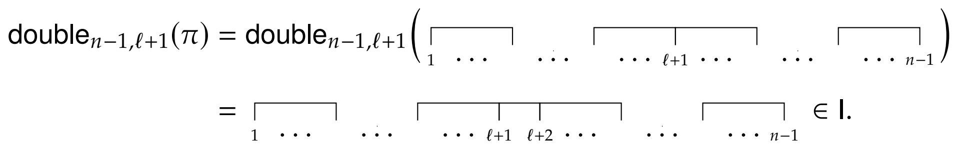
and



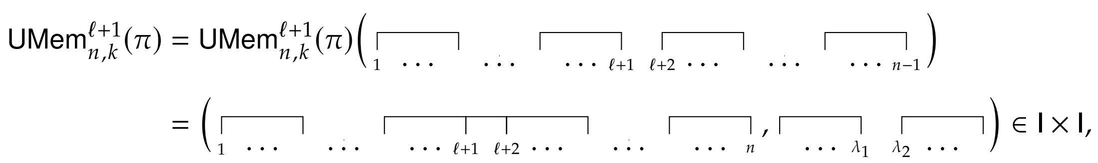
For the other properties is nothing to show, since these make statements about partitions with more than one block. But 1B only has partitions with exactly one block. Hence, 1B is a universal class of partitions.
Ad **(b)**: Of course, $\mathbb{1}_1 \in \mathsf{I}$. Let $\pi \in \mathsf{I}$ with the property that there exists a block b in π such that $\ell+1, \ell+2 \in \mathsf{set}\, b$, then we have



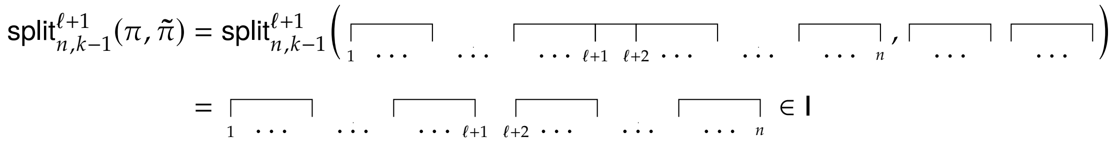
Furthermore, for any partition $\pi \in \mathsf{I}$ we have



Now, consider an arbitrary partition $\pi \in \mathsf{I}$, where $\ell+1$ and $\ell+2$ are in two different blocks of π. Then, we observe



where $\lambda_i = \mathsf{pos}_{b_i}(\ell+i)$ and b_i is the block in π such that $\ell+i \in \mathsf{set}\, b_i$ for each $i \in \{1,2\}$. On the other hand, we have for any $\pi \in \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big) \cap (\mathsf{I}\times\mathsf{I})$

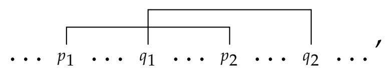


This shows that I satisfies the axioms of a universal class of partitions.

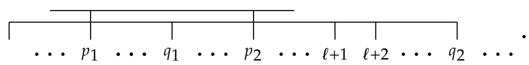
AD **(c)**: Of course, $\mathbb{1}_1 \in \mathsf{NC}$. Next, we assume that $\pi \in \cap(\mathsf{Part}_n^{(\ell+1)\wedge(\ell+2)} \cap \mathsf{NC})$ and $\pi \neq \emptyset$. It immediately follows $\mathsf{delete}_{n,\ell+2}(\pi) \in \mathsf{NC}$, since this "triple legs scenario" of equation (4.1.11) is still satisfied for the remaining legs of $\mathsf{delete}_{n,\ell+2}(\pi)$. Analogously we can show that $\mathsf{double}_{n,\ell+1}(\pi) \in \mathsf{NC}$ for such partition π.

Next, we show that $\mathsf{UMem}_{n,k}^{\ell+1}(\pi) \in \mathsf{NC}$ for any $\pi \in (\mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)} \cap \mathsf{NC})$ with $\pi \neq \emptyset$. For this, we have to show that $\mathsf{pr}_1(\mathsf{UMem}_{n,k}^{\ell+1}(\pi)) \in \mathsf{NC}$ and $\mathsf{pr}_2(\mathsf{UMem}_{n,k}^{\ell+1}(\pi)) \in \mathsf{NC}$. We show both assertions by contradiction. Assume that $\mathsf{pr}_2(\mathsf{UMem}_{n,k}^{\ell+1}(\pi)) \notin \mathsf{NC}$. By definition $\mathsf{pr}_2(\mathsf{UMem}_{n,k}^{\ell+1}(\pi))$ is a two-block partition with blocks b and b'. If this two-block partition has a crossing, then this crossing after the application of the map $\mathsf{split}_{n,k-1}^{\ell+1}$ remains. By Lemma 3.1.7 we have $\mathsf{split}_{n,k-1}^{\ell+1} \circ \mathsf{UMem}_{n,k}^{\ell+1} = \mathsf{id}_{\mathsf{sub}\left(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\right)}$. This would imply that $\pi \notin \mathsf{NC}$, but this is a contradiction to the assumption. Hence, we have shown that $\mathsf{pr}_2(\mathsf{UMem}_{n,k}^{\ell+1}(\pi)) \notin \mathsf{NC}$.

Assume now that $\mathsf{pr}_1(\mathsf{UMem}_{n,k}^{\ell+1}(\pi)) \notin \mathsf{NC}$ holds. Then, a possible crossing can only occur between the "unified block", i. e., the block such that $\ell+1$, $\ell+2$ are elements of this block, and any other block of $\mathsf{pr}_1(\mathsf{UMem}_{n,k}^{\ell+1}(\pi))$. This is justified by the following consideration. If a crossing occurs between two blocks, where none of the two blocks is the unified block, then the crossing remains after the application of $\mathsf{split}_{\cdot,\cdot}^{\ell+1}$. This would imply that $\pi \notin \mathsf{NC}$, which is a contradiction. Therefore, we can assume that the crossing occurs between two blocks, where one of the blocks is the unified block. Denote the unified block in $\mathsf{pr}_1(\mathsf{UMem}_{n,k}^{\ell+1}(\pi))$ by b and the other block in this partition, which causes the crossing, by b'. Then, there exist legs p_1, p_2 and q_1, q_2 in the partition $\mathsf{pr}_1(\mathsf{UMem}_{n,k}^{\ell+1}(\pi))$ such that



i. e. $p_1 < q_1 < p_2 < q_2$. There are two possibilities for the legs to which block they belong. Either the p-legs belong to block b and the q-legs belong to the block b' or the other way around. In the following we want to assume that the q-legs belong to block b and the p-legs to the block b'. The claim for the other possibility is analogously shown. Crucial for the proof is only the relative position of the p- and q-legs to each other, not the affiliation to a certain block. For instance, one possible case of a crossing in the partition $\mathsf{pr}_1(\mathsf{UMem}_{n,k}^{\ell+1}(\pi))$, which is caused by the unified block, is represented by the following diagram



Hence, in the unified block there exists a leg q_1, which lies between two legs p_1 and p_2 of another block which we called b'. Now, the leg q_1 either belongs to a block of π, where the leg $\ell+1$ belongs to or it belongs to the block, where the leg $\ell+2$ belongs to. Assume it belongs to the block of $\ell+1$ and call this block $\tilde{b}$. If we apply $\mathsf{split}_{n,k-1}^{\ell+1}$ to $\mathsf{UMem}_{n,k}^{\ell+1}(\pi)$ a crossing occurs between the blocks $\tilde{b}$ and b', since the unified block b just gets split into two blocks, but the relative position of the p- and q-legs is not changed by the map $\mathsf{split}_{n,k-1}^{\ell+1}$. Since $\mathsf{split}_{n,k-1}^{\ell+1} \circ \mathsf{UMem}_{n,k}^{\ell+1} = \mathsf{id}_{\mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}}$, we obtain the partition π has a crossing which is a contradiction to the assumption. All the other cases have a similar argumentation. Therefore, it must hold that $\mathsf{UMem}_{n,k}^{\ell+1}(\pi) \in (\mathsf{sub}\left(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\right) \cap (\mathsf{NC} \times \mathsf{NC}))$.

Now we want to prove that $\mathsf{split}_{n,k-1}^{\ell+1}(\pi, \tilde{\pi}) \in \mathsf{NC}$ if $(\pi, \tilde{\pi}) \in \mathsf{sub}\left(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\right)$. Assume a crossing in the partition $\mathsf{split}_{n,k-1}^{\ell+1}(\pi, \tilde{\pi})$ stems from the "insertion" of two blocks of $\tilde{\pi}$ into π. Then, this would mean $\tilde{\pi} = \mathsf{pr}_2((\mathsf{UMem}_{n,k}^{\ell+1} \circ \mathsf{split}_{n,k-1}^{\ell+1})(\pi, \tilde{\pi})) \notin \mathsf{NC}$, since the map $\mathsf{UMem}_{n,k}^{\ell+1}$ does not the change the relative position of legs to each other. But $\tilde{\pi} \notin \mathsf{NC}$ is a contradiction to the assumption. Therefore, if $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi})$ has a crossing, then it is not possible that it is caused by the two blocks of $\tilde{\pi}$. Assume $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi})$ has a crossing and is caused by two blocks, where none of the two blocks originated from the splitting or in the words: none of the two blocks in $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi})$ has the legs $\ell+1$ or $\ell+2$ as an element. Similarly, like in the previous case by application of $\mathsf{UMem}^{\ell+1}_{n,k}$ we conclude that $\pi \notin \mathsf{NC}$ which is a contradiction to the assumption. Therefore, if $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi})$ has a crossing, it is not possible that this crossing stems from the above described two blocks. Let us assume that $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi})$ has a crossing which is caused by two blocks, where exactly one of the blocks has the property $\ell+1$ or $\ell+2$ is an element in this block. Assume that the crossing appears between a block b' of $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi})$, where the leg $\ell+2$ does not belong to and a block b, where the leg $\ell+1$ belongs to. Then, there would occur a crossing between the unified block in $\pi = \mathsf{pr}_1((\mathsf{UMem}^{\ell+1}_{n,k} \circ \mathsf{split}^{\ell+1}_{n,k-1})(\pi,\tilde{\pi}))$ and the block b', which remains unchanged. But this is a contradiction to the assumption that $\pi \in \mathsf{NC}$. Therefore, it must hold that $\mathsf{split}^{\ell+1}_{n,k-1}(\pi,\tilde{\pi}) \in \mathsf{NC}$. □

4.1.10 Definition (Reduced partition, Generating set of partitions).

(a) We call a partition $\pi \in \mathsf{Part}$ *reduced* if and only if for each block b in π there do not exist any block neighboring legs (Definition 3.1.1 **(d)**) in the block b. For any given partition $\pi \in \mathsf{Part}$ there exists a unique reduced partition, denoted by $\operatorname{red}\pi \in \mathsf{Part}$ which we obtain by induction and use of the map delete (in order to "delete" all block neighboring legs of a block).

(b) Let $G \subseteq \mathsf{Part}$. Since universal classes of partitions are stable under intersection, we may define another universal class of partitions

$$\mathsf{Gen}(G) := \bigcap_{\substack{\mathcal{P}\supseteq G \\ \mathcal{P} \text{ a UCP}}} \mathcal{P}. \tag{4.1.12}$$

We also say that G *generates a partition* $\pi \in \mathsf{Part}$ if and only if $\pi \in \mathsf{Gen}(G)$. We can extend this terminology to sets of partitions $A \subseteq \mathsf{Part}$ and say that G *generates a set of partitions* A if and only if $A \subseteq \mathsf{Gen}(G)$. If G consists of only one element, i. e., $G = \{g\}$, then we write $\mathsf{Gen}(g)$ instead of $\mathsf{Gen}(\{g\})$.

4.1.11 Remark.

(a) Let $\mathcal{P} \subseteq \mathsf{Part}$ be a universal class of partitions, then for each $n \in \mathbb{N}$ we have a surjective map $\operatorname{red}(\mathcal{P}_{\cdot,2}) \longrightarrow \mathcal{P}_{\cdot,2}$, i. e., by a finite application of the map double we can start from a reduced two-block partition and can generate any two-block partition of $\mathcal{P}$.

(b) This is a rather informal discussion of the expression $\mathsf{Gen}(G)$ if $G \subseteq \mathsf{Part}$. We recursively define for all $n \in \mathbb{N}$

$$G^{(1)} := \bigcup_{\mathsf{f}\in\left\{\substack{\mathsf{delete},\,\mathsf{double},\,\mathsf{pr}_1\circ\mathsf{UMem}, \\ \mathsf{pr}_2\circ\mathsf{UMem},\,\mathsf{id}}\right\}} \operatorname{im}\big(\mathsf{f}\circ\mathsf{inc}_{G,\mathsf{Part}}\big) \cup \operatorname{im}\big(\mathsf{split}\circ\mathsf{inc}_{G\times G,\mathsf{Part}\times\mathsf{Part}}\big) \tag{4.1.13a}$$

$$G^{(n+1)} := \bigcup_{\mathsf{f}\in\left\{\substack{\mathsf{delete},\,\mathsf{double},\,\mathsf{pr}_1\circ\mathsf{UMem},\\ \mathsf{pr}_2\circ\mathsf{UMem},\,\mathsf{id}}\right\}} \mathrm{im}\big(\mathsf{f}\circ\mathsf{inc}_{G^{(n)},\mathsf{Part}}\big) \cup \mathrm{im}\big(\mathsf{split}\circ\mathsf{inc}_{G^{(n)}\times G^{(n)},\mathsf{Part}\times\mathsf{Part}}\big), \tag{4.1.13b}$$

whenever the above maps $\mathsf{delete}_{\cdot,\cdot}$, $\mathsf{double}_{\cdot,\cdot}$, $\mathsf{UMem}^{\cdot}_{\cdot,\cdot}$, $\mathsf{split}^{\cdot}_{\cdot,\cdot}$ might be well-defined for valid index choices. In this setting we can see that

$$\mathsf{Gen}(G) = \bigcup_{n\in\mathbb{N}} G^{(n)} \subseteq \mathsf{Part}\,. \tag{4.1.14}$$

(c) Let $G \subseteq \mathsf{Part}$, then $G \subseteq \mathsf{Gen}(G)$. If $G \subseteq \mathcal{P} \subseteq \mathsf{Part}$, where $\mathcal{P}$ is some universal class of partitions, then we have $\mathsf{Gen}(G) \subseteq \mathcal{P}$.

4.1.12 Lemma. Let G', $G \subseteq \mathsf{Part}$ and assume $G' \subseteq G$.

TFAE: **(a)** $\mathsf{Gen}(G') = \mathsf{Gen}(G)$,

(b) $G \subseteq \mathsf{Gen}(G')$.

PROOF: The equivalence of the assertions follows from Remark 4.1.11 **(c)**. □

4.1.13 Lemma. Let $\mathcal{P} \subseteq \mathsf{Part}$ be an universal class of partitions and assume $G \subseteq \mathcal{P}$ is given.

TFAE: **(a)** $\mathsf{Gen}(G) = \mathcal{P}$,

(b) $\mathrm{red}\big(\mathcal{P}_{\cdot,2}\big) \subseteq \mathsf{Gen}(G)$, i. e., G generates all reduced two-block partitions of $\mathcal{P}$.

PROOF: The equivalence of both assertions follows from Proposition 4.1.5, Remark 4.1.11 **(c)** and **(a)**. □

4.1.14 Lemma. It holds $\mathsf{Gen}(\uparrow\,\uparrow) = \mathsf{I}$.

PROOF: Obviously $\mathrm{red}(\mathsf{I}_{\cdot,2}) = \{\uparrow\,\uparrow\}$ and now the assertion follows from Lemma 4.1.13. □

4.1.15 Lemma.

(a) $\mathsf{Gen}\big(\sqcap\!\!\!|\,\big) = \mathsf{Gen}\big(\{\uparrow\,\uparrow, \sqcap\!\!\!|\,\}\big)$.

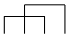
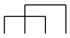
(b) $\mathsf{Gen}\big(\{\uparrow\,\uparrow, \sqcap\!\!\!|\,\}\big) = \mathsf{NC}$.

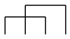
PROOF: AD **(a)**: Set $\mathcal{G} := \mathsf{Gen}\big(\sqcap\!\!\!|\,\big)$. If we can show $\sqcap\!\!\!|\, \in \mathcal{G} \implies \uparrow\,\uparrow \in \mathcal{G}$, then the assertion follows from Lemma 4.1.12. We can calculate

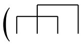
$$\begin{aligned}
&\pi = \sqcap\!\!\!| \in \mathcal{G} \\
\implies\ &\pi_1 = \sqcap\!\!\!|\!\sqcap \in \mathcal{G} \quad [\![\, \pi_1 = \mathsf{double}_{3,3}(\pi) \,]\!] \\
\implies\ &\pi_2 = \sqcap\!\!\!|\,|\!\!\sqcap \in \mathcal{G} \quad [\![\, \pi_2 = \mathsf{split}^3_{4,2}(\pi_1, \pi) \,]\!] \\
\implies\ &\pi_3 = \uparrow\,\uparrow \in \mathcal{G} \quad [\![\, \pi_3 = (\mathsf{pr}_2\circ\mathsf{UMem}^2_{4,3})(\pi_2) \,]\!].
\end{aligned}$$

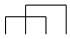
AD **(b)**: Since $\mathrm{red}(\mathsf{NC}_{\cdot,2}) = \{\uparrow\,\uparrow, \sqcap\!\!\!|\,\}$, the assertion follows from Lemma 4.1.13. □

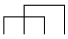
4.1.16 Lemma.

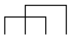
(a) Gen() = Gen({, , }).

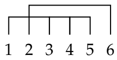
(b) Gen({, , }) = Part.

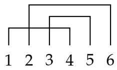
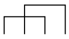
PROOF: AD **(a)**: We set $\mathcal{G} :=$ Gen(). By Lemma 4.1.12 it suffices to show

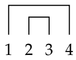
{, , } $\subseteq \mathcal{G}$.

We show  $\in \mathcal{G} \implies$  $\in \mathcal{G}$. For this, consider the following calculation

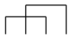
$\pi =$  $\in \mathcal{G}$

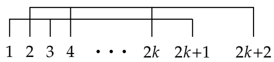
$\implies \pi_1 :=$  $\in \mathcal{G}$ $[\![\, \pi_1 = \mathsf{double}_{\cdot,3}(\pi) \,]\!]$

$\implies \pi_2 :=$  $\in \mathcal{G}$ $[\![\, \pi_2 = \mathsf{split}^3_{6,2}(\pi_1,$ $) \,]\!]$

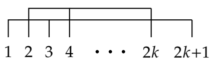
$\implies \pi_3 :=$  $\in \mathcal{G}$ $[\![\, \pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^2_{6,3})(\pi_2) \,]\!]$

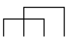
$\implies \pi_4 :=$  $\in \mathcal{G}$ $[\![\, \pi_4 = \mathsf{delete}_{4,3}(\pi_3) \,]\!]$.

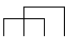
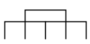
From Lemma 4.1.15 **(a)** follows  $\in \mathcal{G} \implies$  $\in \mathcal{G}$.
AD **(b)**: Put $\mathcal{G} :=$ Gen({, , }). We can determine that {, } is the set of reduced and noncrossing two-block partitions of Part. We need to determine the set of reduced two-block partitions of Part which have at least one crossing. Denote the set of such partitions by X. We can see that the following two types of such partitions exist. Let $\pi \in X$, then either there exists $k \in \mathbb{N}$ such that

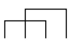
$\pi =$  (I)

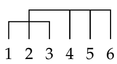
or there exists a $k \in \mathbb{N} \setminus \{1\}$ such that

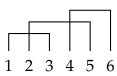
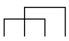
$\pi =$  . (II)

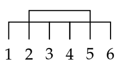
We want to regard  as a partition of type from equation (I) for $k = 1$. Hence, partitions of the type from equation (II) are characterized by the property that the first and the last leg are in one block. Partitions of the type from equation (I) are characterized by the property that the first and the last leg are not in one block.

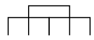
We need to show that $\forall k \in \mathbb{N}$ partitions $\pi \in \mathsf{Part}_{\cdot,2}$ of type from equation (I) and equation (II) are in $\mathcal{G}$. We prove this claim by induction over $k \in \mathbb{N}$. For the proof for the induction base $k = 1$ for partitions of type from equation (II) we notice the following. We need to show  $\in \mathcal{G} \implies$  $\in \mathcal{G}$. This is shown by the following calculation

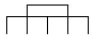
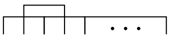
$\pi =$  $\in \mathcal{G}$

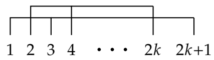
$\implies \pi_1 :=$  $\in \mathcal{G}$ $[\![\, \pi_1 = (\mathsf{double}_{\cdot,\cdot})^2(\pi) \,]\!]$

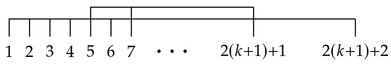
$\implies \pi_2 :=$  $\in \mathcal{G}$ $[\![\, \pi_2 = \mathsf{split}^4_{6,2}(\pi_1,$ $) \,]\!]$

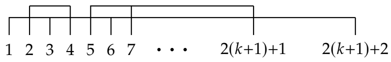
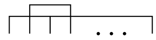
$\implies \pi_3 :=$  $\in \mathcal{G}$ $[\![\, \pi_3 = (\mathsf{pr}_1 \circ \mathsf{UMem}^3_{6,3})(\pi_2) \,]\!]$

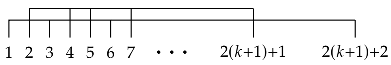
$$\Longrightarrow \pi_4 :=$$  $$\in \mathcal{G} \qquad [\![\, \pi_4 = \mathsf{delete}_{6,4}(\pi_3) \,]\!].$$

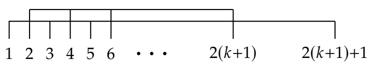
From the above we can conclude by a successive application of $\mathsf{double}_{\cdot,\cdot}$ to  that partitions of the form like  are an element of $\mathcal{G}$. We omit the proof for the induction base for partitions of type from equation (I) and just proof the induction step $k \to k+1$ for partitions of type from equation (II). Let us assume that $\pi \in X$ and π is of type from equation (II) and the above assertion holds for some $k \in \mathbb{N} \setminus \{1\}$. Then, we calculate

$$\pi =$$  $$\in \mathcal{G} \qquad [\![\, \text{IH} \,]\!]$$

$$\Longrightarrow \pi_1 :=$$  $$\in \mathcal{G} \quad [\![\, \pi_1 = (\mathsf{double}_{\cdot,1})^3(\pi) \,]\!]$$

$$\Longrightarrow \pi_2 :=$$  $$\in \mathcal{G} \quad [\![\, \pi_2 = \mathsf{split}^3_{\cdot,2}(\pi_1,$$  $$) \,]\!]$$

$$\Longrightarrow \pi_3 :=$$  $$\in \mathcal{G} \quad [\![\, \pi_3 = (\mathsf{pr}_1 \circ \mathsf{UMem}^4_{\cdot,3})(\pi_2) \,]\!]$$

$$\Longrightarrow \pi_4 :=$$  $$\in \mathcal{G} \quad [\![\, \pi_4 = \mathsf{delete}_{\cdot,5}(\pi_3) \,]\!].$$

The proof for partitions $\pi \in X$ which are of type from equation (I) is proven analogously. The only difference is that there exists an additional leg in the higher block according to the diagrams from above, which denotes the last leg in the partition. □

4.1.17 Theorem. Let $m = 1$ and $\mathcal{P}$ be a universal class of partitions. Then, the only possible cases for $\mathcal{P}$ are

- the u.c.p. of one block partitions 1B,
- the u.c.p. of interval partitions I,
- the u.c.p. of noncrossing partitions NC,
- the u.c.p. of all partitions Part.

These universal classes of partitions satisfy

$$\mathsf{1B} \subsetneq \mathsf{I} \subsetneq \mathsf{NC} \subsetneq \mathsf{Part}\,. \tag{4.1.15}$$

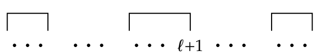
Proof: We go through all the possible cases. We start with the following claim.

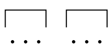
Claim 1. Assume that $\mathcal{P}$ is a universal class of partitions and $\mathcal{P} \subseteq \mathsf{1B}$ and $\mathcal{P} \neq \emptyset$, then $\mathcal{P} = \mathsf{1B}$.

Since $\mathcal{P}$ is a universal class of partitions we have that the reduced one-block part $\mathbb{1}_1 = | \in \mathcal{P}$. But $\mathsf{Gen}(|) = \mathsf{1B}$ and now the result follows from Remark 4.1.11 **(c)**.

Claim 2. Assume that $\mathcal{P}$ is universal class of partitions such that $\mathcal{P} \subseteq \mathsf{I}$ and $\mathcal{P} \setminus \mathsf{1B} \neq \emptyset$, then $\mathcal{P} = \mathsf{I}$.

Let π denote a partition such that $\pi \in \mathcal{P} \setminus \mathsf{1B} \subseteq \mathsf{I}$. Then, there needs to exist a $k \in \mathbb{N} \setminus \{1\}$ such that $\pi \in \mathcal{P}_{\cdot,k}$, i. e., π has k blocks as an interval partition. Now, we can calculate with the existence of $\ell \in \{0, \ldots, n-2\}$ such that

$$\pi =$$  $$\in \mathcal{P}$$

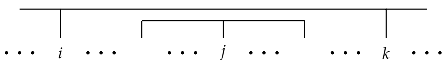
$$\Longrightarrow (\mathsf{pr}_2 \circ \mathsf{UMem}^{\ell+1}_{n,k})(\pi) =$$  $$\in \mathcal{P} \quad [\![\, \mathcal{P} \text{ is UCP} \,]\!]$$

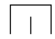
$$\Longrightarrow \; |\;| \in \mathcal{P} \quad [\![\, \text{successive application of } \mathsf{delete}_{\cdot,\cdot},\, \mathcal{P} \text{ is UCP} \,]\!]$$

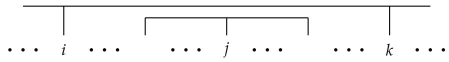
From Remark 4.1.11 **(c)** we obtain Gen() $\subseteq \mathcal{P}$. In Lemma 4.1.14 we have stated that Gen() $=$ I and therefore I $\subseteq \mathcal{P}$, which shows Claim 2.

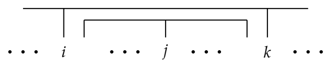
Claim 3. Assume that $\mathcal{P}$ is universal class of partitions, $\mathcal{P} \subseteq$ NC and $\mathcal{P} \setminus$ I $\neq \emptyset$, then $\mathcal{P} =$ NC.

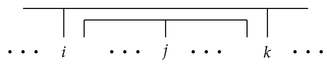
Let π denote a partition such that $\pi \in \mathcal{P} \setminus$ I $\subseteq$ NC. Since π is not an interval partition, but is noncrossing, we can see from equation (4.1.11) that there must exist a so-called "nesting" for π, i. e.,

$\pi =$ 

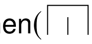
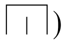
Now, we show if π has such a nesting, then this implies  $\in \mathcal{P}$. We calculate

$\pi =$  $\in \mathcal{P}$

$\Longrightarrow \pi_1 =$  $\in \mathcal{P}$ 〚 successive application of $\mathsf{pr}_1 \circ \mathsf{UMem}^{\cdot}_{\cdot,\cdot}$ 〛

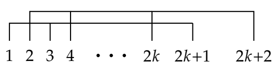
$\Longrightarrow \pi_2 =$  $\in \mathcal{P}_{\cdot,2}$ 〚 apply $\mathsf{pr}_2 \circ \mathsf{UMem}^{i}_{\cdot,\cdot}$ to π_1 〛

$\Longrightarrow \pi_3 =$  $\in \mathcal{P}$ 〚 successive application of $\mathsf{delete}_{\cdot,\cdot}$ 〛.

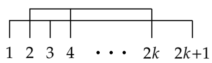
From Remark 4.1.11 **(c)** we obtain Gen() $\subseteq \mathcal{P}$. In Lemma 4.1.15 we have stated that Gen() $=$ NC and therefore NC $\subseteq \mathcal{P}$ which shows Claim 3.

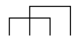
Claim 4. Assume that $\mathcal{P}$ is universal class of partitions and $\mathcal{P} \subseteq$ Part and $\mathcal{P} \setminus$ NC $\neq \emptyset$, then $\mathcal{P} =$ Part.

Let $\pi \in \mathcal{P}$ denote a partition from the above prerequisite such that $\pi \in \mathcal{P} \setminus$ NC. Then, π needs to have a crossing, since $\pi \notin$ NC. We show for such $\pi \in \mathcal{P}$ that this implies the existence of a partition $\tilde{\pi} \in \mathcal{P}$ of the following two types. Either there exists some $k \in \mathbb{N}$ such that

$\tilde{\pi} =$  $\in \mathcal{P}$ (I)

or there exists some $k \in \mathbb{N} \setminus \{1\}$ such that

$\tilde{\pi} =$  $\in \mathcal{P}$. (II)

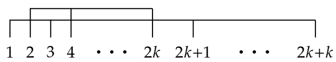
We want to regard  as a partition of type from equation (I) for $k = 1$. We want to describe a so-called "unification procedure" which is build upon a finite number of operations within the universal class of partitions $\mathcal{P}$. Whenever this unification procedure is applied to a partition π with a crossing it gives us the existence of a partition $\tilde{\pi} \in \mathcal{P}$ which is either type from equation (II) or from equation (I). In fact behind this unification procedure is nothing but a recursive definition of a certain map, but we prefer a somehow algorithmic or informal description of this recursive map to better illustrate its behavior.

Assume that a partition $\pi \in \mathcal{P}$ has a crossing. Then, according to equation (4.1.8) the set Cross(π) is not empty. Hence, there exist $\big((p_1,p_2),(q_1,q_2)\big) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $\big((p_1,p_2),(q_1,q_2)\big) \in$ Cross(π). This implies the existence of two blocks b_p and b_q such that b_p, $b_q \in \pi$ and $\{p_1,p_2\} \subseteq$ set(b_p) and $\{q_1,q_2\} \subseteq$ set(b_q). If the partition π has only two blocks, then $\tilde{\pi} :=$ red π gives the desired result. If the partition π has more than two blocks, then proceed as follows. We have an order for the blocks of π due to the description of this order in Definition 4.1.2. We are able to unify the first and the second block in the partition π by UMem. If the block b_p has been unified and all block neighboring legs have been deleted by delete in $(\mathsf{pr}_1 \circ \mathsf{UMem})(\pi)$, then address this block by b_p and set $\pi := \mathrm{red}(\pi)$. The same holds for the block b_q. We repeat this procedure until either b_p is the first block of π and the second leg of π is the minimal leg of b_q or either b_q is the first block π and the second leg of π is the minimal leg of b_p. The procedure needs to terminate since in each step we reduce the amount of blocks in π. After this procedure has terminated we unify the first and the second leg in the partition red π and then we put $\tilde{\pi} := (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,\cdot})(\pi)$. We want to refer to this procedure as the *unification procedure*. Also compare this unification procedure against the map GenerateTwoBlocks defined in Definition 4.1.2. We end up with a partition $\tilde{\pi} \in \mathcal{P}$ which has two blocks and has at least one crossing and which is either of type from equation (II) or equation (I).

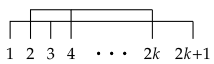
Now, let $\tilde{\mathcal{P}}$ denote the set of all two-block partitions $\pi \in \mathcal{P}_{\cdot,2}$ such that π is of type from equation (II) or equation (I) for some $k \in \mathbb{N}$. So far we have shown

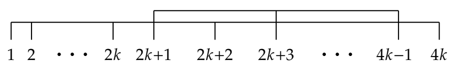
$$\mathcal{P} \setminus \mathsf{NC} \neq \emptyset \implies \widetilde{\mathcal{P}} \neq \emptyset.$$

We want to show $\widetilde{\mathcal{P}} \neq \emptyset \implies$  $\in \mathcal{P}$. Let $\pi \in \tilde{\mathcal{P}}$ and assume it is of type from equation (II) for some $k \in \mathbb{N} \setminus \{1\}$. By a $(k-1)$-times application of $\mathsf{double}_{\cdot,\cdot}$ to π, we obtain that

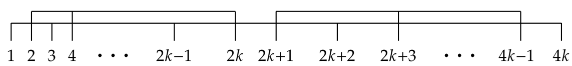
$\pi_1 :=$  $\in \mathcal{P}.$

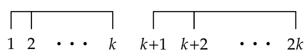
By this we calculate

$\pi =$  $\in \mathcal{P}$

$\implies \pi_2 :=$  $\in \mathcal{P}$

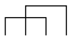
$[\![\, \pi_2 \mathrel{\hat{=}} (2k-1)\text{-application of } \mathsf{double}_{\cdot,1} \text{ to } \pi \,]\!]$

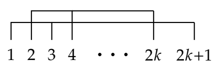
$\implies \pi_3 :=$  $\in \mathcal{P}$ $[\![\, \pi_3 = \mathsf{split}^1_{4k,2}(\pi_2, \pi_1) \,]\!]$

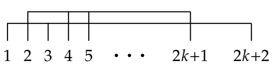
$\implies \pi_4 :=$  $[\![\, \pi_4 = (\mathsf{pr}_2 \circ \mathsf{UMem}^{2k}_{4k,3})(\pi_3) \,]\!]$

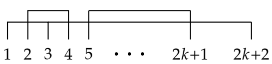
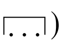
$\implies$  $\in \mathcal{P}$ $[\![$ successive application of $\mathsf{delete}_{\cdot,\cdot}$ to π_4 $]\!]$.

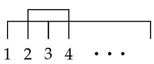
There is an analogous proof in the case that π is of type from equation (I).

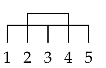
Next, we want to show $\tilde{\mathcal{P}} \neq \emptyset \implies$  $\in \mathcal{P}$. For this, consider the following calculation for a partition $\pi \in \tilde{\mathcal{P}}$ of type from equation (II).

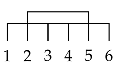
$\pi =$  $\in \mathcal{P}$

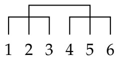
$\implies \pi_1 :=$  $\in \mathcal{P}$ $[\![\, \pi_1 = \mathsf{double}_{2k,4}(\pi) \,]\!]$

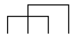
$\implies \pi_2 :=$  $\in \mathcal{P}$ $[\![\, \pi_2 = \mathsf{split}^4_{2k+2,2}(\pi_1,$ $) \,]\!]$

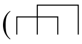
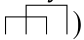
$\implies \pi_3 :=$  $\in \mathcal{P}$ $[\![\, \pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{2k+2,2})(\pi_2) \,]\!]$

$\implies \pi_4 :=$  $\in \mathcal{P}$ $[\![$ successive application of $\mathsf{delete}_{\cdot,\cdot}$ to π_3 $]\!]$

$\implies \pi_5 :=$  $\in \mathcal{P}$ $[\![\, \pi_5 = \mathsf{double}_{5,3}(\pi_4) \,]\!]$

$\implies \pi_6 :=$  $\in \mathcal{P}$ $[\![\, \pi_6 = \mathsf{split}^3_{6,2}(\pi_5,$  $) \,]\!]$

$\implies$  $\in \mathcal{P}$ $[\![\, \pi_6 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{6,3})(\pi_5) \,]\!]$.

Thus, we have shown that  $\in \mathcal{P}$ and by Remark 4.1.11 **(c)** we obtain $\mathsf{Gen}($$) \subseteq \mathcal{P}$. In Lemma 4.1.16 we have shown that $\mathsf{Gen}($$) = \mathsf{Part}$ and therefore $\mathsf{Part} \subseteq \mathcal{P}$ and this finishes the proof of Claim 4. □

4.1.18 Lemma. Let $\mathcal{P}$ be a Universal Class of Partitions and $n \in \mathbb{N}$, then $\mathcal{P}_n$ is a complete lattice by reversed refinement.

Proof: If $\mathcal{P} = \mathsf{NC}$, then we refer for the proof to [NS06, Prop. 9.17]. For the other cases of $\mathcal{P}$ the proof is similar and can also be compared with the proof of Lemma 4.2.48 in the case of a two-colored universal class of partitions. □

4.2 Classification of universal classes of partitions: two-colored case

4.2.1 Convention.

(a) We set for any $k \in \mathbb{N}$, $m \in \mathbb{N}$ and any m-colored universal class of partitions $\mathcal{P}$

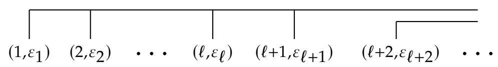
$$\mathcal{P}_{\cdot,k} := \bigcup_{n\in\mathbb{N}} \bigcup_{\varepsilon\in[m]^{\times n}} \mathcal{P}_{\varepsilon,k}. \tag{4.2.1}$$

(b) Let $m \in \mathbb{N}$. We extend the Convention 4.1.1 **(c)** for the single-colored case to the m-colored case. Furthermore, it can occur that the maps mirror and cCol can be applied to partitions where the maps do not carry any indices or just have some placeholders attached to them. Which indices need to be chosen is determined by the partition to which we apply these maps.

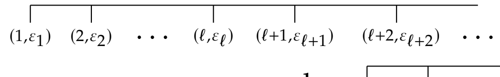
4.2.2 Definition. Let $m \in \mathbb{N}$ and $\varepsilon \in [m]^{\times n}$. In Definition 4.1.2 we have defined a natural order between blocks of a single-colored partition and we can do this similarly in the multi-colored case. Let $\pi \in \mathcal{P}_\varepsilon$. For two blocks $b, b' \in \pi$ we can put $b < b'$ if and only if $\lambda < \lambda'$, where $\lambda = \min\big(\mathsf{set}\big(\mathsf{type}(b)\big)\big)$ and $\lambda' = \min\big(\mathsf{set}\big(\mathsf{type}(b')\big)\big)$. With respect to this order we have a first block $b_1 \in \pi$ and a second block $b_2 \in \pi$. In the first block b_1 there exists a leg $\ell+1 \in \mathsf{set}\big(\mathsf{type}(b_1)\big)$ such that $\ell + 2 = \min\big(\mathsf{set}\big(\mathsf{type}(b_2)\big)\big)$. By these notations we put

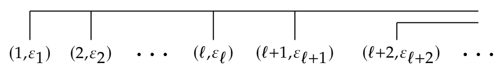
$$\mathsf{BorderLeg}(\pi) := \ell + 1 \tag{4.2.2}$$

4.2.3 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $\pi \in \mathcal{P}_{\varepsilon,k}$ for some $\varepsilon \in [m]^{\times n}$ and $k \in \mathbb{N} \setminus \{1\}$. If $\ell + 1 := \mathsf{BorderLeg}(\pi)$ and

$\pi =$  $\in \mathcal{P}_{\varepsilon,k}$, (4.2.3)

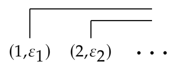
then

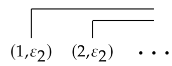
 $\in \mathcal{P}_{\varepsilon,k-1}$

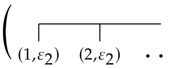
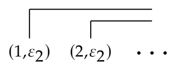
and  $\in \mathcal{P}_{(\dots,\varepsilon_{\ell+1},\varepsilon_{\ell+2},\dots),2}$ (4.2.4) which is equivalent to

$$\mathsf{UMem}^{\ell+1}_{\varepsilon,k}(\pi) \in \operatorname{sub}\big(\mathcal{P}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big). \tag{4.2.5}$$

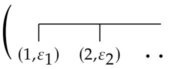
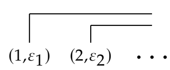
Proof: We prove this assertion by induction over $\ell \in \mathbb{N}$. For the induction base $\ell = 0$ we have a block $b \in \pi$ such that $1 \in \mathsf{set}\big(\mathsf{type}(b)\big)$ and a block $b' \in \pi \setminus \{b\}$ such that $2 \in \mathsf{set}\big(\mathsf{type}(b')\big)$. Assume that $\varepsilon_1 \neq \varepsilon_2$, then we calculate

$\pi =$  $\in \mathcal{P}_{\varepsilon,k}$

$\Longrightarrow \pi_1 :=$  $\in \mathcal{P}_{(\varepsilon_2,\varepsilon_2,\varepsilon_3,\ldots,),k}$ $\llbracket\, \pi_1 = \mathsf{cCol}_{\varepsilon,(\varepsilon_2,\varepsilon_n)}(\pi) \,\rrbracket$

$\Longrightarrow (\pi_2, \tilde{\pi}_2) := \Big($  ,  $\Big) \in \operatorname{sub}\big(\mathcal{P}^{1\wedge 2}_{(\varepsilon_2,\varepsilon_2,\varepsilon_3,\ldots,),k-1}\big)$

$\llbracket\, (\pi_2, \tilde{\pi}_2) = \mathsf{UMem}^{1}_{(\varepsilon_2,\varepsilon_2,\varepsilon_3,\ldots,),k}(\pi_1) \,\rrbracket$

$\Longrightarrow (\pi_3, \tilde{\pi}_3) := \Big($  ,  $\Big) \in \operatorname{sub}\big(\mathcal{P}^{1\wedge 2}_{\varepsilon,k-1}\big)$

$\llbracket\, (\pi_3, \tilde{\pi}_3) = \big(\mathsf{cCol}_{\varepsilon,(\varepsilon_1,\varepsilon_n)}(\pi_2), \mathsf{cCol}_{(\varepsilon_2,\varepsilon_2,\ldots,\varepsilon_{\ell+1},\varepsilon_{\ell+2},\ldots,\varepsilon_r),(\varepsilon_1,\varepsilon_r)}(\tilde{\pi}_2)\big) \,\rrbracket.$

In the case $\varepsilon_1 = \varepsilon_2$ we can directly apply $\mathsf{UMem}^{1}_{\varepsilon,k}$ to π and obtain the desired result. For the induction step $\ell \longrightarrow \ell + 1$ we assume that the assertion holds for $\ell \in \mathbb{N}$. Let $\pi \in \mathcal{P}_\varepsilon$ and assume $\varepsilon_1 = \varepsilon_2$. Then, the induction hypothesis holds for $\pi_1 := \mathsf{delete}_{\varepsilon,2}(\pi) \in \mathcal{P}_{(\varepsilon_1,\varepsilon_3,\ldots)}$ since we have deleted the second leg. This means $(\pi_2, \tilde{\pi}_2) := \mathsf{UMem}^{\ell+1}_{(\varepsilon_1,\varepsilon_3,\ldots),k}(\pi_1) \in \operatorname{sub}\big(\mathcal{P}^{(\ell+1)\wedge(\ell+2)}_{(\varepsilon_1,\varepsilon_3,\ldots),k-1}\big)$. Then,

$$(\pi_3, \tilde{\pi}_3) := (\mathsf{double}_{(\varepsilon_1,\varepsilon_3,\ldots),1}(\pi_2), \mathsf{double}_{(\varepsilon_1,\varepsilon_3,\ldots),1}(\tilde{\pi}_2)) \in \operatorname{sub}\big(\mathcal{P}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k-1}\big).$$

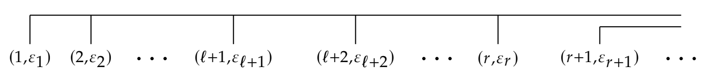
Hence, we have doubled the first leg in π_2 resp. $\tilde{\pi}_2$.

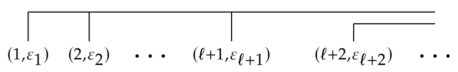
In the case $\varepsilon_1 \neq \varepsilon_2$, we can achieve the same color for the first and the second leg by $\mathsf{cCol}_{\varepsilon,(\varepsilon_2,\varepsilon_n)}(\pi) \in \mathcal{P}_{(\varepsilon_2,\varepsilon_2,\ldots)}$. Then, the argumentation from above applies and for the last step we can change the color of the first leg back to ε_1. □

4.2.4 Remark. The importance of Lemma 4.2.3 relies in the fact that it allows us to unify the first and the second block of a partition $\pi \in \mathcal{P}_\varepsilon$ by $\mathsf{UMem}^{\ell+1}_{\varepsilon,k}$ although $\varepsilon_{\ell+1} \neq \varepsilon_{\ell+2}$ and the image is an element of $\mathcal{P} \times \mathcal{P}$. This is an essential implication of the axiom of Definition 3.4.9 **(f)** since we can change the color of the first leg of a partition and we stay in $\mathcal{P}$. A similar assertion to Lemma 4.2.3 holds for two blocks at the end of a partition $\pi \in \mathsf{Part}_\varepsilon$. By using the axiom of "mirror symmetry" from Definition 3.4.9 **(g)** the leg $\mathsf{BorderLeg}(\pi)$, which was at the beginning of a partition π, gets shifted towards the end of π after application of $\mathsf{mirror}_\varepsilon$ to π.

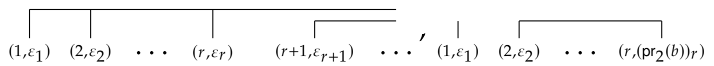
4.2.5 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions for some $m \in \mathbb{N}$. Let $\pi \in \mathcal{P}_{\varepsilon,k}$ for some $\varepsilon \in [m]^{\times n}$ and $k \in \mathbb{N}\setminus\{1\}$. If $r := \mathsf{BorderLeg}(\pi) \in \mathbb{N}$, then for any $\ell \in \{0, \ldots, r-2\} \subseteq \mathbb{N}_0$ and $\sigma \in \mathcal{P}_{\mathsf{col}(b),2}$ such that $(\pi, \sigma) \in \operatorname{sub}\big(\mathcal{P}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big)$ we have

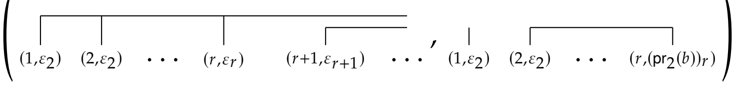
$$\mathsf{split}^{\ell+1}_{\varepsilon,k-1}(\pi, \sigma) \in \mathcal{P}_{\varepsilon,k} \tag{4.2.6a}$$

$= \mathsf{split}^{\ell+1}_{\varepsilon,k-1}\Big($  ,

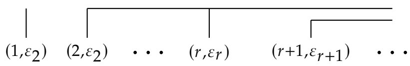
$$\Big($$  $$\Big) \in \mathcal{P}_{\varepsilon,k} \qquad (4.2.6b)$$

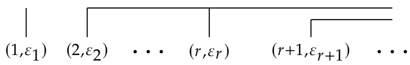
PROOF: We prove this claim by induction over $\ell \in \{0,\dots,r-2\} \subseteq \mathbb{N}_0$ for any $r \in \mathbb{N}$. For the induction base $\ell = 0$ we assume $\varepsilon_1 \neq \varepsilon_2$ and calculate

$$(\pi,\sigma) = \Big($$  $$\Big) \in \mathrm{sub}\big(\mathcal{P}^{1\wedge 2}_{\varepsilon,k}\big)$$

$$\Longrightarrow (\pi_1,\tilde{\pi}_1) := \Big($$  $$\Big) \in \mathcal{P}\times\mathcal{P}$$

$$[\![\, (\pi_1,\tilde{\pi}_1) = \big(\mathsf{cCol}_{\varepsilon,(\varepsilon_2,\varepsilon_n)}(\pi), \mathsf{cCol}_{\mathrm{pr}_2(b),(\varepsilon_2,(\mathrm{pr}_2(b)))}(\sigma)\big) \,]\!]$$

$$\Longrightarrow \pi_2 :=$$  $$[\![\, \mathsf{split}^1_{\varepsilon,k-1}(\pi_1,\tilde{\pi}_1) \,]\!] \in \mathcal{P}$$

$$\Longrightarrow \pi_3 :=$$  $$\in \mathcal{P} \quad [\![\, \pi_3 = \mathsf{cCol}_{\varepsilon,(\varepsilon_1,\varepsilon_n)}(\pi_2) \,]\!].$$

In the case $\varepsilon_1 = \varepsilon_2$ we can directly apply $\mathsf{split}^1_{\varepsilon,k-1}$ to (π,σ) and obtain the desired result. For the induction step $\ell \to \ell+1$ we assume that the assertion holds for some $\ell \in [r-3]$. Let $(\pi,\sigma) \in \mathrm{sub}\big(\mathcal{P}^{(\ell+2)\wedge(\ell+3)}_{\varepsilon,k}\big)$. Then, the induction hypothesis holds for $(\pi_1,\tilde{\pi}_1) := (\mathsf{delete}_{\varepsilon,2}(\pi), \mathsf{delete}_{\varepsilon,2}(\sigma)) \in \mathrm{sub}\big(\mathcal{P}^{(\ell+1)\wedge(\ell+2)}_{(\varepsilon_1,\varepsilon_3,\dots),k}\big)$ because we have deleted the second leg in each partition. This means $\pi_2 := \mathsf{split}^{\ell+1}_{(\varepsilon_1,\varepsilon_3,\dots),k-1}(\pi_1,\tilde{\pi}_1) \in \mathcal{P}_{(\varepsilon_1,\varepsilon_3,\dots),k}$. Then, $\pi_3 := \mathsf{double}_{\varepsilon,1}(\pi_2) \in \mathcal{P}_{\varepsilon,k}$ since we have doubled the first leg. The existence of $\pi_3 \in \mathcal{P}$ proves the induction step.

In the case $\varepsilon_1 \neq \varepsilon_2$, we can achieve the same color for the first and the second leg by $\mathsf{cCol}_{\varepsilon,(\varepsilon_2,\varepsilon_n)}(\pi) \in \mathcal{P}_{(\varepsilon_2,\varepsilon_2,\dots)}$. Then, the argumentation from above applies and for the last step we can change the color of the first leg back to ε_1. □

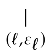
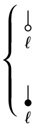
4.2.6 Remark. The importance of Lemma 4.2.5 relies in the fact that it allows us to split block neighboring legs $\ell+1$ and $\ell+2$ in the first block although $\varepsilon_{\ell+1} \neq \varepsilon_{\ell+2}$ and the image is an element of $\mathcal{P}$. This is an essential implication of the axiom of Definition 3.4.9 **(f)** since we can change the color of the first leg of a partition and we stay in $\mathcal{P}$. A similar assertion to Lemma 4.2.5 holds for the two blocks at the end of a partition $\pi \in \mathsf{Part}_\varepsilon$. By using the axiom of "mirror symmetry" from Definition 3.4.9 **(g)** the leg $\mathsf{BorderLeg}(\pi)$, which was at the beginning of a partition π, gets shifted towards the end of π after application of $\mathsf{mirror}_\varepsilon$ to π.

4.2.7 Convention.

(a) For any m-colored partition $\pi = \{b_1,\dots,b_k\}$ with $\mathbb{N} \ni k$ blocks we put by Convention 2.3.5

$$\mathsf{type}(\pi) := \big\{\mathsf{type}(b_1),\dots,\mathsf{type}(b_k)\big\} \qquad (4.2.7)$$

(b) We want to introduce the following convention to display legs with their attached color label in a partition for the two-colored case. We set

 $$\longmapsto \begin{cases} & \text{for } \varepsilon_\ell = 1 \\ & \text{for } \varepsilon_\ell = 2. \end{cases} \qquad (4.2.8)$$ 

Whenever a leg in a partition can take both colors of white ∘ and black • we also draw it with a gray sublabel •. For instance the color of the first leg respectively the last leg of a partition can be independently chosen to be either white or black by Definition 3.4.9 **(g)**. Therefore, we can draw them both with a gray sublabel.

4.2.8 Definition. Let $m,\ k \in \mathbb{N} \setminus \{1\},\ n \in \mathbb{N} \setminus [k-1]$ and $\varepsilon \in [m]^{\times n}$. We want to recursively define a map $\mathsf{GenerateTwoBlocks}$ which assigns to each partition $\pi \in \mathsf{Part}_{\varepsilon,k}$ with k Blocks a $(k-1)$-tuple of two-block partitions. If $k = 2$ and $\pi \in \mathsf{Part}_{\varepsilon,k}$, then set

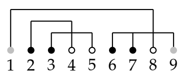
$$\mathsf{GenerateTwoBlocks}(\pi) = (\pi). \tag{4.2.9}$$

Now, assume $k \in \mathbb{N} \setminus \{2\}$ and $\mathsf{GenerateTwoBlocks}$ has been defined for each $\pi \in \mathsf{Part}_{\cdot,k-1}$. Then, for any $\varepsilon \in [m]^{\times n}$ and $\pi \in \mathsf{Part}_{\varepsilon,k}$ the ordered tuple

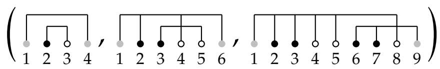
$$\Big((\mathsf{pr}_2 \circ \mathsf{UMem}^{\ell+1}_{\varepsilon,k})(\pi), \mathsf{GenerateTwoBlocks}\big((\mathsf{pr}_1 \circ \mathsf{UMem}^{\ell+1}_{\cdot,k})(\pi)\big)\Big) \tag{4.2.10}$$

defines an $(k-1)$-tuple of two-block partitions. The assignment is well-defined by Lemma 4.2.3.

4.2.9 Example. Let $\mathcal{P}$ be a two-colored universal class of partitions. Assume

$\pi =$  $\in \mathcal{P}_{\cdot,4}.$ (4.2.11)

Then, $\mathsf{GenerateTwoBlocks}(\pi)$ leads to

 $\in \mathcal{P}_{\cdot,2} \times \mathcal{P}_{\cdot,2} \times \mathcal{P}_{\cdot,2}.$ (4.2.12)

4.2.10 Lemma.

(a) Let $m \in \mathbb{N}$ and define for any $k \in \mathbb{N}$ the set $\mathsf{Part}_{\cdot,k} := \bigcup_{n\in\mathbb{N}} \bigcup_{\varepsilon\in[m]^{\times n}} \mathsf{Part}_{\varepsilon_k}$. The map $\mathsf{GenerateTwoBlocks}\colon \mathsf{Part}_{\cdot,k} \longrightarrow (\mathsf{Part}_{\cdot,2})^{\times(k-1)}$ is injective.

(b) Let $\mathcal{P} \subseteq \mathsf{Part}$ be an m-colored universal class of partitions for $m \in \mathbb{N}$, then $\mathsf{GenerateTwoBlocks}\restriction_{\mathcal{P}}\colon \mathcal{P} \longrightarrow (\mathcal{P}_{\cdot,2})^{\times(k-1)}$.

Proof: Ad **(a)**: The proof is technical but can be directly done. Like in the single-colored case we notice that we can "reconstruct" a partition $\pi \in \mathsf{Part}_{\cdot,k}$ from $\mathsf{GenerateTwoBlocks}(\pi)$. In other words, we can give a left inverse to $\mathsf{GenerateTwoBlocks}$ by a successive application of Lemma 4.2.5.

Ad **(b)**: This is a consequence from Lemma 4.2.3. □

We have an analogous result to Proposition 4.1.5 for the multi-colored case.

4.2.11 Proposition. Let $\mathcal{P}$ and $\mathcal{R}$ be two m-colored universal classes of partitions for $m \in \mathbb{N}$. Denote the set of all two-block partitions of $\mathcal{P}$ by $\mathcal{P}_{\cdot,2}$ resp. the set of all two-block partitions of $\mathcal{R}$ by $\mathcal{R}_{\cdot,2}$.

Tfae: **(a)** $\mathcal{R} = \mathcal{P}$,

(b) $\mathcal{R}_{\cdot,2} = \mathcal{P}_{\cdot,2}$.

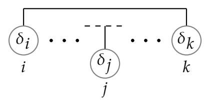
Proof: The proof in this multi-colored case follows by an analogous reasoning from the single-colored case (Proposition 4.1.5). □

4.2.12 Definition (Crossing partition). Let $m,\, n \in \mathbb{N}$ and $\varepsilon \in [n]$. For any partition $\pi \in \mathsf{Part}_\varepsilon$ we set

$$\mathsf{Cross}(\pi) := \left\{ \big((p_1, p_2), (q_1, q_2)\big) \in \mathbb{N}^{\times 2} \times \mathbb{N}^{\times 2} \;\middle|\; \begin{array}{l} p_1 < q_1 < p_2 < q_2,\ \exists b,\, b' \in \pi: \\ b \neq b',\ \{p_1, p_2\} \subseteq \mathsf{set}\big(\mathsf{type}(b)\big), \\ \{q_1, q_2\} \subseteq \mathsf{set}\big(\mathsf{type}(b')\big) \end{array} \right\}. \tag{4.2.13}$$

We call elements of the set $\mathsf{Cross}(\pi)$ *crossings*. For partition $\pi \in \mathsf{Part}_\varepsilon$ we say it is a *partition with crossing* or just *crossing* if and only if $\mathsf{Cross}(\pi) \neq \emptyset$. If $c := \big((p_1, p_2), (q_1, q_2)\big) \in \mathsf{Cross}(\pi)$, then we say that p_1 is the *first outer leg* of the crossing c, q_1 is the *first inner leg* of the crossing c, p_2 is the *second inner leg* of the crossing c and q_2 is the *second outer leg* of the crossing c.

Next, we are going to define certain types of two-colored partitions. It will turn out that these in fact satisfy the properties of a universal class of partitions. To decide if a given partition π is an element of a certain type of two-colored partitions, we need to check what we call a "*triple legs scenario*" within this partition π



(4.2.14)

depending on the colors $\delta_i,\, \delta_j,\, \delta_k \in \{\circ, \bullet\}$. This means that, whenever there are legs i and k, which belong to one block of the partition π, and there is a leg j, then we need to answer the question "what is the allowed relative position of the legs i, j, k in the partition π depending on the colors $\delta_i,\, \delta_j,\, \delta_k \in \circ, \bullet$?".

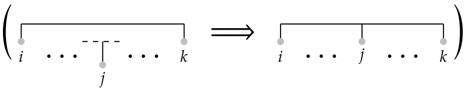
4.2.13 Definition. Let $m = 2$. For any $n \in \mathbb{N}$ we define the following types of partitions

(a) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be a *one block* for $\varepsilon = (\varepsilon_i)_{i \in [n]} \in \{\circ, \bullet\}^{\times n}$ if and only if $|\pi| = 1 \iff \mathsf{type}(\pi) \in \mathsf{1B}$. The set of all partitions which are one block for any ε is denoted by $\mathsf{1B}_{\{\circ,\bullet\}}$.

(b) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be an *interval* for $\varepsilon = (\varepsilon_i)_{i \in [n]} \in \{\circ, \bullet\}^{\times n}$ if and only if $\mathsf{type}(\pi) \in \mathsf{I} \iff$

$\forall j \in [n],\ \forall i \in [j-1],\ \forall k \in [n] \setminus [j],\ \forall b \in \pi:$

$$\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big) \implies j \in \mathsf{set}\big(\mathsf{type}(b)\big)\Big) \tag{4.2.15a}$$

$\widehat{=}$ 

(4.2.15b)

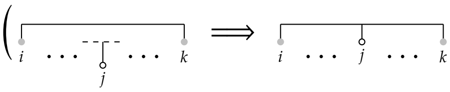
The set of all partitions which are an interval for any ε is denoted by $\mathsf{I}_{\{\circ,\bullet\}}$.

(c) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *interval-noncrossing* for $\varepsilon = (\varepsilon_i)_{i \in [n]} \in \{\circ, \bullet\}^{\times n}$ if and only if

$$\mathsf{type}(\pi) \in \mathsf{NC} \tag{4.2.16a}$$

and $\forall j \in [n],\ \forall i \in [j-1],\ \forall k \in [n] \setminus [j],\ \forall b \in \pi:$

$$\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \circ \implies (j, \varepsilon_j) \in \mathsf{set}(b)\Big) \tag{4.2.16b}$$

$\widehat{=}$ 

(4.2.16c)

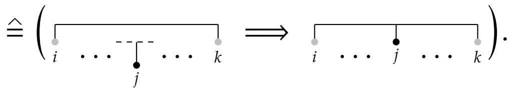
The set of all partitions which are interval-noncrossing for any ε is denoted by $\mathsf{I}_\circ\mathsf{NC}_\bullet$.

(d) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *noncrossing-interval* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if

$$\mathsf{type}(\pi) \in \mathsf{NC} \tag{4.2.17a}$$

and $\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

$$\Big(i, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \bullet \implies (j, \varepsilon_j) \in \mathsf{set}(b)\Big) \tag{4.2.17b}$$



(4.2.17c)

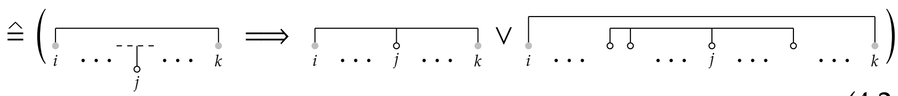
The set of all partitions which are interval-noncrossing for any ε is denoted by $\mathsf{NC}_\circ\mathsf{I}_\bullet$.

(e) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *pure noncrossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if

$$\mathsf{type}(\pi) \in \mathsf{NC} \tag{4.2.18a}$$

and $\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

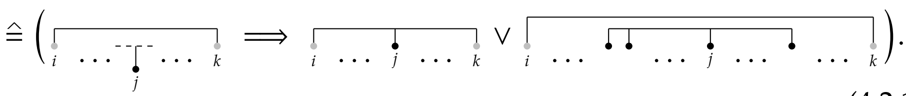
$$\begin{aligned}\Big(i, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \circ \implies &\big((j, \varepsilon_j) \in \mathsf{set}(b)\big) \\ &\vee \big(\exists b' \in \pi \setminus \{b\} \colon j \in \mathsf{set}(\mathsf{type}(b')), \\ &\quad \mathsf{set}(\mathsf{type}(b')) \subseteq [i+1, k-1] \subseteq \mathbb{N}, \\ &\quad \mathsf{set}(\mathsf{col}(b')) = \circ\big)\end{aligned} \tag{4.2.18b}$$



(4.2.18c)

and $\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

$$\begin{aligned}\Big(i, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \bullet \implies &\big((j, \varepsilon_j) \in \mathsf{set}(b)\big) \\ &\vee \big(\exists b' \in \pi \setminus \{b\} \colon j \in \mathsf{set}(\mathsf{type}(b')), \\ &\quad \mathsf{set}(\mathsf{type}(b')) \subseteq [i+1, k-1] \subseteq \mathbb{N}, \\ &\quad \mathsf{set}(\mathsf{col}(b')) = \bullet\big)\end{aligned} \tag{4.2.18d}$$



(4.2.18e)

The set of all partitions which are pure noncrossing for any ε is denoted by pureNC.

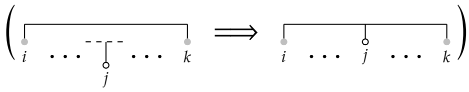
(f) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *noncrossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if

$$\mathsf{type}(\pi) \in \mathsf{NC}. \tag{4.2.19}$$

The set of all partitions which are noncrossing for any ε is denoted by $\mathsf{NC}_{\{\circ,\bullet\}}$.

(g) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *interval-crossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if $\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

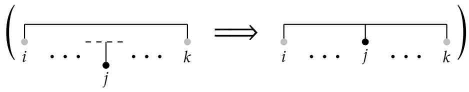
$$\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \circ \implies (j, \varepsilon_j) \in \mathsf{set}(b)\Big) \tag{4.2.20a}$$

$\widehat{=}$  (4.2.20b)

The set of all partitions which are interval-crossing for any ε is denoted by $\mathsf{I}_\circ\mathsf{A}_\bullet$.

(h) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *crossing-interval* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if $\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

$$\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \bullet \implies (j, \varepsilon_j) \in \mathsf{set}(b)\Big) \tag{4.2.21a}$$

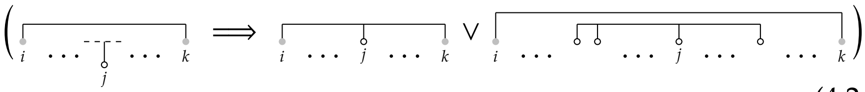
$\widehat{=}$  (4.2.21b)

The set of all partitions which are interval-crossing for any ε is denoted by $\mathsf{A}_\circ\mathsf{I}_\bullet$.

(i) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *noncrossing-crossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if

$\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

$$\begin{aligned}\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \circ \implies &\big((j, \varepsilon_j) \in \mathsf{set}(b)\big)\\ &\vee \big(\exists b' \in \pi \setminus \{b\}\colon j \in \mathsf{set}(\mathsf{type}(b')),\\ &\quad \mathsf{set}(\mathsf{type}(b')) \subseteq [i+1, k-1] \subseteq \mathbb{N},\\ &\quad \mathsf{set}(\mathsf{col}(b')) = \circ\big)\end{aligned} \tag{4.2.22a}$$

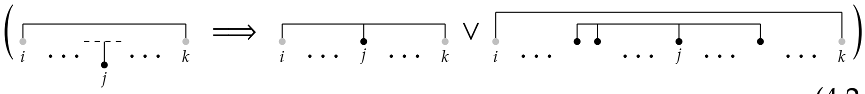
$\widehat{=}$  (4.2.22b)

The set of all partitions which are noncrossing-crossing for any ε is denoted by $\mathsf{NC}_\circ\mathsf{A}_\bullet$.

(j) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *crossing-noncrossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if

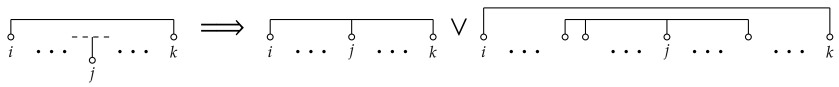
$\forall j \in [n], \forall i \in [j-1], \forall k \in [n] \setminus [j], \forall b \in \pi$:

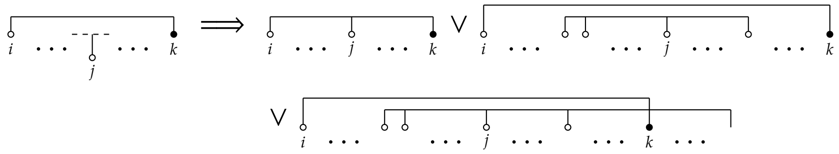
$$\begin{aligned}\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \bullet \implies &\big((j, \varepsilon_j) \in \mathsf{set}(b)\big)\\ &\vee \big(\exists b' \in \pi \setminus \{b\}\colon j \in \mathsf{set}(\mathsf{type}(b')),\\ &\quad \mathsf{set}(\mathsf{type}(b')) \subseteq [i+1, k-1] \subseteq \mathbb{N},\\ &\quad \mathsf{set}(\mathsf{col}(b')) = \bullet\big)\end{aligned} \tag{4.2.23a}$$

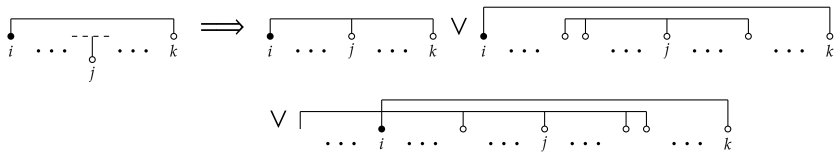
$\widehat{=}$  (4.2.23b)

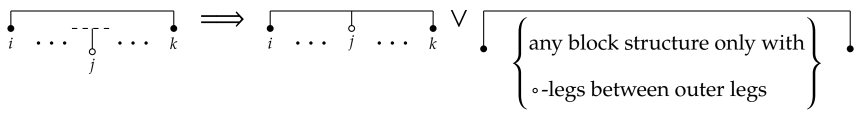
The set of all partitions which are noncrossing-crossing for any ε is denoted by $\mathsf{A}_\circ\mathsf{NC}_\bullet$.

(k) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *binoncrossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if $\forall j \in [n]$, $\forall i \in [j-1]$, $\forall k \in [n] \setminus [j]$, $\forall b \in \pi$:

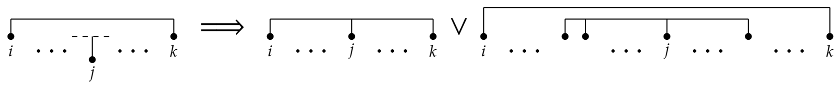

(4.2.24a)

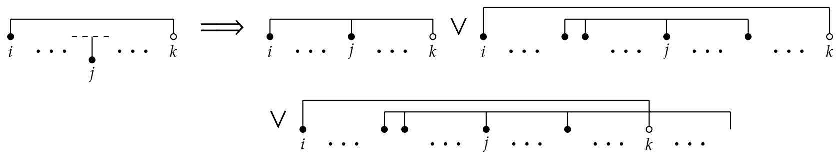

(4.2.24b)

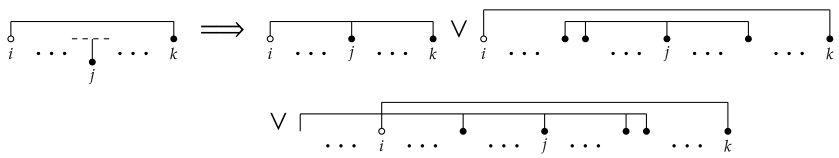

(4.2.24c)

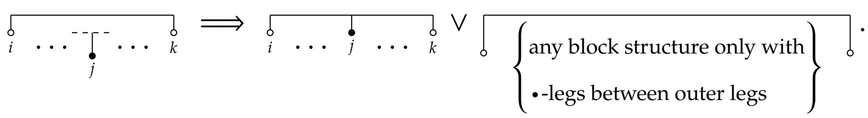


(4.2.24d)


(4.2.24e)


(4.2.24f)

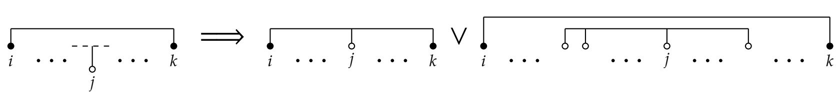

(4.2.24g)

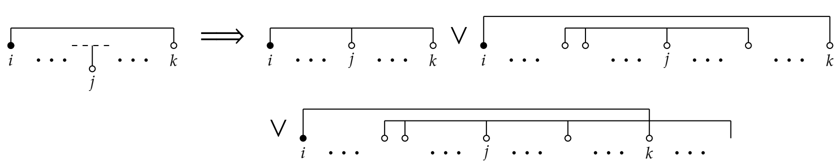


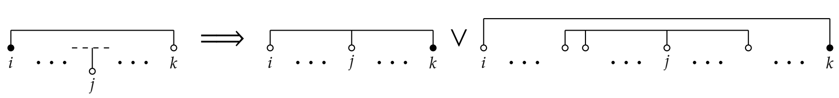
(4.2.24h)

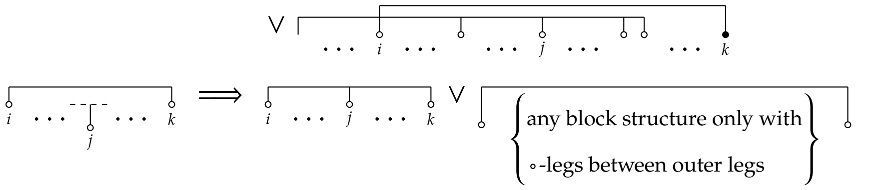
The set of all partitions which are binoncrossing for any ε is denoted by biNC.

(l) A partition $\pi \in \mathsf{Part}_\varepsilon$ is said to be *pure crossing* for $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$ if and only if $\forall j \in [n]$, $\forall i \in [j-1]$, $\forall k \in [n] \setminus [j]$, $\forall b \in \pi$:

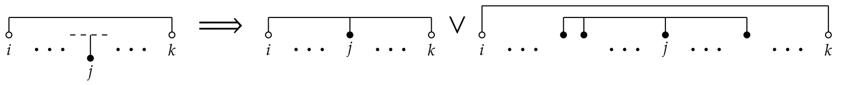

(4.2.25a)


(4.2.25b)

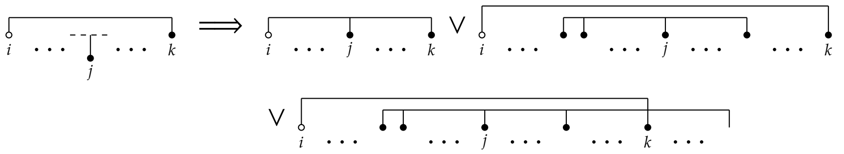

(4.2.25c)



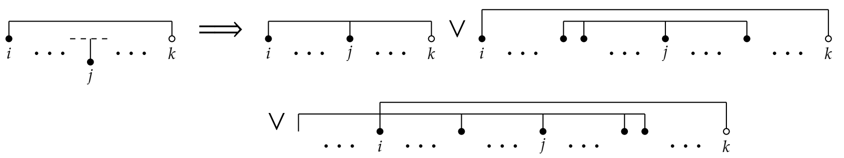
(4.2.25d)



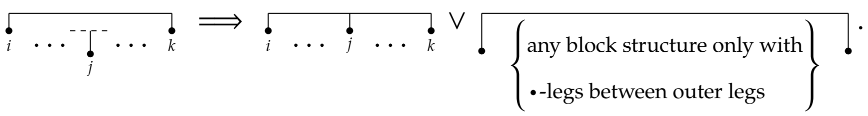
(4.2.25e)



(4.2.25f)



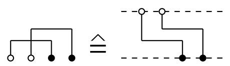
(4.2.25g)



(4.2.25h)

The set of all partitions which are pure crossing for any ε is denoted by pureC.

4.2.14 Remark. It can be shown that our definition of interval-noncrossing is equivalent to the definition provided by [Liu19]. Furthermore, it can be shown that our definition of binoncrossing partitions is equivalent to the usual one, for instance used in [CNS15]. We can actually see what literally makes these partitions binoncrossing, if we connect white labeled legs to an upper line. As an example we have



(4.2.26)

4.2.15 Lemma. The set of all one block partitions $\mathsf{1B}_{\{\circ,\bullet\}}$ is a two-colored universal class of partitions.

Proof: We need to check the properties for a two-colored universal class of partitions from Definition 3.4.9 for $m = 2$. By definition of $\mathsf{1B}_{\{\circ,\bullet\}}$ in Definition 4.2.13 **(a)** it is clear that Definition 3.4.9 **(a)** is satisfied. It remains to check the properties of **(b)** – **(g)**. Therefore let $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ and let π be a one block partition for some $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times m}$. Denote the left inverse of f by g. The existence of g is well defined by the assertions of Lemma 3.4.4, Lemma 3.4.7 and Lemma 3.4.13. Assume that $\mathsf{f}(\pi) \notin \mathsf{1B}_{\{\circ,\bullet\}}$, then this means that $|\mathsf{f}(\pi)| \geq 2$. For each left inverse g of f we have $|(\mathsf{g}\circ\mathsf{f})(\pi)| \geq 2$. But this would imply that $\pi \notin \mathsf{1B}_{\{\circ,\bullet\}}$, since $\mathsf{g}\circ\mathsf{f} = \mathsf{id}$ which is a contradiction to the assumption $\pi \in \mathsf{1B}_{\{\circ,\bullet\}}$. Similar statements hold for UMem and split. By proof of contradiction the properties of Definition 3.4.9 **(b)** – **(g)** are fulfilled. □

4.2.16 Lemma. The set of all interval partitions $\mathsf{I}_{\{\circ,\bullet\}}$ is a two-colored universal class of partitions.

Proof: Since $\mathsf{1B}_{\{\circ,\bullet\}} \subseteq \mathsf{I}_{\{\circ,\bullet\}}$ we have that Definition 3.4.9 **(a)** is satisfied. It remains to check the properties of Definition 3.4.9 **(b)** – **(g)**. This means we have to prove for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double},$

$\mathsf{pr}_1 \circ \mathsf{UMem}, \mathsf{pr}_2 \circ \mathsf{UMem}, \mathsf{cCol}, \mathsf{mirror}\}$

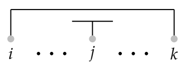
$$\pi \in \mathsf{1B}_{\{\circ,\bullet\}} \implies \mathsf{f}(\pi) \in \mathsf{1B}_{\{\circ,\bullet\}}$$

and

$$(\pi, \tilde{\pi}) \in \Big(\mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big) \cap (\mathsf{1B}_{\{\circ,\bullet\}} \times \mathsf{1B}_{\{\circ,\bullet\}})\Big) \implies \mathsf{split}(\pi, \tilde{\pi}) \in \mathsf{1B}_{\{\circ,\bullet\}}.$$

If a partition $\pi \notin \mathsf{I}_{\{\circ,\bullet\}}$, then according to equation (4.2.15a) this is equivalent to the following statement

$$\exists j \in [n], \exists i \in [j-1], \exists k \in [n] \setminus [j], \exists \beta \in \pi :$$

$$\Big(i,\ k \in \mathsf{set}\big(\mathsf{type}(\beta)\big)\Big) \wedge \Big(j \notin \mathsf{set}\big(\mathsf{type}(\beta)\big)\Big) \mathrel{\hat{=}}$$  . (I)

If a partition is not element of $\mathsf{I}_{\{\circ,\bullet\}}$, then according to the diagram of equation (I) we let b_j denote the block of π such that $j \in \mathsf{set}(\mathsf{pr}_1(b_j))$.

Let $\pi \notin \mathsf{I}_{\{\circ,\bullet\}}$, then we claim $\mathsf{f}(\pi) \notin \mathsf{I}_{\{\circ,\bullet\}}$ for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}\}$. The relative order of the blocks of $\mathsf{f}(\beta)$ and $\mathsf{f}(b_j)$ remains in $\mathsf{f}(\pi)$ as β and b_j have in π. This holds independently of the fact if a leg has been deleted respectively doubled before the position i, between i and j, between j and k or after k. Since delete and double are inverse to each other, it follows by proof of contradiction that $\pi \in \mathsf{I}_{\{\circ,\bullet\}} \implies \mathsf{f}(\pi) \in \mathsf{I}_{\{\circ,\bullet\}}$.

Next, we want to show $\tilde{\pi} \in \mathsf{I}_{\{\circ,\bullet\}} \implies \mathsf{UMem}(\tilde{\pi}) \in \mathsf{I}_{\{\circ,\bullet\}} \times \mathsf{I}_{\{\circ,\bullet\}}$. We show this proof of contradiction. For this, assume $\tilde{\pi} \in \mathsf{I}_{\{\circ,\bullet\}}$ and $\pi' := (\mathsf{pr}_2 \circ \mathsf{UMem}^{\,\cdot}_{\cdot,\cdot})(\tilde{\pi})$. Hence, π' is a two block partition. Furthermore, set $\pi := (\mathsf{pr}_1 \circ \mathsf{UMem}^{\,\cdot}_{\cdot,\cdot})(\tilde{\pi})$. By definition of the image of the map UMem, we have $(\pi, \pi') \in \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big)$, hence there exists a block b in π, which can get split by $\{\beta, b_j\}$. Then, the two blocks of the two-block partition $b \Vdash \{\beta, b_j\}$ have the same relative order in $\mathsf{split}(\pi, \pi')$ as β and b_j in π'. Assume now $\pi \notin \mathsf{I}_{\{\circ,\bullet\}}$ and $(\pi, \pi') \in \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big)$. We make the following case consideration. A block b in π gets split by π' which is neither β nor b_j. Then, the relative order of β and b_j remains in $\mathsf{split}(\pi, \pi')$ as β and b_j have in π. Assume the block β gets split. Only block neighboring legs (with the same color) can get split up and the split point can be before or after the position j. Then, it is obvious that $\mathsf{split}(\pi, \pi') \notin \mathsf{I}_{\{\circ,\bullet\}}$. For the last case assume that the block b_j gets split. In this case also follows that $\mathsf{split}(\pi, \pi') \notin \mathsf{I}_{\{\circ,\bullet\}}$. Now, since split is the left inverse of UMem it follows by contradiction that $\tilde{\pi} \in \mathsf{I}_{\{\circ,\bullet\}} \implies \mathsf{UMem}(\tilde{\pi}) \in \mathsf{I}_{\{\circ,\bullet\}} \times \mathsf{I}_{\{\circ,\bullet\}}$.

Next, we want to show $(\rho, \rho') \in ((\mathsf{I}_{\{\circ,\bullet\}} \times \mathsf{I}_{\{\circ,\bullet\}}) \cap \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big) \implies \pi := \mathsf{split}(\rho, \rho') \in \mathsf{I}_{\{\circ,\bullet\}}$. Assume therefor that $\pi \notin \mathsf{I}_{\{\circ,\bullet\}}$. Assume two neighboring legs with the same color of different blocks get unified such that not both blocks are β and b_j. Then, the relative order of β and b_j remains in $(\mathsf{pr}_1 \circ \mathsf{UMem})(\pi)$ as of β and b_j in π. Assume now that β and b_j get unified. Then, the relative order of β and b_j remains in $(\mathsf{pr}_2 \circ \mathsf{UMem})(\pi)$ as of β and b_j in π. Since UMem is the left inverse to split it follows by contradiction that $(\rho, \rho') \in (\mathsf{I}_{\{\circ,\bullet\}} \times \mathsf{I}_{\{\circ,\bullet\}}) \cap \mathrm{sub}\big(\mathsf{Part}^{(\ell+1)\wedge(\ell+2)}_{\varepsilon,k}\big) \implies \pi \in \mathsf{I}_{\{\circ,\bullet\}}$.

It is clear that if $\mathsf{f} \in \{\mathsf{cCol}, \mathsf{mirror}\}$, then $\pi \in \mathsf{I}_{\{\circ,\bullet\}} \implies \mathsf{f}(\pi) \in \mathsf{I}_{\{\circ,\bullet\}}$. □

4.2.17 Remark. The above proof could also be heavily shortened by referencing to the fact that the set I is a universal class of partitions. Instead we have chosen going a longer way, since the concept of the above proof will be a role model for the following proofs. We use a proof of contradiction because it is much easier to track a "defect" in a partition which prevents it from being an element in a certain two-colored universal class of partitions.

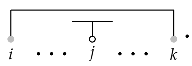
4.2.18 Lemma. The set of all interval-noncrossing partitions $\mathsf{I}_\circ\mathsf{NC}_\bullet$ is a two-colored universal class of partitions.

Proof: It is clear that Definition 3.4.9 **(a)** is satisfied. We prove the remaining axioms by proof of contradiction. Therefor we need an equivalent characterization for the fact that a partition $\pi \notin \mathsf{I}_\circ\mathsf{NC}_\bullet$. The negation of statements of equations (4.2.16a) and (4.2.16b) lead to $\pi \notin \mathsf{I}_\circ\mathsf{NC}_\bullet \iff$

$$\mathsf{type}(\pi) \notin \mathsf{NC} \tag{I}$$

or $\exists j \in [n]$, $\exists i \in [j-1]$, $\exists k \in [n] \setminus [j]$, $\exists \beta \in \pi$:

$$\Big(i,\, k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \circ\Big) \wedge \big((j, \varepsilon_j) \notin \mathsf{set}(\beta)\big) \tag{II}$$

$\widehat{=}$  .

Hence, we have to make two case considerations: equation (I) or equation (II). Assume that $\mathsf{type}(\pi) \notin \mathsf{NC}$. We have already proven that the class NC is closed under the operations $\mathsf{delete}, \mathsf{double}, \mathsf{UMem}$ and split. This is equivalent to say that if a partition is noncrossing, then it remains noncrossing under the operations $\mathsf{delete}, \mathsf{double}, \mathsf{UMem}$ and split. We easily convince ourselves that the same is true for the operations cCol and mirror. Now, we perform the proof of contradiction. Assume $\pi \in \mathsf{I}_\circ\mathsf{NC}_\bullet$ and $\mathsf{type}(\mathsf{f}(\pi)) \notin \mathsf{NC}$ for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$. Let g denote the left inverse for $\mathsf{type} \circ \mathsf{f}$. The left inverse satisfies $g \in \{\mathsf{delete}, \mathsf{double}, \mathsf{UMem}\}$ in the case $m = 1$. Then, $g \circ (f \circ \mathsf{type}) = \mathsf{id}$ and we obtain $\mathsf{type}(\pi) \notin \mathsf{NC}$, which is a contradiction to the assumption. A similar statement holds for UMem and split.

Now, we claim that if a partition $\pi \in \mathsf{Part}_\varepsilon$ satisfies the statement of equation (II), then the partition $\mathsf{f}(\pi)$ for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ satisfies it too. A similar statement holds for UMem and split. The proof is performed analogously like in the proof of Lemma 4.2.16 because the color of the leg j is now specified to $\circ$. Assuming that $\pi \in \mathsf{I}_\circ\mathsf{NC}_\bullet$ and $\mathsf{f}(\pi)$ satisfies equation (II) leads to a contradiction using the left inverse of f. A similar statement holds for UMem and split. This shows that $\mathsf{I}_\circ\mathsf{NC}_\bullet$ is a two-colored universal class of partitions. □

4.2.19 Lemma. The set of all interval-noncrossing partitions $\mathsf{NC}_\circ\mathsf{I}_\bullet$ is a two-colored universal class of partitions.

Proof: The proof is similar to the proof of Lemma 4.2.18. □

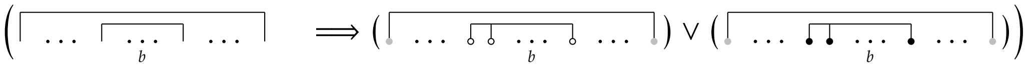
4.2.20 Lemma. The set of all pure noncrossing partitions pureNC is a two-colored universal class of partitions.

Proof: We prove this statement by contradiction. We first give an equivalent characterization of equation (4.2.18) for a partition to be pure noncrossing, which is $\pi \in \mathsf{pureNC} \iff$

$$\mathsf{type}(\pi) \in \mathsf{NC} \tag{Ia}$$

and $\forall b \in \pi$:

$$\Big(\mathsf{type}(b) \text{ is nested block } \Longrightarrow \big(\mathsf{set}(\mathsf{col}(b)) = \{\circ\}\big) \qquad \vee \big(\mathsf{set}(\mathsf{col}(b)) = \{\bullet\}\big)\Big) \tag{Ib}$$

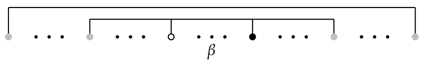
$\widehat{=}$  .

Starting from the above we want to formulate an equivalent characterization for a partition π not being pure noncrossing. According to equation (Ia) and equation (Ib) we have $\pi \notin \mathsf{pureNC} \iff$

$$\mathsf{pr}_1(\pi) \notin \mathsf{NC} \tag{II}$$

or $\exists \beta \in \pi$:

$$\big(\mathsf{type}(\beta) \text{ is nested block}\big) \wedge \Big(\big(\mathsf{set}(\mathsf{col}(\beta)) \neq \{\circ\}\big) \wedge \big(\mathsf{set}(\mathsf{col}(\beta)) \neq \{\bullet\}\big)\Big) \tag{III}$$

$\widehat{=}$ .

Assuming that $\pi \notin \mathsf{pureNC}$ means considering two cases. In the first case we assume that term (II) holds. But we have already seen that the set of all noncrossing partitions is a universal class of partitions. Therefore, by proof of contradiction we obtain $\pi \in \mathsf{pureNC} \implies \mathsf{type}(\pi) \in \mathsf{NC}$.

Now, let us assume that the term in (III) holds for a partition $\pi \in \mathsf{Part}_\varepsilon$. Then, for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ the partition $\mathsf{f}(\pi)$ fulfills the term in (III) too. The proof is performed analogously like the proof of Lemma 4.2.16, where the unique block of the leg j has to be replaced by the block β, which is an inner block and has at least two legs, where one leg has the color $\circ$ and the other one has the color $\bullet$. Assuming that $\pi \in \mathsf{pureNC}$ and $\mathsf{f}(\pi)$ satisfies the term in (III) leads to a contradiction using the left inverse of f. A similar statement holds for UMem and split. This shows that pureNC is a two-colored universal class of partitions. □

4.2.21 Lemma. The set of all noncrossing partitions $\mathsf{NC}_{\{\circ,\bullet\}}$ is a two-colored universal class of partitions.

Proof: The proof relies on the fact that the set of all noncrossing partitions NC is a universal class of partitions, which we have already proven. □

4.2.22 Lemma. The set of all interval-crossing partitions $\mathsf{I}_\circ\mathsf{A}_\bullet$ is a two-colored universal class of partitions.

Proof: The proof is similar to the proof of Lemma 4.2.18. □

4.2.23 Lemma. The set of all crossing-interval partitions $\mathsf{A}_\circ\mathsf{I}_\bullet$ is a two-colored universal class of partitions.

Proof: The proof is similar to the proof of Lemma 4.2.18. □

4.2.24 Lemma. The set of all noncrossing-crossing partitions $\mathsf{NC}_\circ\mathsf{A}_\bullet$ is a two-colored universal class of partitions.

Proof: We want to prove this statement by proof of contradiction. For this, we need an equivalent characterization for a partition $\pi \notin \mathsf{NC}_\circ\mathsf{A}_\bullet$. For $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ and $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ and from equation (4.2.22a) we obtain that $\mathsf{f}(\pi) \notin \mathsf{NC}_\circ\mathsf{A}_\bullet \iff$

$$\exists j \in [n],\ \exists i \in [j-1],\ \exists k \in [n] \setminus [j],\ \exists \beta \in \mathsf{f}(\pi)\colon \Big(i,\ k \in \mathsf{set}\big(\mathsf{type}(\beta)\big),\ \varepsilon_j = \circ\Big) \tag{Ia}$$

$$\wedge \Big(\big(j, \varepsilon_j\big) \notin \mathsf{set}(\beta) \tag{Ib}$$

$$\wedge \big(\forall b' \in \pi \setminus \{\beta\}\colon j \notin \mathsf{set}(\mathsf{type}(b'))\big) \tag{Ic}$$

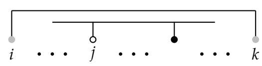
$$\vee\ \mathsf{set}(\mathsf{type}(b')) \not\subseteq [i+1, k-1] \subseteq \mathbb{N}\big) \tag{Id}$$

$$\vee \ \mathsf{set}(\mathsf{col}(b')) \neq \circ)\Big). \tag{Ie}$$

Hence, if a partition $\mathsf{f}(\pi) \notin \mathsf{NC}_\circ\mathsf{A}_\bullet$, it can be satisfied by 3 different case, i. e., the terms in (Ia), (Ib) and at least one of the 3 terms in (Ic), (Id) or (Ie) is true.

Assume for a partition $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ and for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ that the following statement is true

$$(\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet) \wedge \underbrace{\big((\mathrm{Ia}) \wedge (\mathrm{Ib}) \wedge (\mathrm{Ie})\big)} \tag{II}$$

$\widehat{=}$  .

Hence, the block in $\mathsf{f}(\pi)$ which inherits leg j but not i and k, has the property that at least two legs have different colors. It can be shown that this also holds for the partition of $(\mathsf{g} \circ \mathsf{f})(\pi)$, where g is the left inverse of f. The proof is analogous to the proof of Lemma 4.2.18 and Lemma 4.2.20. Finally, this leads to a contradiction to the assumption that $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$. Therefore, the statement of (II) is false. A similar statement holds for UMem and split.

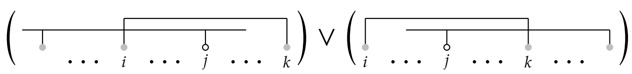
Assume for a partition $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ and for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ that the following statement is true

$$(\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet) \wedge \big((\mathrm{Ia}) \wedge (\mathrm{Ib}) \wedge (\mathrm{Ic})\big). \tag{III}$$

In other words, in $\mathsf{f}(\pi)$ there is no block, such that the leg j is an element of this block. This also holds for the partition $(\mathsf{g} \circ \mathsf{f})(\pi)$, where g is the left inverse of f. This leads to a contradiction to the assumption that $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$. Therefore the statement of (III) is false. A similar statement holds for UMem and split.

Assume for a partition $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ and for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ that the following statement is true

$$(\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet) \wedge \underbrace{\big((\mathrm{Ia}) \wedge (\mathrm{Ib}) \wedge (\mathrm{Id})\big)} \tag{IV}$$

$\widehat{=}$  . (V)

This means in the partition $\mathsf{f}(\pi)$ the block which inherits the leg j but not the legs i and k has the property, that it has a crossing with the block of the legs i and k. If we track the relative order of the legs i, j and k in the partition $(\mathsf{g} \circ \mathsf{f})(\pi)$, where g is the left inverse of f, then the crossing of (V) remains. This is a contradiction to the assumption that $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$. A similar statement holds for UMem and split.

So far we have shown that (II) ∨ (III) ∨ (IV) is false, therefore its negation must be true, which is $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet \implies \mathsf{f}(\pi) \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ for $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$. Analogously for UMem and split. This shows that $\mathsf{NC}_\circ\mathsf{A}_\bullet$ is a two-colored universal class of partitions. □

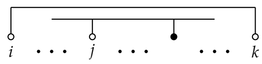
4.2.25 Lemma. The set of all crossing-noncrossing partitions $\mathsf{A}_\circ\mathsf{NC}_\bullet$ is a two-colored universal class of partitions.

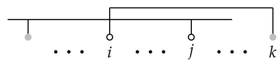
Proof: The is similar to the proof of Lemma 4.2.24. □

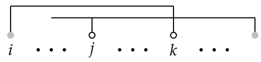
4.2.26 Lemma. The set of all binoncrossing partitions biNC is a two-colored universal class of partitions.

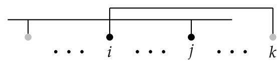
Proof: We prove this property by contradiction. Therefor we need implications for a partition not being an element in the set biNC. Let $\pi \in \mathsf{biNC}$ and $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$. If we perform the negation of the expressions of (4.2.24), then we obtain $\mathsf{f}(\pi) \notin \mathsf{biNC} \iff$

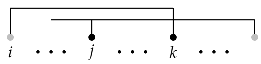
$$\exists j \in [n],\ \exists i \in [j-1],\ \exists k \in [n] \setminus [j]: \tag{I}$$

 occurs in $\mathsf{f}(\pi)$ (II)

or  occurs in $\mathsf{f}(\pi)$ (III)

or  occurs in $\mathsf{f}(\pi)$ (IV)

or  occurs in $\mathsf{f}(\pi)$ (V)

or  occurs in $\mathsf{f}(\pi)$. (VI)

Assume $\pi \in \mathsf{biNC}$ and the situation characterized in the diagram in (II). This is a special case of a case which has already been treated in the proof of Lemma 4.2.24. The difference is that the color of the legs i and k have been specified to $\circ$. Therefore, it can be shown that a similar diagram as (II) occurs in the partition $(\mathsf{g} \circ \mathsf{f})(\pi)$, where g is the left inverse of f. But this leads to a contradiction to the assumption that $\pi \in \mathsf{biNC}$. Thus, the statement $\pi \in \mathsf{biNC} \wedge$ (I) $\wedge$ (II) is false. A similar statement holds for UMem and split.

Assume now $\pi \in \mathsf{biNC}$ and the situation characterized in the diagram in (III). This case is characterized by the property that in the partition $\mathsf{f}(\pi)$ exists a crossing, where the interior legs of the crossing both share the same color, i. e., in this case they have the color $\circ$. We have to show that this property also holds for the partition $(\mathsf{g} \circ \mathsf{f})(\pi)$, where g is the left inverse of f. We already know that from the case $m = 1$ that the following implication is true

$$\pi \in \mathsf{NC} \implies \mathsf{f}(\pi) \in \mathsf{NC}.$$

This is equivalent to the implication

$$\mathsf{f}(\pi) \notin \mathsf{NC} \implies \pi \notin \mathsf{NC}.$$

Let $\pi \in \mathsf{biNC}$ and $\mathsf{f}(\pi) \notin \mathsf{biNC}$, then by the above implication we obtain

$$\mathsf{f}\big(\mathsf{type}(\pi)\big) \notin \mathsf{NC} \implies (\mathsf{g} \circ \mathsf{f})\big(\mathsf{type}(\pi)\big) \notin \mathsf{NC},$$

where the crossing in $(\mathsf{g} \circ \mathsf{f})\big(\mathsf{type}(\pi)\big)$ is caused by the blocks of the legs i and j in the partition π. It is clear that the colors of the legs i and j can not be changed by any left inverse g of f. This shows that in the partition $(\mathsf{g} \circ \mathsf{f})(\pi)$ a similar situation as in (III) occurs which is a contradiction to the assumption $\pi \in \mathsf{biNC}$. Therefore the statement $\pi \in \mathsf{biNC} \wedge$ (I) $\wedge$ (III) is false. A similar statement holds for UMem and split.

Analogously we can show that the statements

$$\pi \in \mathsf{biNC} \wedge \text{(I)} \wedge \text{(IV)},$$

$$\pi \in \mathsf{biNC} \wedge \text{(I)} \wedge \text{(V)},$$

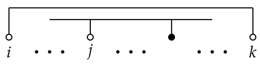
$$\pi \in \mathsf{biNC} \wedge \text{(I)} \wedge \text{(VI)}$$

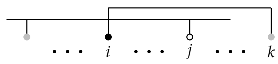
are false. A similar statement holds for UMem and split. In all cases we have led the assumption $\pi \in \mathsf{biNC} \wedge \mathsf{f}(\pi) \notin \mathsf{biNC}$ to a contradiction and the assertion follows by proof of contradiction. □

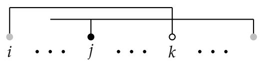
4.2.27 Lemma. The set of all pure crossing partitions pureC is a two-colored universal class of partitions.

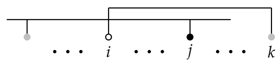
Proof: The proof of this assertion is similar to the proof of Lemma 4.2.26. W use the notation from there. This means for any $\mathsf{f} \in \{\mathsf{delete}, \mathsf{double}, \mathsf{cCol}, \mathsf{mirror}\}$ that a partition $\mathsf{f}(\pi) \notin \mathsf{pureC}$ is equivalently characterized by the following properties

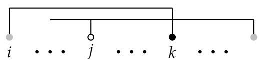
$$\exists j \in [n], \exists i \in [j-1], \exists k \in [n] \setminus [j]:$$

 occurs in $\mathsf{f}(\pi)$

or  occurs in $\mathsf{f}(\pi)$

or  occurs in $\mathsf{f}(\pi)$

or  occurs in $\mathsf{f}(\pi)$

or  occurs in $\mathsf{f}(\pi)$.

The last four diagrams show that there are two blocks which lead to a crossing and the interior legs of this crossing have different colors. This can be seen in similarity to the proof of Lemma 4.2.26, where an occurring crossing with same colored inner legs forbids the belonging of a partition to the set biNC. Therefore, the proof is analogously done and the assertion follows. □

4.2.28 Definition (Reduced partition, Generating set of partitions). Let $n \in \mathbb{N}$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$.

(a) We call a partition $\pi \in \mathsf{Part}_{\{\circ,\bullet\}}$ *reduced* if and only if for each block b in π there do not exist block neighboring legs (with the same color) in the block b. For any given partition $\pi \in \mathsf{Part}_{\{\circ,\bullet\}}$ there exists a unique reduced partition denoted by $\operatorname{red}\pi \in \mathsf{Part}$, which we obtain by induction and the map delete and cCol in order to "delete" all block neighboring legs within a block.

(b) Let $\mathsf{Part}_{\{\circ,\bullet\}}$ be the set of all two-colored partitions. Let $G \subseteq \mathsf{Part}_{\{\circ,\bullet\}}$. Since two-colored universal classes of partitions are stable under intersection, we may define another two-colored universal class of partitions

$$\mathsf{Gen}(G) := \bigcap_{\substack{\mathcal{P}\supseteq G \\ \mathcal{P} \text{ a 2-col. UCP}}} \mathcal{P}. \tag{4.2.27}$$

We also say that G *generates a partition* $\pi \in \mathsf{Part}$ if and only if $\pi \in \mathsf{Gen}(G)$. We can extend this terminology to sets of partitions $A \subseteq \mathsf{Part}$ and say that G *generates a set of partitions* A if and only if $A \subseteq \mathsf{Gen}(G)$. If G consists of only one element, i. e., $G = \{g\}$, then we write $\mathsf{Gen}(g)$ instead of $\mathsf{Gen}(\{g\})$). By this we define

4.2.29 Remark.

(a) Let $\mathcal{P} \subseteq \mathsf{Part}_{\{\circ,\bullet\}}$ be a universal class of partitions. Then, for each $n \in \mathbb{N}$ and $\varepsilon \in \{\circ, \bullet\}^{\times n}$ we have a surjective map $\operatorname{red}(\mathcal{P}_{\varepsilon,2}) \longrightarrow \mathcal{P}_{\varepsilon,2}$, i. e., by a finite application of the map double and cCol we can start from a reduced two-block partition and can generate any two-block partition of $\mathcal{P}_{\varepsilon,2}$.

(b) This is a rather informal discussion of the expression $\mathsf{Gen}(G)$ if $G \subseteq \mathsf{Part}_{\{\circ,\bullet\}}$. We recursively define for all $n \in \mathbb{N}$

$$\begin{aligned} G^{(1)} &:= \bigcup_{\mathsf{f} \in \left\{ \substack{\mathsf{delete},\, \mathsf{double},\, \mathsf{pr}_1 \circ \mathsf{UMem}, \\ \mathsf{pr}_2 \circ \mathsf{UMem},\, \mathsf{cCol},\, \mathsf{mirror},\, \mathsf{id}} \right\}} \mathrm{im}\big(\mathsf{f} \circ \mathsf{inc}_{G,\mathsf{Part}}\big) \\ &\qquad \cup \mathrm{im}\big(\mathsf{split} \circ \mathsf{inc}_{G \times G, \mathsf{Part} \times \mathsf{Part}}\big) \end{aligned} \tag{4.2.28a}$$

$$\begin{aligned} G^{(n+1)} &:= \bigcup_{\mathsf{f} \in \left\{ \substack{\mathsf{delete},\, \mathsf{double},\, \mathsf{pr}_1 \circ \mathsf{UMem}, \\ \mathsf{pr}_2 \circ \mathsf{UMem},\, \mathsf{cCol},\, \mathsf{mirror},\, \mathsf{id}} \right\}} \mathrm{im}\big(\mathsf{f} \circ \mathsf{inc}_{G^{(n)},\mathsf{Part}}\big) \\ &\qquad \cup \mathrm{im}\big(\mathsf{split} \circ \mathsf{inc}_{G^{(n)} \times G^{(n)}, \mathsf{Part} \times \mathsf{Part}}\big), \end{aligned} \tag{4.2.28b}$$

wherever the above maps $\mathsf{delete}_{\cdot,\cdot}$, $\mathsf{double}_{\cdot,\cdot}$, $\mathsf{UMem}^{\cdot}_{\cdot,\cdot}$, $\mathsf{split}^{\cdot}_{\cdot,\cdot}$, $\mathsf{cCol}_{\cdot,\cdot}$, $\mathsf{mirror}_{\cdot}$ might be well-defined for valid index choices. In this setting we can see that

$$\mathsf{Gen}(G) = \bigcup_{n \in \mathbb{N}} G^{(n)}. \tag{4.2.29}$$

(c) Let $G \subseteq \mathsf{Part}_{\{\circ,\bullet\}}$, then $G \subseteq \mathsf{Gen}(G)$. If $G \subseteq \mathcal{P} \subseteq \mathsf{Part}$, where $\mathcal{P}$ is some universal class of partitions, then we have $\mathsf{Gen}(G) \subseteq \mathcal{P}$.

4.2.30 Lemma. Let G', $G \subseteq \mathsf{Part}_{\{\circ,\bullet\}}$ and assume $G' \subseteq G$.

TFAE: **(a)** $\mathsf{Gen}(G') = \mathsf{Gen}(G)$,

(b) $G \subseteq \mathsf{Gen}(G')$.

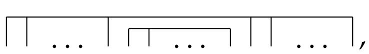
PROOF: The equivalence of the assertions follows from Remark 4.2.29 **(c)**. □

For the following proofs of lemmas we need to determine the set of all two-block partitions for a given two-colored universal class of partitions. For $\varepsilon = (\varepsilon_i)_{i \in [n]} \in \{\circ, \bullet\}^{\times n}$ and a partition $\pi \in \mathsf{Part}_\varepsilon$ with two blocks we can see from Definition 4.2.13 that we need to distinct between three cases for $\mathsf{type}(\pi)$:

- $\mathsf{type}(\pi)$ can be an interval partition, i. e.,

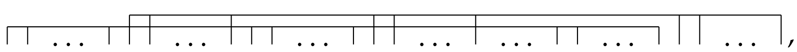
(4.2.30)

- $\mathsf{type}(\pi)$ can be a noncrossing partition which is not an interval partition, i. e.,

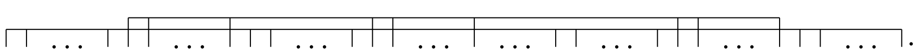


(4.2.31)

- $\mathsf{type}(\pi)$ can be a crossing partition, i. e.,



(4.2.32a)



(4.2.32b)

The difference between diagram (4.2.32a) and diagram (4.2.32b) is that the first and the last leg lie in different blocks or the first and the last leg are in the same block.

If we want to determine all two-block partitions for a two-colored universal class of partitions, we need to decide how we are allowed to "decorate" the legs by $\circ$ or $\bullet$ in the diagrams (4.2.30), (4.2.31) and (4.2.32). This is one of the tasks we have to look in the following.

4.2.31 Lemma. Let $\mathcal{P} \subseteq \mathsf{Part}_{\{\circ,\bullet\}}$ be an universal class of partitions and assume $G \subseteq \mathcal{P}$ is given.

TFAE: **(a)** $\mathsf{Gen}(G) = \mathcal{P}$,

(b) $\mathrm{red}(\mathcal{P}_{\cdot,2}) \subseteq \mathsf{Gen}(G)$, i. e., G generates all reduced two-block partitions of $\mathcal{P}$.

PROOF: The equivalence of both assertions follows from Proposition 4.2.11, Remark 4.2.29 **(c)** and **(a)**. □

4.2.32 Lemma. It holds that $\mathsf{Gen}(\,\downarrow\,\downarrow\,) = \mathsf{I}_{\{\circ,\bullet\}}$.

PROOF: We need to determine all reduced two-block partitions of $\mathsf{I}_{\{\circ,\bullet\}}$. The first step is determine a general form for an arbitrary two-block partition of $\mathsf{I}_{\{\circ,\bullet\}}$. Therefor we have to decide how we can attach colors to the legs of a two-block partition π of equations (4.2.30), (4.2.31) and (4.2.32) such that the expression of (4.2.15a) is fulfilled. But in this case it is trivial because according to Definition 4.2.13 **(b)** we have $\pi \in \mathsf{I}_{\{\circ,\bullet\}} :\Longleftrightarrow \mathsf{type}(\pi) \in \mathsf{I}$. Therefore, an arbitrary two-block partition of $\mathsf{I}_{\{\circ,\bullet\}}$ has the form

$(1,\varepsilon_1)\ \cdots\ (\ell,\varepsilon_\ell)\ \ (\ell+1,\varepsilon_{\ell+1})\ \cdots\ (n,\varepsilon_n)$, (I)

where $(\varepsilon_i)_{i\in[n]} \in \{\circ,\bullet\}^{\times n}$. By Definition 3.4.9 **(f)** we know that the first and the last leg in each partition can take any color and by a finite application of double we see that $\downarrow\,\downarrow$ can generate any partition of the form from equation (I). We can now use Lemma 4.2.31 and the assertion follows. □

4.2.33 Lemma.

(a) $\mathsf{Gen}(\sqcap\!\sqcap) = \mathsf{Gen}(\{\downarrow\,\downarrow, \sqcap\!\sqcap\})$,

(b) $\mathsf{Gen}(\{\downarrow\,\downarrow, \sqcap\!\sqcap\}) = \mathsf{I}_\circ\mathsf{NC}_\bullet$.

PROOF: AD **(a)**: From Lemma 4.2.30 we can see that this statement is shown by proving the equation

$$\downarrow\,\downarrow \in \mathsf{Gen}(\sqcap\!\sqcap). \tag{I}$$

For this, we consider the following calculation, where we abbreviate $\mathcal{G} := \mathsf{Gen}(\sqcap\!\sqcap)$

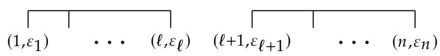
$$\begin{aligned} &\pi = \sqcap\!\sqcap \in \mathcal{G} \\ \Longrightarrow\ &\pi_1 = \sqcap\!\sqcap\!\sqcap \in \mathcal{G} \quad [\![\, \pi_1 = \mathsf{double}_{\cdot,3}(\pi) \,]\!] \\ \Longrightarrow\ &\pi_2 = \sqcap\!\sqcap\!\sqcap \in \mathcal{G} \quad [\![\, \pi_2 = \mathsf{split}^3_{\cdot,2}(\pi_1,\pi) \,]\!] \\ \Longrightarrow\ &\pi_3 = \downarrow\,\downarrow \in \mathcal{G} \quad [\![\, \pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^2_{\cdot,3})(\pi_2) \,]\!]. \end{aligned}$$

This proves the statement of equation (I).

AD **(b)**: We claim

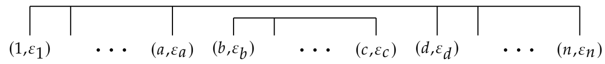
$$\mathrm{red}\big((\mathsf{I}_\circ\mathsf{NC}_\bullet)_{\cdot,2}\big) \subseteq \mathsf{Gen}(\{\downarrow\,\downarrow, \sqcap\!\sqcap\}). \tag{II}$$

We determine therefor all reduced two-block partitions of $\mathsf{I}_\circ\mathsf{NC}_\bullet$. Let $\pi \in \mathrm{red}(\mathsf{I}_\circ\mathsf{NC}_\bullet)$ be a partition consisting of two blocks. According to equation (4.2.16a) we have $\mathsf{type}(\pi) \in \mathsf{NC}$. Then, type($\pi$) is either of type from diagram (4.2.30) or of type from diagram (4.2.31). If type(π) is an interval partition, then π has the form



, (III)

where $(\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$. This is in accordance to diagram (4.2.16c). If type(π) is noncrossing, but not of the type from equation (III), then we have to decide how we can attach colors to the legs. Hence in this case π has the form



. (IV)

Here we have $\forall i \in \{1,\ldots,a\} \cup \{d,\ldots,n\} \subseteq \mathbb{N}\colon \varepsilon_i \in \{\circ, \bullet\}$, which is in accordance to diagram (4.2.16c). We are left to determine possible cases for $i \in \{b,\ldots,c\} \subseteq \mathbb{N}$. We obtain that $\forall i \in \{b,\ldots,c\} \subseteq \mathbb{N}\colon \varepsilon_i = \bullet$. Any other choice for ε_i leads to a contradiction of diagram (4.2.16c). By Definition 3.4.9 **(f)** we know that the first and the last leg in each partition can take any color and by finite application of double we see that can generate any partition of the form from equation (III). Moreover by the same reasoning we can see that generates any partition of the form from (IV), in other words we have shown equation (II). We can now use Lemma 4.2.31 and the assertion follows. □

4.2.34 Lemma.

(a) Gen( ) = Gen({ , }),

(b) Gen({ , }) = $\mathsf{NC}_\circ\mathsf{I}_\bullet$.

PROOF: The proof is similar to the proof of Lemma 4.2.33. □

4.2.35 Lemma.

(a) Gen({ , }) = Gen({ , , }),

(b) Gen({ , , }) = pureNC.

PROOF: AD **(a)**: By application of Lemma 4.2.30 we can see that this statement is similar to Lemma 4.2.33 **(a)**.
AD **(b)**: We claim

$$\mathsf{red}\big((\mathsf{pureNC})_{\cdot,2}\big) \subseteq \mathsf{Gen}\big(\{\ ,\ ,\ \}\big). \tag{I}$$

We determine therefor the set of all reduced two-block partitions of pureNC. Let $\pi \in$ red(pureNC) which has two blocks. Then due to equation (4.2.18a) we have type(π) $\in$ NC. If type(π) is an interval partition, then from the above we already know that generates π. From the proof of Lemma 4.2.33 it is clear that or generates π in the case that type(π) is noncrossing but not an interval. This is in accordance to diagram (4.2.18c). Thus, we have shown equation (I) and the assertion now follows from Lemma 4.2.31. □

4.2.36 Lemma.

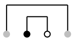
(a) Gen( ) = Gen({ , , , }),

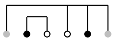
(b) Gen({ , , , }) = $\mathsf{NC}_{\{\circ,\bullet\}}$.

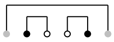
PROOF: AD **(a)**: Set $\mathcal{G} :=$ Gen( ). By application of Lemma 4.2.30 it suffices to show that

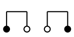
$$\{\ ,\ ,\ \} \subseteq \mathcal{G}.$$

Therefor we calculate

$\pi :=$  $\in \mathcal{G}$

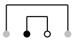
$\Longrightarrow \pi_1 :=$  $\in \mathcal{G}$ ⟦ two times application of double ⟧

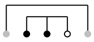
$\Longrightarrow \pi_2 :=$  $\in \mathcal{G}$ ⟦ $\pi_2 = \mathsf{split}^3_{\cdot,2}(\pi_1, \mathsf{mirror}(\pi))$ ⟧

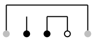
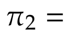
$\Longrightarrow \pi_3 :=$  $\in \mathcal{G}$ ⟦ $\pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^3_{\cdot,3})(\pi)$ ⟧

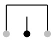
$\Longrightarrow \pi_4 :=$  $\in \mathcal{G}$ ⟦ 2 times application of delete ⟧

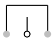
Furthermore, we have

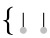
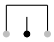
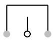
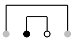
$\pi :=$  $\in \mathcal{G}$

$\Longrightarrow \pi_1 :=$  $\in \mathcal{G}$ ⟦ $\pi_1 = \mathsf{double}_{\cdot,3}(\pi)$ ⟧

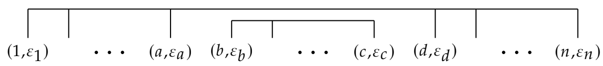
$\Longrightarrow \pi_2 :=$  $\in \mathcal{G}$ ⟦ $\pi_2 = \mathsf{split}^2_{\cdot,2}(\pi_1,$ $)$ ⟧

$\Longrightarrow \pi_3 :=$  $\in \mathcal{G}$ ⟦ $\pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,3})(\pi_2)$ ⟧

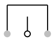
Likewise, we can prove  $\in \mathcal{G}$, which finishes the proof of **(a)**.
Ad **(b)**: We claim

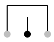
$\mathrm{red}\big((\mathsf{NC}_{\{\circ,\bullet\}})_{\cdot,2}\big) \subseteq \mathsf{Gen}\big(\{$ , , ,  $\}\big).$ (I)

We determine therefor the set of all reduced two-block partitions of $\mathsf{NC}_{\{\circ,\bullet\}}$. Let $\pi \in \mathrm{red}(\mathsf{NC}_{\{\circ,\bullet\}})$ and assume it has two blocks. Due to equation (4.2.19) we have $\mathsf{pr}_1(\pi) \in \mathsf{NC}$. If $\mathsf{pr}_1(\pi)$ is an interval, then we already know that  generates π. In the other case π can have the form

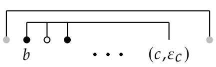
 . (II)

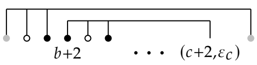
Here we have $(\varepsilon_i)_{i\in[n]} \in \{\circ, \bullet\}^{\times n}$. We make the following case considerations.

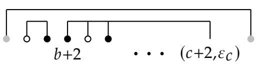
Assume $\forall\, i \in \{b, \dots, c\} \subseteq \mathbb{N}\colon \varepsilon_i = \circ$, then we already know that  generates π.

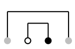
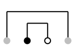
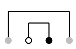
Assume $\forall\, i \in \{b, \dots, c\} \subseteq \mathbb{N}\colon \varepsilon_i = \bullet$, then we already know that  generates π.

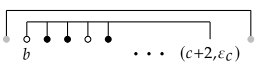
Assume ε_i is alternating between $\bullet$ and $\circ$ and starts with $\varepsilon_b = \bullet$. Let n' denote the number of legs with color $\bullet$ in the interval $[b, c] \subseteq \mathbb{N}$, then we prove by induction over $n' \in \mathbb{N}$ that we can generate any noncrossing, non-interval partition with the above properties. We only prove the induction step, since the induction base is analogously proven. Therefore, assume the assertion is true for some $n' \in \mathbb{N}$. Then, for $n \to n+1$ we consider

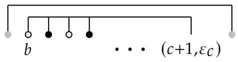
$\pi =$  $\in \mathsf{NC}_{\{\circ,\bullet\}}$ ⟦ induction hypothesis ⟧

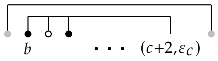
$\Longrightarrow \pi_1 =$  $\in \mathsf{NC}_{\{\circ,\bullet\}}$ ⟦ $\pi_1 = (\mathsf{double}_{\cdot,1})^2(\pi)$ ⟧

$\Longrightarrow \pi_2 =$  $\in \mathsf{NC}_{\{\circ,\bullet\}}$

⟦  $= \mathsf{mirror}\Big($  $\Big) \in \mathsf{NC}_{\{\circ,\bullet\}},\ \pi_2 = \mathsf{split}^1_{\cdot,2}\Big(\pi_1,$  $\Big)$ ⟧

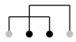
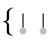
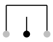
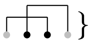
$\Longrightarrow \pi_3 =$  $\in \mathsf{NC}_{\{\circ,\bullet\}}$ ⟦ $\pi_3 = (\mathsf{pr}_1 \circ \mathsf{UMem}^{b+1}_{\cdot,3})(\pi_2)$ ⟧

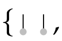
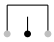
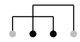
$\Longrightarrow \pi_4 =$  $\in \mathsf{NC}_{\{\circ,\bullet\}}$ $[\![\, \pi_4 = \mathsf{delete}_{\cdot,b+2}(\pi_2) \,]\!]$

$\Longrightarrow \pi_5 =$  $\in \mathsf{NC}_{\{\circ,\bullet\}}$ $[\![$ analogous reasons from above $]\!]$

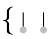
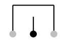
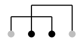
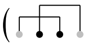
In a similar way we can generate any partition of type from equation (II), where ε_i is alternating between $\bullet$ and $\circ$ and starts with $\varepsilon_b = \circ$. This characterizes all reduced two-block partitions of $\mathsf{NC}_{\{\circ,\bullet\}}$ because the first and the last leg in a partition can take any color, which allows us to generate an arbitrary sequence of legs with $\bullet$ and $\circ$ attached to them. Thus, we have shown equation (I). We can now use Lemma 4.2.31 and the assertion follows. □

4.2.37 Lemma.

(a) Gen() = Gen({ , ,  }),

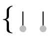
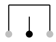
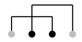
(b) Gen({ , ,  }) = $\mathsf{I}_\circ\mathsf{A}_\bullet$.

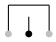
Proof: Ad **(a)**: By application of Lemma 4.2.30 it suffices to show that

{ , ,  } $\subseteq$ Gen().

For the above reduced partitions of consideration, legs with the color $\circ$ can only occur as the first or last leg. Therefore, it is clear that we can apply the same methods as in the case $m = 1$, since we can assume that the first and last leg have the color $\bullet$. We refer for a proof to the proof of Theorem 4.1.17.

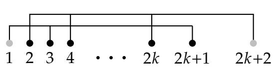
Ad **(b)**: We claim

red$\big((\mathsf{I}_\circ\mathsf{A}_\bullet)_{\cdot,2}\big)$ $\subseteq$ Gen({ , ,  }). (I)

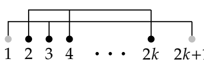
We determine the set of all reduced two-block partitions of $\mathsf{I}_\circ\mathsf{A}_\bullet$. Therefor we note that a partition π which is interval-crossing shares the same properties of a partition, which is interval-noncrossing except that $\mathsf{type}(\pi)$ does not need to be noncrossing. Hence, it is clear that the set { ,  } generates all reduced two-block partitions π such that $\mathsf{type}(\pi) \in \mathsf{NC}$. It remains to determine two-block partitions of $\mathsf{I}_\circ\mathsf{A}_\bullet$ which are of type from equation (4.2.32), i. e., which possess a crossing. According to Definition 4.2.12 we need to consider the set $\mathsf{Cross}(\pi)$. From equation (4.2.20a) we obtain

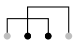
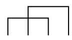
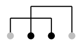
$$\forall \pi \in \mathsf{I}_\circ\mathsf{A}_\bullet : \Big(\big((p_1,p_2),(q_1,q_2)\big) \in \mathsf{Cross}(\pi) \implies \varepsilon_{p_2} = \varepsilon_{q_1} = \bullet \Big).$$

We also say that if a crossing in a partition π occurs, then the inner legs of any crossing need to have the color $\bullet$. Legs with color $\circ$ can not participate in a crossing as inner legs. Therefore the most general forms for a reduced partition $\pi \in \mathsf{I}_\circ\mathsf{A}_\bullet$ with crossing are the following 2 types

$\forall k \in \mathbb{N}:$  $\in \mathsf{I}_\circ\mathsf{A}_\bullet$ (II)

or

$\forall k \in \mathbb{N} \setminus \{1\}:$  $\in \mathsf{I}_\circ\mathsf{A}_\bullet$.

We want to regard  as a partition of type from equation (II) for $k = 1$. In the single-colored case we can see from Lemma 4.1.16 that the partition  generates all reduced two-block partitions with crossing. For the proof of the claim that the partition  generates all reduced two-block partitions of $\mathsf{I}_\circ\mathsf{A}_\bullet$ with crossing, we refer therefore to the above $m = 1$ case. We can do this because the first and the last leg in such partitions can take any color and only legs with the color $\bullet$ can participate as inner legs of crossings. This shows the statement of equation (I) and the assertion now follows from Lemma 4.2.31. □

4.2.38 Lemma.

(a) Gen( ) = Gen({ , , }),

(b) Gen({ , , }) = $\mathsf{A}_\circ\mathsf{I}_\bullet$.

PROOF: The proof is similar to the proof of Lemma 4.2.37. □

4.2.39 Lemma.

(a) Gen({ , }) = Gen({ , , , }),

(b) Gen({ , , , }) = $\mathsf{NC}_\circ\mathsf{A}_\bullet$.

PROOF: AD **(a)**: By application of Lemma 4.2.30 it suffices to show that

$$\{ \ , \ , \ , \ \} \subseteq \mathsf{Gen}(\{ \ , \ \})$$

The proof can be taken from the proof of Lemma 4.2.37 **(a)**.
AD **(b)**: Since { , , , } ⊆ $\mathsf{NC}_\circ\mathsf{A}_\bullet$ we may apply Lemma 4.2.31 to prove this assertion. Thus, we need to determine the set of reduced two-block partitions of $\mathsf{NC}_\circ\mathsf{A}_\bullet$. We can see from equation (4.2.22a) that the condition for legs with color $\circ$ of a partition π needs to satisfy such that $\pi \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ is the same condition as in equation (4.2.18b) for a partition π to be pure noncrossing. Equation (4.2.22a) does not impose any condition on legs with color $\bullet$. Therefore, reduced two-block partitions of $\mathsf{NC}_\circ\mathsf{A}_\bullet$, which are noncrossing, are generated by { , , } according to the proof of Lemma 4.2.35. Like in the case of partitions, which are interval-crossing, inner legs of an occurring crossing in a noncrossing-crossing partition need to be of color $\bullet$. Therefore, we can show as in the proof of Lemma 4.2.37, that the partition generates all reduced two-block partitions of $\mathsf{NC}_\circ\mathsf{A}_\bullet$ with crossing. □

4.2.40 Lemma.

(a) Gen({ , }) = Gen({ , , , }),

(b) Gen({ , , , }) = $\mathsf{A}_\circ\mathsf{NC}_\bullet$.

PROOF: The proof is similar to the proof of Lemma 4.2.39. □

4.2.41 Lemma.

(a)) Gen( ) = Gen{ , , , }),

(b) Gen({ , , , }) = biNC.

PROOF: AD **(a)**: Set $\mathcal{G} := \mathsf{Gen}(\ )$. By application of Lemma 4.2.30 it suffices to show that

$$\{ \ , \ , \ \} \subseteq \mathcal{G}. \tag{I}$$

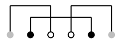
We set

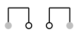
$$\mathcal{G} \ni \pi_1 := \mathsf{mirror}(\ ) = \ = \ .$$

By this we calculate

$$\pi = $$

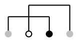
$$\Longrightarrow \pi_2 := \ \in \mathcal{G} \quad [\![\ \pi_2 \mathrel{\widehat{=}} \text{two times application of } \mathsf{double}_{\cdot,1} \text{ to } \pi\ ]\!]$$

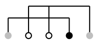
$\Longrightarrow \pi_3 :=$  $\in \mathcal{G}$ $[\![\, \pi_3 = \mathsf{split}^1_{\cdot,2}(\pi_2, \pi_1) \,]\!]$

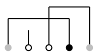
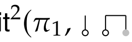
$\Longrightarrow \pi_4 :=$  $\in \mathcal{G}$ $[\![\, (\mathsf{pr}_2 \circ \mathsf{UMem}^2_{\cdot,3})(\pi_3) \,]\!]$

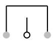
$\Longrightarrow$  $\in \mathcal{G}$ $[\![$ two times application of $\mathsf{delete}_{\cdot,\cdot}$ to π_4 $]\!]$

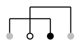
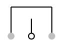
Furthermore, we calculate

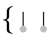
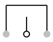
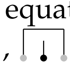
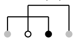
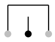
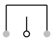
$\pi =$  $\in \mathcal{G}$

$\Longrightarrow \pi_1 :=$  $\in \mathcal{G}$ $[\![\, \pi_1 = \mathsf{double}_{\cdot,2}(\pi) \,]\!]$

$\Longrightarrow \pi_2 :=$  $\in \mathcal{G}$ $[\![\, \pi_2 = \mathsf{split}^2(\pi_1,$  $) \,]\!]$

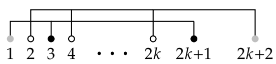
$\Longrightarrow$  $\in \mathcal{G}$

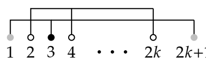
We have an analogous calculation for the statement  $\in \mathcal{G} \Longrightarrow$  $\in \mathcal{G}$. The above calculations show that equation (I) holds.

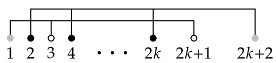
Ad **(b)**: Since $\{$ , , ,  $\} \subseteq \mathsf{biNC}$ we shall apply Lemma 4.2.31 to prove this assertion. Therefor we need to determine the set of all reduced two-block partitions of biNC. As in the previous cases it is clear that $\{$ , ,  $\}$ generates all reduced two-block partitions of biNC which are noncrossing. It remains to determine reduced two-block partitions of biNC which are crossing. From Definition 4.2.12 **(k)** we obtain

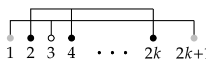
$$\forall \pi \in \mathsf{biNC}\colon \Big(\big((p_1, p_2), (q_1, q_2)\big) \in \mathsf{Cross}(\pi) \implies (\varepsilon_{p_2} = \circ,\ \varepsilon_{q_1} = \bullet) \vee (\varepsilon_{p_2} = \bullet,\ \varepsilon_{q_1} = \circ) \Big).$$

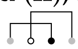
Hence, we can say that whenever a crossing in a partition $\pi \in \mathsf{biNC}$ occurs, the inner legs need to differ in their colors. This gives us four types of reduced two-block partitions of biNC which are crossing, namely

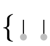
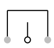
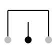
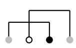
$\forall k \in \mathbb{N}\colon$  $\in \mathsf{biNC},$ (II)

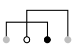
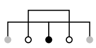
$\forall k \in \mathbb{N} \setminus \{1\}\colon$  $\in \mathsf{biNC},$ (III)

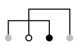
$\forall k \in \mathbb{N}\colon$  $\in \mathsf{biNC},$

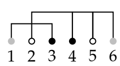
$\forall k \in \mathbb{N} \setminus \{1\}\colon$  $\in \mathsf{biNC}.$

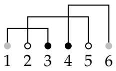
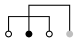
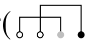
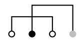
It suffices to show that we can generate partitions of the type from equation (III) and (II), since the other two types can be obtained by application of mirror. We want to regard  as a partition of type from equation (II) for $k = 1$. Put

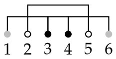
$G := \{$ , , ,  $\}.$

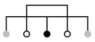
First, we show  $\in G \implies \exists n \in \mathbb{N}\colon$  $\in G^{(n)}$ with notation of $G^{(n)}$ taken from equation (4.2.28b). Consider therefore the following calculation

$\pi =$  $\in G$

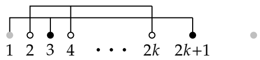
$\Longrightarrow \pi_1 :=$  $\in \mathsf{Gen}(G)$ $[\![\, \pi_1 = (\mathsf{double}_{\cdot,\cdot})^2(\pi) \,]\!]$

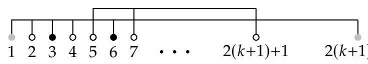
$\Longrightarrow \pi_2 :=$  $\in \mathsf{Gen}(G)$ ⟦  = mirror(), $\pi_2 = \mathsf{split}^4_{\cdot,2}(\pi_1,$  $)$ ⟧

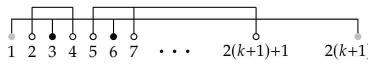
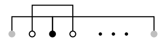
$\Longrightarrow \pi_3 :=$  $\in \mathsf{Gen}(G)$ ⟦ $\pi_3 = (\mathsf{pr}_1 \circ \mathsf{UMem}^3_{\cdot,3})(\pi_2)$ ⟧

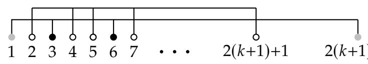
$\Longrightarrow \pi_4 :=$  $\in \mathsf{Gen}(G)$ ⟦ $\pi_4 = \mathsf{delete}_{\cdot,4}(\pi_3)$ ⟧.

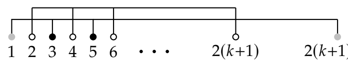
We prove that we can generate partitions of the type from equation (II) by induction over $k \in \mathbb{N}$. The proof that we can generate partitions of the type from equation (III) is done similarly. We just give a proof for the induction step $k \to k+1$ because the proof for the induction base is similar. Hence,

$\pi =$  $\in G$

$\Longrightarrow \pi_1 :=$  $\in \mathsf{Gen}(G)$ ⟦ $\pi_1 = (\mathsf{double}_{\cdot,1})^3(\pi)$ ⟧

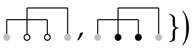
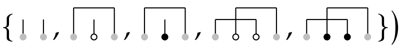
$\Longrightarrow \pi_2 :=$  $\in \mathsf{Gen}(G)$ ⟦ $\pi_2 = \mathsf{split}^3_{\cdot,2}(\pi_1,$  $)$ ⟧

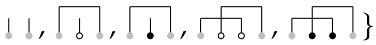
$\Longrightarrow \pi_3 :=$  $\in \mathsf{Gen}(G)$ ⟦ $\pi_3 = (\mathsf{pr}_1 \circ \mathsf{UMem}^4_{\cdot,3})(\pi_2)$ ⟧

$\Longrightarrow \pi_4 :=$  $\in \mathsf{Gen}(G)$ ⟦ $\pi_4 = \mathsf{delete}_{\cdot,5}(\pi_3)$ ⟧.

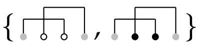
We have shown that the set $\mathsf{Gen}(G)$ contains all reduced two-block partitions of biNC which proves the assertion by Lemma 4.2.31. □

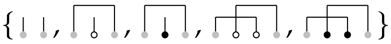
4.2.42 Lemma.

(a) Gen({  }) = Gen({  }),

(b) Gen({  }) = pureC.

Proof: Ad **(a)**: By application of Lemma 4.2.30 it suffices to show that

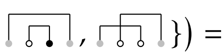
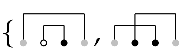
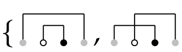
{  } ⊆ Gen({  }).

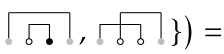
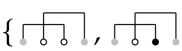
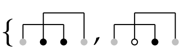
The proof is similar to Lemma 4.2.37 **(a)** and is therefore omitted.
Ad **(b)**: Since {  } ⊆ pureC we shall apply Lemma 4.2.31 to prove the assertion. We need to determine the set of all reduced two-block partitions of pureC. The reasoning is similar to the proof of Lemma 4.2.41 **(b)**. The difference is that now

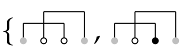
$$\forall \pi \in \mathsf{biNC}\colon \Big(\big((p_1,p_2),(q_1,q_2)\big) \in \mathsf{Cross}(\pi) \implies (\varepsilon_{p_2} = \varepsilon_{q_1} = \circ) \vee (\varepsilon_{p_2} = \varepsilon_{q_1} = \bullet)\Big) \qquad \text{(I)}$$

holds. This means that inner legs of a crossing need to have the same color. The steps from the proof of Lemma 4.2.41 can be easily modified such that they do not violate equation (I) and we obtain that $\mathsf{Gen}(G)$ contains all reduced two-block partitions of pureC which proves the assertion. □

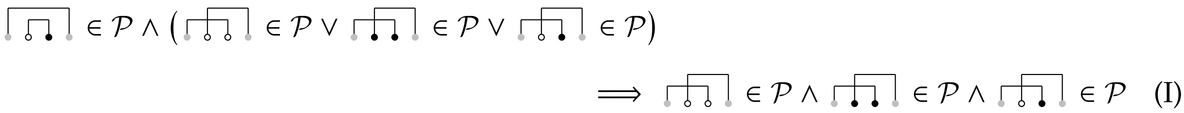
4.2.43 Lemma.

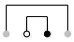
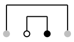
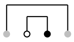
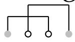
(a) Gen({  }) = Gen({  }) = Gen({  }),

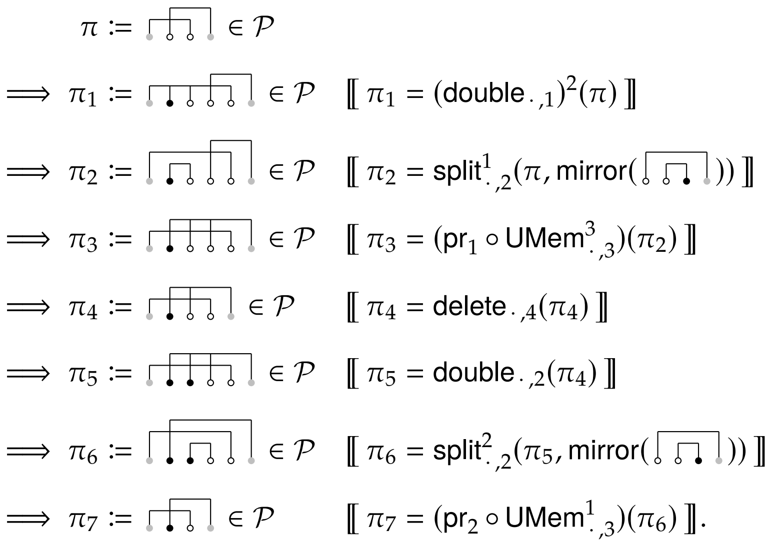
(b) Gen({  }) = Gen({  }) = Gen({  }),

(c) $\mathcal{G} :=$ Gen({  }) = $\mathsf{Part}_{\{\circ,\bullet\}}$.

Proof: Ad **(a)**: We claim that if $\mathcal{P}$ denotes a two-colored universal class of partitions, then

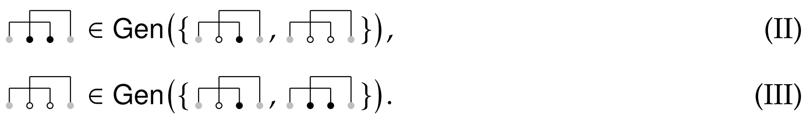


The above equation tells us that whenever  is partition of a universal class of partitions $\mathcal{P}$ and in $\mathcal{P}$ is at least one crossing partition with four legs, then $\mathcal{P}$ inherits all crossing partitions with four legs. Hence the partition  has the potential to change the color of a leg at position 2 and 3 in a crossing partition with four legs. We demonstrate this for the leg at position 2 and assume it has color $\circ$. Hence, if we assume  $\in \mathcal{P}$ and for instance  $\in \mathcal{P}$, then

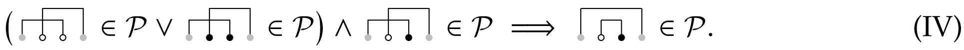


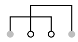
Assume we want to change the color of the leg at position 3 of a crossing partition with four legs. Then, we can apply the above procedure, we just have to additionally use mirror in the first and last step. The above argumentation proves the statement of equation (I). The assertion now follows from equation (I), from the fact that Gen(·) is a universal class of partitions and from Lemma 4.2.30.

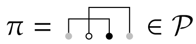
Ad **(b)**: For the proof of the last equality in **(b)** by Lemma 4.2.30 it suffices to show

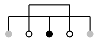


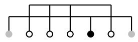
Let $\mathcal{P}$ denote any two-colored universal class of partitions, then we claim

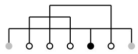
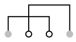


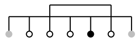
For the proof of the above implication let us assume that  $\in \mathcal{P}$, then we calculate

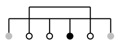


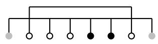
$\implies \pi_1 :=$  $\in \mathcal{P}$ ⟦ proof of Lemma 4.2.41 **(b)** ⟧

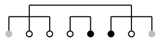
$\implies \pi_2 :=$  $\in \mathcal{P}$ ⟦ $\pi_2 = (\mathsf{double}_{\cdot,2})^2(\pi_1)$ ⟧

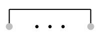
$\implies \pi_3 :=$  $\in \mathcal{P}$ ⟦ $\pi_3 = \mathsf{split}^2_{\cdot,2}(\pi_3,$  $)$ ⟧

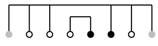
$\implies \pi_4 :=$  $\in \mathcal{P}$ ⟦ $\pi_4 = (\mathsf{pr}_1 \circ \mathsf{UMem}^1_{\cdot,3})(\pi_3)$ ⟧

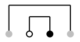
$\implies \pi_5 :=$  $\in \mathcal{P}$ ⟦ $\pi_5 = \mathsf{delete}_{\cdot,2}(\pi_4)$ ⟧

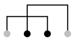
$\implies \pi_6 :=$  $\in \mathcal{P}$ ⟦ $\pi_6 = (\mathsf{double}_{\cdot,5} \circ \mathsf{double}_{\cdot,3})(\pi_5)$ ⟧

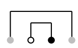
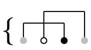
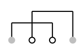
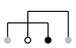
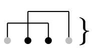
$\implies \pi_7 :=$  $\in \mathcal{P}$

⟦ 2 times application of split(· ,  ),
 $\in \mathcal{P}$ because of Lemma 4.2.41 **(a)** ⟧

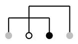
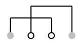
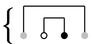
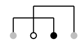
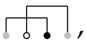
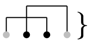
$\implies \pi_8 :=$  $\in \mathcal{P}$ ⟦ 2 times application of $\mathsf{pr}_1 \circ \mathsf{UMem}$ ⟧

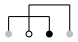
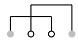
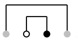
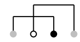
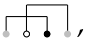
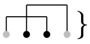
$\implies \pi_9 :=$  $\in \mathcal{P}$ ⟦ successive application of delete ⟧.

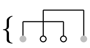
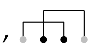
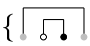
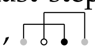
If  $\in \mathcal{P}$, then the proof of equation (IV) is analogous to the above one because $\mathcal{P}$ is closed under application of mirror. Therefore, we omit the proof in this case. Since Gen(·) is a two-colored universal class of partitions, we obtain from equation (IV) that

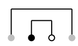
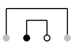
 $\in \mathsf{Gen}(\{$  ,  $\}) \cap \mathsf{Gen}(\{$  ,  $\})$.

This implies

$\mathsf{Gen}(\{$  ,  $\}) \supseteq$ $\{$  ,  $\}$ $\subseteq \mathsf{Gen}(\{$  ,  $\})$

$\implies \mathsf{Gen}(\{$  ,  $\}) \supseteq \mathsf{Gen}(\{$  ,  $\}) \subseteq \mathsf{Gen}(\{$  ,  $\})$, (V)

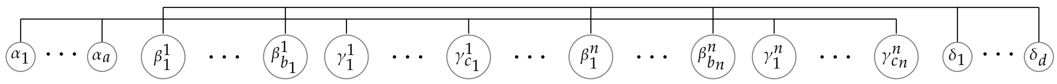
where the last step follows from Remark 4.2.29 **(c)**. From **(a)** we obtain $\{$  ,  $\} \subseteq$ Gen({  ,  }). By this, by an application of Lemma 4.2.30 and by equation (V), we finally obtain the statements of equations (II) and (III).

Ad **(c)**: In order to prove this assertion, we shall use Lemma 4.2.31. Hence, we will show that all reduced two-block partitions of $\mathsf{Part}_{\{\circ,\bullet\}}$ form a subset of $\mathcal{G}$. The first observation is that any partition $\pi \in \mathsf{Part}_{\{\circ,\bullet\}}$ which is noncrossing and where the color of the legs can be freely put needs to be an element of $\mathsf{NC}_{\{\circ,\bullet\}}$. From **(b)** we can conclude that  $\in \mathcal{G}$. This shows that $\mathcal{G}$ inherits all two-colored noncrossing partitions because in Lemma 4.2.36 we have shown that Gen() $= \mathsf{NC}_{\{\circ,\bullet\}}$.

It remains to show that a partition with an "arbitrary" crossing is an element of $\mathcal{G}$. The classes pureC and biNC inherit crossing partitions which need to respect colorizing the legs in a certain way. Therefore, we need to determine the most general form of a reduced, crossing two-block partition where we do not impose any restrictions on the color of the legs. For the most general form of a reduced two-colored two-block partition with a nonzero number of crossings we need to distinct between two cases, namely

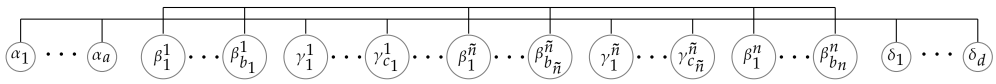
$\forall n \in \mathbb{N},\ \forall a, d \in \mathbb{N},\ (b_i)_{i\in[n]}, (c_i)_{i\in[n]} \in \mathbb{N}^{\times n},\ \forall j \in [n],\ \forall (\alpha_i)_{i\in[a]}, (\beta^j_i)_{i\in[b_j]}, (\gamma^j_i)_{i\in[c_j]}, (\delta_i)_{i\in[d]} \in$

$\mathbb{A}(\{\circ, \bullet\}$:

 $\in \mathcal{G}$ (VI)

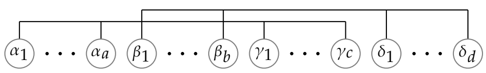
and

$\forall n \in \mathbb{N} \setminus \{1\}$, $\forall a, d \in \mathbb{N}$, $(b_i)_{i \in [n]}, (c_i)_{i \in [n-1]} \in \mathbb{N}^{\times(n-1)}$, $\forall j \in [n]$, $\forall (\alpha_i)_{i \in [a]}, (\beta_i^j)_{i \in [b_j]}, (\gamma_i^j)_{i \in [c_j]}, (\delta_i)_{i \in [d]} \in \mathbb{A}(\{\circ, \bullet\}$:

 $\in \mathcal{G}$. (VII)

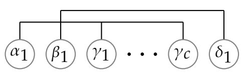
In equation (VII) we have set $\tilde{n} := n - 1$. We have to show that above two types of crossing partitions are contained in $\mathcal{G}$ for any $n \in \mathbb{N}$. First, we prove the existence of crossing partitions of type from equation (VI) in $\mathcal{G}$ by induction over $n \in \mathbb{N}$. For the induction base $n = 1$ we have to show that

$\forall a, b, c, d \in \mathbb{N}$, $\forall \alpha := (\alpha_i)_{i \in [a]}, \beta := (\beta)_{i \in [b]}, \gamma := (\gamma_i)_{i \in [c]}, \delta := (\delta_i)_{i \in [d]} \in \mathbb{A}(\{\circ, \bullet\})$:

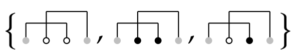
$\mathsf{m2bC}(\alpha, \beta, \gamma, \delta) :=$  $\in \mathcal{G}$. (VIII)

We can think of $\mathsf{m2bC}(\cdot)$ as an abbreviation of "minimal two-block crossing partition". We show the statement of equation (VIII) by induction in several steps. We first show by induction over $c \in \mathbb{N}$ that

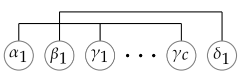
$\forall c \in \mathbb{N}$, $\forall \alpha_1, \beta_1, \gamma := (\gamma_i)_{i \in [c]}, \delta_1 \in \mathbb{A}(\{\circ, \bullet\})$:

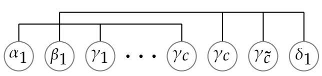
$\mathsf{m2bC}(\alpha_1, \beta_1, \gamma, \delta_1) =$  $\in \mathcal{G}$. (IX)

We notice that **(a)** and **(b)** imply

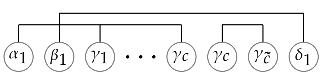
 $\subseteq \mathcal{G}$.

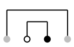
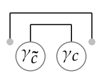
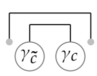
From this we can conclude that the induction base $c = 1$ for equation (IX) holds. Now, we perform the induction step $c \to c + 1 =: \tilde{c}$ and calculate

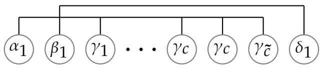
$\pi_1 :=$  $\in \mathcal{G}$

$\implies \pi_2 :=$  $\in \mathcal{G}$

⟦ change color of last leg in π_1 by cCol, double last leg by double, change color of last leg ⟧

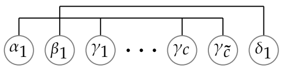
$\implies \pi_3 :=$  $\in \mathcal{G}$

⟦ by **(b)** we have  $\in \mathcal{G}$, therefore  $\in \mathcal{G}$, then use Lem. 4.2.5 to split the legs with color δ_1 and $\gamma_{\tilde{c}}$ in mirror(π_2) by , afterwards apply mirror once more to the split partition ⟧

$\implies \pi_4 :=$ 

$\in \mathcal{G}$

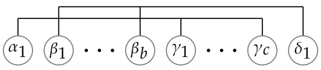
⟦ unify the legs with color γ_c in π_3 by $\mathsf{pr}_1 \circ \mathsf{UMem}$ ⟧

$\implies \pi_5 :=$ 

$\in \mathcal{G}$ ⟦ delete leg with γ_c in π_4 by delete ⟧

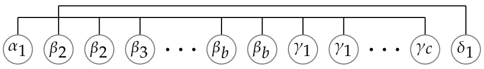
Now, we show that

$\forall b, c \in \mathbb{N}, \forall \alpha_1, \beta := (\beta_i)_{i \in [b]}, \gamma := (\gamma_i)_{i \in [c]}, \delta_1 \in \mathbb{A}(\{\circ, \bullet\})$:

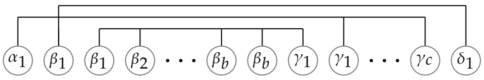


$\in \mathcal{G}.$ (X)

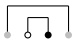
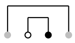
To show this, we consider the following calculation

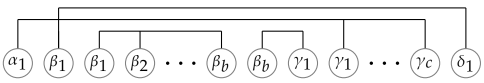
$\pi_1 :=$ 

$\in \mathcal{G}$ ⟦ eq. (IX) ⟧

$\implies \pi_2 :=$ 

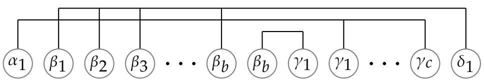
$\in \mathcal{G}$

⟦ by **(b)** we have  $\in \mathcal{G}$, and by Lem. 4.2.36  generates any noncrossing two-colored partition, then split the block neighboring legs with color γ_1 in π_1 with the regarding noncrossing partition ⟧

$\implies \pi_3 :=$ 

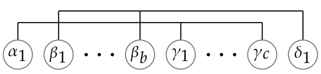
$\in \mathcal{G}$

⟦ $\eta \in \mathcal{G} \implies$  $\in \mathcal{G}$,
split the block neighboring legs with β_b by the relevant interval partition ⟧

$\implies \pi_4 :=$ 

$\in \mathcal{G}$

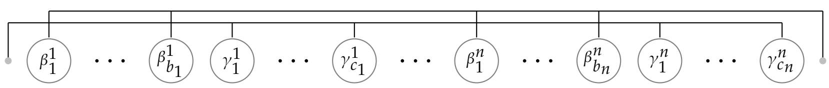
⟦ unify the legs with β_1, then delete block neighboring legs ⟧

$\implies \pi_5 :=$ 

$\in \mathcal{G}$ ⟦ apply $\mathsf{pr}_1 \circ \mathsf{UMem}$ to the first leg in π_4 ⟧.

Thus, we have proven equation (X). This also shows that equation (VIII) holds, since we can generate any sequence of legs at the beginning and the end of a partition by a finite application of cCol and double.

So far we have shown the induction base $n = 1$ of equation (VI). Now we perform the induction step $n \to n + 1 =: \tilde{n}$ and calculate

$\pi_1 :=$ 

$\in \mathcal{G}$

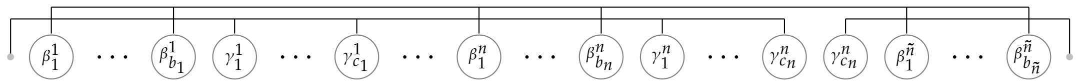
⟦ induction hypothesis ⟧

$\implies \pi_2 :=$

$\in \mathcal{G}$

⟦ insert sequence of legs at the end
by a finite application of cCol and double ⟧

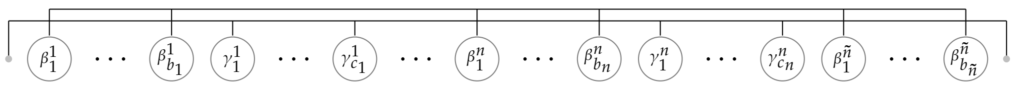
$\Longrightarrow \pi_3 :=$



$\in \mathcal{G}$

⟦ by a combination of mirror and Lem. 4.2.5 we can
split at the forelast leg in π_2 by partition m2bC from eq. (IX) ⟧

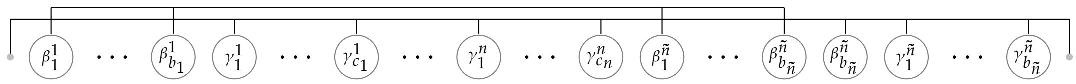
$\Longrightarrow \pi_4 :=$



$\in \mathcal{G}$

⟦ unify the legs with $\gamma_{c_n}^n$ in π_6, then delete block neighboring legs ⟧

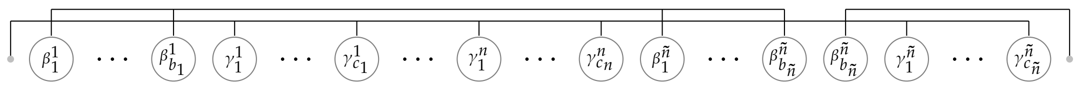
$\Longrightarrow \pi_5 :=$



$\in \mathcal{G}$

⟦ insert sequence of legs at the end
by a finite application of cCol and double ⟧

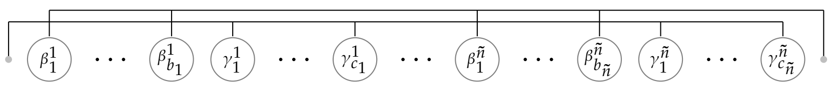
$\Longrightarrow \pi_6 :=$



$\in \mathcal{G}$

⟦ by a combination of mirror and Lem. 4.2.5 we can
split at the last leg in π_5 by partition m2bC from eq. (IX) ⟧

$\Longrightarrow \pi_7 :=$



$\in \mathcal{G}$

⟦ unify the legs with $\beta_{b_{\tilde n}}^{\tilde n}$,
then delete block neighboring legs of the same block and the same color ⟧

This shows the induction step $n \to n+1$ of equation (VI). The induction base of equation (VII) for $n = 2$ can be obtained from the "minimal two-block crossing partition" $\mathsf{m2bC}(\alpha, \beta, \gamma, \delta) \in \mathcal{G}$ by a similar reasoning as in the previous induction base made. We can also compare this to a similar argument used in the proof of Lemma 4.1.16 **(b)** for the single-colored case. The proof of the induction step $n \to n+1$ of equation (VII) is also similar to the proof of the induction step $n \to n+1$ of equation (VI). Therefore, we omit these proofs. □

4.2.44 Theorem. Let $m = 2$. The sets of partitions from Definition 4.2.13 are all possible two-colored universal classes of partitions.

PROOF: In the above lemmas we have shown that all the sets of partitions defined in Definition 4.2.13 satisfy the axioms for a two-colored universal class of partitions. It remains to show that these define all possible choices of a two-colored universal class of partitions. The proper subset relation $\subsetneq$ defines a (strict) partial order on the set of all two-colored universal classes of partitions. We recall that if $(X, \leq)$ is a partially ordered set, then we say that an element $x \in X$ is an immediate predecessor of $y \in X$ if and only if $x < y$, i. e., $x \leq y$, $x \neq y$, and if there is no element $z \in X$ such that $x < z < y$. We use the notation $\mathsf{ipred}(y)$ to denote the set of all immediate predecessors of y.

With respect to this partial order we list for each universal class of partition $\mathcal{P}$ from Definition 4.2.13 all possible immediate predecessors, denoted by $\mathsf{ipred}(\mathcal{P})$, among the set of universal classes of partition from Definition 3.4.9.

$$
\begin{array}{llll}
\mathsf{ipred}\big(\mathsf{1B}_{\{\circ,\bullet\}}\big) & \supseteq \{\emptyset\}, & \mathsf{ipred}\big(\mathsf{I}_{\{\circ,\bullet\}}\big) & \supseteq \big\{\mathsf{1B}_{\{\circ,\bullet\}}\big\}, \\
\mathsf{ipred}\big(\mathsf{I}_\circ\mathsf{NC}_\bullet\big) & \supseteq \big\{\mathsf{I}_{\{\circ,\bullet\}}\big\}, & \mathsf{ipred}\big(\mathsf{NC}_\circ\mathsf{I}_\bullet\big) & \supseteq \big\{\mathsf{I}_{\{\circ,\bullet\}}\big\}, \\
\mathsf{ipred}\big(\mathsf{pureNC}\big) & \supseteq \big\{\mathsf{I}_\circ\mathsf{NC}_\bullet, \mathsf{NC}_\circ\mathsf{I}_\bullet\big\}, & \mathsf{ipred}\big(\mathsf{biNC}\big) & \supseteq \big\{\mathsf{pureNC}\big\}, \\
\mathsf{ipred}\big(\mathsf{NC}_{\{\circ,\bullet\}}\big) & \supseteq \big\{\mathsf{pureNC}\big\}, & \mathsf{ipred}\big(\mathsf{I}_\circ\mathsf{A}_\bullet\big) & \supseteq \big\{\mathsf{I}_\circ\mathsf{NC}_\bullet\big\}, \\
\mathsf{ipred}\big(\mathsf{A}_\circ\mathsf{I}_\bullet\big) & \supseteq \big\{\mathsf{NC}_\circ\mathsf{I}_\bullet\big\}, & \mathsf{ipred}\big(\mathsf{NC}_\circ\mathsf{A}_\bullet\big) & \supseteq \big\{\mathsf{pureNC}, \mathsf{I}_\circ\mathsf{A}_\bullet\big\}, \\
\mathsf{ipred}\big(\mathsf{A}_\circ\mathsf{NC}_\bullet\big) & \supseteq \big\{\mathsf{pureNC}, \mathsf{A}_\circ\mathsf{I}_\bullet\big\}, & \mathsf{ipred}\big(\mathsf{pureC}\big) & \supseteq \big\{\mathsf{NC}_\circ\mathsf{A}_\bullet, \mathsf{A}_\circ\mathsf{NC}_\bullet\big\}, \\
\mathsf{ipred}\big(\mathsf{Part}_{\{\circ,\bullet\}}\big) & \supseteq \big\{\mathsf{biNC}, \mathsf{NC}_{\{\circ,\bullet\}}, \mathsf{pureC}\big\}. & &
\end{array}
$$

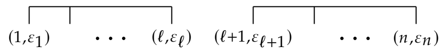
Using the defining properties in Definition 3.4.9 for each universal class of partitions $\mathcal{P}$ from above it is not hard to check the above claims. If we are able to prove that for a given universal class of partitions $\mathcal{P}$ from Definition 3.4.9 in the above list the relation $\supseteq$ can be replaced by $=$, then we have shown that Definition 3.4.9 defines all possibilities of two-colored universal classes of partitions. We want to present our strategy for the proof that in the above equations holds $=$ instead of $\supseteq$. Assume for a set C holds $\{A, B\} \subseteq \mathsf{ipred}(C)$ and we want to show that this implies $\{A, B\} = \mathsf{ipred}(C)$. Assume there exists another set $D \neq A$ and $D \neq B$ such that $D \in \mathsf{ipred}(C)$. It is not possible that $D \subseteq A$, because this would contradict the assumption that $D \in \mathsf{ipred}(C)$. By the same reasoning we obtain that $D \not\subseteq B$. If we are able to show that $D = C$, then this would contradict the assumption that $D \in \mathsf{ipred}(C)$ and we obtain that $\{A, B\} = \mathsf{ipred}(C)$.

Claim 1. $\mathsf{ipred}\big(\mathsf{1B}_{\{\circ,\bullet\}}\big) = \{\emptyset\}$

We already know that $\mathsf{ipred}\big(\mathsf{1B}_{\{\circ,\bullet\}}\big) \supseteq \{\emptyset\}$. Assume $\mathcal{P} \neq \emptyset$ is a universal class of partitions for $m = 2$ and $\mathcal{P} \in \mathsf{ipred}\big(\mathsf{1B}_{\{\circ,\bullet\}}\big)$, then $\mathcal{P} = \mathsf{1B}_{\{\circ,\bullet\}}$. From Definition 3.4.9 **(b)** we know that all alternating one block partitions are in $\mathcal{P}$. By a finite application of double we can generate any one block partition, i. e., any element of $\mathsf{1B}_{\{\circ,\bullet\}}$. This proves $\mathcal{P} = \mathsf{1B}_{\{\circ,\bullet\}}$ and contradicts the assumption that $\mathcal{P} \in \mathsf{ipred}\big(\mathsf{1B}_{\{\circ,\bullet\}}\big)$. This proves Claim 1

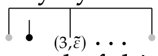
Claim 2. $\mathsf{ipred}\big(\mathsf{I}_{\{\circ,\bullet\}}\big) = \big\{\mathsf{1B}_{\{\circ,\bullet\}}\big\}$

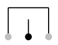
We already know that $\mathsf{ipred}\big(\mathsf{I}_{\{\circ,\bullet\}}\big) \supseteq \big\{\mathsf{1B}_{\{\circ,\bullet\}}\big\}$. Let $\mathcal{P} \neq \mathsf{1B}_{\{\circ,\bullet\}}$ be a two-colored universal class of partitions and $\mathcal{P} \in \mathsf{ipred}\big(\mathsf{I}_{\{\circ,\bullet\}}\big)$. We have $\mathcal{P} \setminus \mathsf{1B}_{\{\circ,\bullet\}} \neq \emptyset$, i. e., there exists a partition $\pi \in \mathcal{P} \setminus \mathsf{1B}_{\{\circ,\bullet\}}$. From Lemma 4.2.32 we know that the set $\{\,|\;|\,\}$ generates $\mathsf{I}_{\{\circ,\bullet\}}$. Therefore, it suffices to show $\pi \in \mathcal{P} \implies$  $\in \mathcal{P}$. Since $\pi \notin 1\mathsf{B}_{\{\circ,\bullet\}}$ it must have at least two blocks. But we have assumed that $\pi \in \mathsf{I}_{\{\circ,\bullet\}}$ and this yields for each block b of π that $\mathsf{set}(b)$ is an interval. This is shown by contradiction to Definition 4.2.13 **(b)**. According to Lemma 4.2.3 we can apply UMem to the first leg of the second block and to its predecessor and obtain that

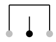
 $\in \mathcal{P}$.

Since the first and the last leg can take both colors, we can apply delete to the first and the last leg. A finite application of delete on these legs yields that  $\in \mathcal{P}$ and implies $\mathcal{P} = \mathsf{I}_{\{\circ,\bullet\}}$. This contradicts the assumption $\mathcal{P} \in \mathsf{ipred}(\mathsf{I}_{\{\circ,\bullet\}})$ and the Claim 2 follows.

Claim 3. $\mathsf{ipred}(\mathsf{I}_\circ\mathsf{NC}_\bullet) = \{\mathsf{I}_{\{\circ,\bullet\}}\}$

We already know that $\mathsf{ipred}(\mathsf{I}_\circ\mathsf{NC}_\bullet) \supseteq \{\mathsf{I}_{\{\circ,\bullet\}}\}$. Let $\mathcal{P}$ be a two-colored universal class of partitions, $\mathcal{P} \in \mathsf{ipred}(\mathsf{I}_\circ\mathsf{NC}_\bullet)$ and $\mathcal{P} \neq \mathsf{I}_{\{\circ,\bullet\}}$. We have $\mathcal{P} \setminus \mathsf{I}_{\{\circ,\bullet\}} \neq \emptyset$, i. e., $\exists \pi \in \mathcal{P} \setminus \mathsf{I}_{\{\circ,\bullet\}}$. π has more than one block because we already know that $1\mathsf{B}_{\{\circ,\bullet\}} \subseteq \mathsf{I}_{\{\circ,\bullet\}}$ and we have assumed that $\pi \notin \mathsf{I}_{\{\circ,\bullet\}}$. Therefore, π has at least two blocks. In π there is at least one block which is not of an interval type. Because if every block in π would be of interval type, then this would imply that $\pi \in \mathsf{I}_{\{\circ,\bullet\}}$. But this would again contradict our assumption that $\pi \notin \mathsf{I}_{\{\circ,\bullet\}}$. Choose a block of π which is not of interval type and denote the block by b. The block b has a "gap", i. e., there exist natural numbers $\beta_\ell < \beta_{\ell+1} \in \mathbb{N}$ such that $\mathsf{set}(b) \setminus [\beta_\ell, \beta_{\ell+1}] \subseteq \mathsf{set}(b)$. In the interval $[\beta_\ell, \beta_{\ell+1}]$ are legs which belong to a block $b' \neq b$. Since $\pi \in \mathsf{I}_\circ\mathsf{NC}_\bullet$, the partition needs to be noncrossing. Therefore, $\mathsf{set}(b') \subseteq [\beta_\ell, \beta_{\ell+1}]$. We need to determine the color of the legs in block b'. Block b' can not have any legs of color $\circ$ because this would violate the condition of equation (4.2.16c). Hence, block b' can only have legs of color $\bullet$. By Lemma 4.2.3 we can successively apply UMem to π until block b is the first block in the partition. By a finite application of delete, we can achieve that after the first leg of the partition comes the first and only leg of the second block. Denote the partition obtained in such a way by π'. If we unify the first and second leg of the partition π' by $\mathsf{pr}_2 \circ \mathsf{UMem}$, then we obtain  $\in \mathcal{P}$. By a finite application of delete we can delete all block neighboring legs at the end of this partition and obtain

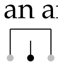
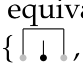
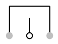
 $\in \mathcal{P}$.

In Lemma 4.2.33 we have shown that $\mathsf{Gen}($$) = \mathsf{I}_\circ\mathsf{NC}_\bullet$. This implies $\mathsf{I}_\circ\mathsf{NC}_\bullet \subseteq \mathcal{P}$ which is a contradiction to the assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{I}_\circ\mathsf{NC}_\bullet)$. Hence, by proof of contradiction we have shown the assertion of Claim 3.

Claim 4. $\mathsf{ipred}(\mathsf{NC}_\circ\mathsf{I}_\bullet) = \{\mathsf{I}_{\{\circ,\bullet\}}\}$

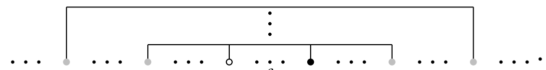
The proof of this claim is similar to the proof of Claim 3 and is omitted.

Claim 5. $\mathsf{ipred}(\mathsf{pureNC}) = \{\mathsf{I}_\circ\mathsf{NC}_\bullet, \mathsf{NC}_\circ\mathsf{I}_\bullet\}$

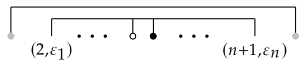
We already know that $\mathsf{ipred}(\mathsf{pureNC}) \supseteq \{\mathsf{I}_\circ\mathsf{NC}_\bullet, \mathsf{NC}_\circ\mathsf{I}_\bullet\}$. We prove the other direction by contradiction. We assume therefor that $\mathcal{P}$ is a two-colored universal class of partitions and $\mathcal{P} \in \mathsf{ipred}(\mathsf{pureNC})$, $\mathcal{P} \neq \mathsf{I}_\circ\mathsf{NC}_\bullet$ and $\mathcal{P} \neq \mathsf{NC}_\circ\mathsf{I}_\bullet$. By this we have $\mathcal{P} \setminus \mathsf{I}_\circ\mathsf{NC}_\bullet \neq \emptyset$ and $\mathcal{P} \setminus \mathsf{NC}_\circ\mathsf{I}_\bullet \neq \emptyset$. We obtain $\exists \pi_1 \in \mathcal{P}$ such that $\pi_1 \notin \mathsf{I}_\circ\mathsf{NC}_\bullet$ and $\exists \pi_2 \in \mathcal{P}$ such that $\pi_2 \notin \mathsf{NC}_\circ\mathsf{I}_\bullet$. By an analogous proof of Claim 3 we conclude that $\pi_1 \in \mathcal{P} \implies$  $\in \mathcal{P}$ and likewise $\pi_2 \in \mathcal{P} \implies$ . Also compare this with the proof of Lemma 4.2.18, where we have provided an equivalent characterization of the fact $\pi \notin \mathsf{I}_\circ\mathsf{NC}_\bullet$. In Lemma 4.2.35 we have shown that $\mathsf{Gen}(\{$, $\}) = \mathsf{pureNC}$ and this implies $\mathsf{pureNC} \subseteq \mathcal{P}$. This contradicts our assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{pureNC})$. By proof of contradiction we have shown that Claim 5 holds.

Claim 6. $\mathsf{ipred}\big(\mathsf{NC}_{\{\circ,\bullet\}}\big) = \big\{\mathsf{pureNC}\big\}$

We already know $\mathsf{ipred}\big(\mathsf{NC}_{\{\circ,\bullet\}}\big) \supseteq \big\{\mathsf{pureNC}\big\}$. We prove the other direction by proof of contradiction. We assume therefor that $\mathcal{P}$ is a two-colored universal class of partitions, $\mathcal{P} \in \mathsf{ipred}(\mathsf{NC}_{\{\circ,\bullet\}})$ and $\mathcal{P} \neq \mathsf{pureNC}$. By this we have $\mathcal{P} \setminus \mathsf{pureNC} \neq \emptyset$. Thus, we obtain that a partition $\pi \in \mathcal{P}$ needs to exists such that $\pi \notin \mathsf{pureNC}$. In the proof of Lemma 4.2.20 we have provided an equivalent characterization of the fact $\pi \notin \mathsf{pureNC}$. From there we can convince ourselves that since $\pi \in \mathsf{NC}_{\{\circ,\bullet\}}$, in particular it is non-crossing, we have an inner block β which has at least two legs with different colors. In other words, we have the following occurrence in the partition π

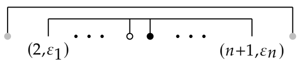


By a successive application of Lemma 4.2.3 and by a further application of delete to π, we can achieve that the block β becomes the second block in the partition and its minimal leg is the second leg of the partition π. Then, we can apply $\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,\cdot}$ and obtain that

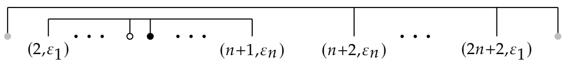


$\in \mathcal{P},$

where $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}(\{\circ,\bullet\})$ for some $n \geq 2$. We claim that the existence of the above two-block partition in $\mathcal{P}$, implies that  $\in \mathcal{P}$. Consider therefor the following calculation

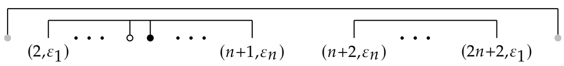
$\pi :=$ 

$\in \mathcal{P}$

$\Longrightarrow \pi_1 :=$ 

$\in \mathcal{P}$

$[\![$ n-times application of double $]\!]$

$\Longrightarrow \pi_2 :=$ 

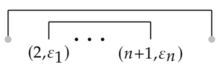
$\in \mathcal{P}$

$[\![\ \pi_2 = \mathsf{split}^{2n+2}_{\cdot,2}(\pi_1, \mathsf{mirror}(\pi))\ ]\!]$

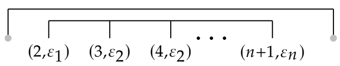
$\Longrightarrow \pi_3 :=$  $\in \mathcal{P}$

$[\![\ \pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^{n+1}_{\cdot,3})(\pi_2)$ & successive application of delete $]\!]$

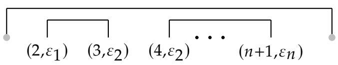
Now we calculate

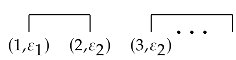
$\pi :=$ 

$\in \mathcal{P}$

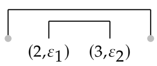
$\Longrightarrow \pi_1 :=$ 

$\in \mathcal{P}$ $[\![\ \pi_1 = \mathsf{double}_{\cdot,3}(\pi)\ ]\!]$

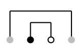
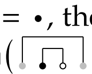
$\Longrightarrow \pi_2 =$ 

$\in \mathcal{P}$ $[\![\ \pi_2 = \mathsf{split}^3_{\cdot,2}(\pi_1,$ 

$)\ ]\!]$

$\Longrightarrow \pi_3 :=$ 

$\in \mathcal{P}$ $[\![\ \pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,3})(\pi_2)\ ]\!]$

If $\varepsilon_1 = \bullet$ and $\varepsilon_2 = \circ$ we conclude that  $\in \mathcal{P}$. If $\varepsilon_1 = \circ$ and $\varepsilon_2 = \bullet$, then by application of mirror we conclude the same. In Lemma 4.2.36 we have shown $\mathsf{Gen}\big($$\big) = \mathsf{NC}_{\{\circ,\bullet\}}$ and this implies $\mathsf{NC}_{\{\circ,\bullet\}} \subseteq \mathcal{P}$ which contradicts our assumption $\mathcal{P} \in \mathsf{ipred}(\mathsf{NC}_{\{\circ,\bullet\}})$. Therefore, we have shown that Claim 6 holds.

Next, we show that

$$\mathsf{biNC} \cap \mathsf{NC}_{\{\circ,\bullet\}} = \mathsf{pureNC}. \tag{I}$$

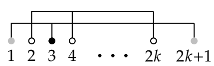
Assume $\pi \in \mathsf{pureNC}$. Then, from the defining properties of π for being pure noncrossing in Definition 4.2.13 **(e)**, we can see that these are shared by the defining properties for being binoncrossing (Definition 4.2.13 **(k)**) and for being noncrossing (Definition 4.2.13 **(f)**). For the other direction we assume that $\pi \in \mathsf{biNC} \cap \mathsf{NC}_{\{\circ,\bullet\}}$. Since $\pi \in \mathsf{NC}_{\{\circ,\bullet\}}$, the partition π needs to be noncrossing. Hence, any occurrences of possible relative positions of legs to each other, which lead to a crossing in π, are ruled out. If we do this in Definition 4.2.13 **(k)** we are left with conditions which have been imposed in Definition 4.2.13 **(e)**, i. e., which are the defining properties for a partition for being pure crossing. This shows equation (I).

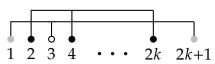
Claim 7. $\mathsf{ipred}(\mathsf{biNC}) = \{\mathsf{pureNC}\}$

We already know that $\mathsf{ipred}(\mathsf{biNC}) \supseteq \{\mathsf{pureNC}\}$. We prove the other direction by proof of contradiction. Hence, we assume there exists another two-colored universal class of partitions denoted by $\mathcal{P}$ such that $\mathcal{P} \in \mathsf{ipred}(\mathsf{biNC})$ and $\emptyset \neq \mathcal{P} \neq \mathsf{pureNC}$. Therefore, we have $\mathcal{P} \setminus \mathsf{pureNC} \neq \emptyset$. Hence, there exists a partition $\pi \in \mathcal{P}$ such that $\pi \notin \mathsf{pureNC}$. From equation (I) we can see that $\pi \notin \mathsf{NC}_{\{\circ,\bullet\}}$ because $\pi \in \mathcal{P} \subseteq \mathsf{biNC}$. The fact $\pi \notin \mathsf{NC}_{\{\circ,\bullet\}}$ implies that π needs to have a crossing. Therefore, the set $\mathsf{Cross}(\pi)$, defined in equation (4.2.13), is not empty. Hence, there exist $((p_1, p_2), (q_1, q_2)) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $((p_1, p_2), (q_1, q_2)) \in \mathsf{Cross}(\pi)$. Because $\pi \in \mathsf{biNC}$ and because of Definition 4.2.13 **(k)** we have

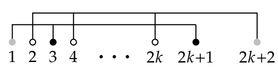
$$(\varepsilon_{p_2} = \circ,\ \varepsilon_{q_1} = \bullet) \vee (\varepsilon_{p_2} = \bullet,\ \varepsilon_{q_1} = \circ). \tag{II}$$

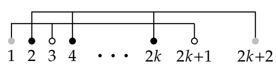
This implies the existence of two blocks b_p and b_q such that $b_p,\ b_q \in \pi$ and $\{p_1, p_2\} \subseteq \mathsf{set}(b_p)$ and $\{q_1, q_2\} \subseteq \mathsf{set}(b_q)$. If the partition π only has two blocks, then we make π reduced by $\pi := \mathrm{red}\,\pi$. If the partition π has more than two blocks, then proceed as follows. By a successive application of Lemma 4.2.3 to π we are able to unify the first and the second block in the partition π. If the block b_p has been unified in π and all block neighboring legs with the same color have been deleted by delete, address this block by b_p and set $\pi := \mathrm{red}(\pi)$ afterwards. The same holds for the block b_q. We repeat this procedure until either b_p is the first block of π and the second leg of π is the minimal leg of b_q or either b_q is the first block π and the second leg of π is the minimal leg of b_p (look at Definition 4.2.2 for an order on the blocks of an m-colored partition). The procedure needs to terminate since in each step we reduce the amount of blocks in π. After this procedure has terminated we unify the first and the second leg in the partition π by $(\mathsf{pr}_2 \circ \mathsf{UMem})$. We want to refer to this procedure as the *unification procedure*. Also compare this unification procedure against the map *GenerateTwoBlocks* defined in Definition 4.2.8. We already had a similar unification procedure in the single-colored case, introduced in the proof of Claim 4 of Theorem 4.1.17. Whenever this unification procedure is applied to a partition with a crossing, it gives us the existence of a partition which is one of the following four types of two-block partition. We denote the resulting partition again by π. In particular, this means that there either exists a natural number $k \in \mathbb{N} \setminus \{1\}$ such that $\pi \in \mathcal{P}$ is exactly one of the following two types

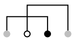
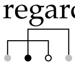
$\pi =$  (III)

or $\pi =$  , (IV)

or $\exists k \in \mathbb{N}$ such that $\pi \in \mathcal{P}$ is exactly one of the following two types

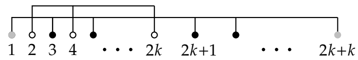
$\pi =$  (V)

or $\pi =$  . (VI)

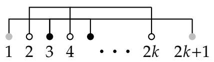
Like in the $m = 1$ case we regard the partition  as a partition of type from equation (V) for $k = 1$ and the partition  as a partition of type from equation (VI) for $k = 1$. Because of equation (II) the color of the legs in the above four types must be alternating.

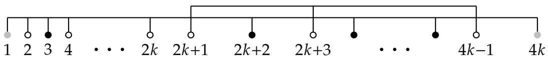
Claim 8. If $\pi \in \mathcal{P}$ is of type from equation (III), equation (IV), equation (V) or equation (VI), then this implies  $\in \mathcal{P}$.

For the proof of the above claim let $\pi \in \mathcal{P}$ be a partition of type from equation (III) for some $k \in \mathbb{N} \setminus \{1\}$. By $(k-1)$-times application of double to π we can insert legs at the end only with color $\bullet$, i. e.,

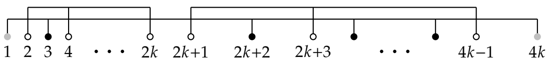
$\pi_1 :=$  $\in \mathcal{P}.$ (VII)

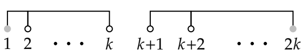
By this we calculate

$\pi =$  $\in \mathcal{P}$

$\Longrightarrow \pi_2 :=$  $\in \mathcal{P}$

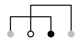
$[\![\pi_2 \mathrel{\hat{=}} (2k-1)\text{-application of } \mathsf{double}_{\cdot,1} \text{ to } \pi]\!]$

$\Longrightarrow \pi_3 :=$  $\in \mathcal{P}$ $[\![\pi_3 = \mathsf{split}^1_{\cdot,2}(\pi_2, \pi_1)]\!]$

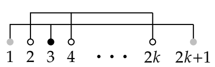
$\Longrightarrow \pi_4 :=$  $\in \mathcal{P}$ $[\![\pi_4 = (\mathsf{pr}_2 \circ \mathsf{UMem}^{2k}_{\cdot,\cdot})(\pi_3)]\!]$

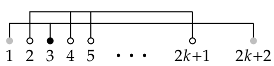
$\Longrightarrow$  $\in \mathcal{P}$ $[\![$ successive application of $\mathsf{delete}_{\cdot,\cdot}$ to π_4 $]\!]$

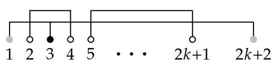
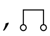
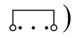
We can apply the above proof of Claim 8 in the case that $\pi \in \mathcal{P}$ is of type from equation (IV), where we only need to swap the occurring colors. There is a similar proof for this claim in the case that the partition $\pi \in \mathcal{P}$ is of type from equation (V). The only difference is that $\pi \neq \mathsf{mirror}(\pi)$ and therefore $\pi_1 \in \mathcal{P}$ from equation (VII) is obtained by inserting $(k-1)$ legs of color $\bullet$ at the end of the partition $\mathsf{mirror}(\pi)$. Then, we may apply the same steps from the proof above. For a partition of type from equation (VI) we only need to swap the colors. Now, that we know that  $\in \mathcal{P}$, we can deduce that any two-block interval partition must be an element of $\mathcal{P}$.

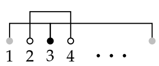
Claim 9. If $\pi \in \mathcal{P}$ is of type from equation (III), equation (IV), equation (V) or equation (VI), then this implies  $\in \mathcal{P}$.

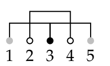
For the proof of above claim we let $\pi \in \mathcal{P}$ be of type from equation (III) for some $k \in \mathbb{N} \setminus \{1\}$ and calculate

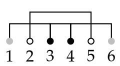
$\pi =$  $\in \mathcal{P}$

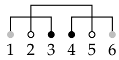
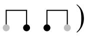
$\Longrightarrow \pi_1 :=$  $\in \mathcal{P}$ $[\![\pi_1 = \mathsf{double}_{\cdot,4}(\pi)]\!]$

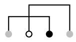
$\Longrightarrow \pi_2 :=$  $\in \mathcal{P}$ $[\![\pi_2 = \mathsf{split}^4_{2k+2,2}(\pi_1,$  $)]\!]$

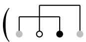
$\Longrightarrow \pi_3 :=$  $\in \mathcal{P}$ $[\![\pi_3 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,2})(\pi_2)]\!]$

$\Longrightarrow \pi_4 :=$  $\in \mathcal{P}$ $[\![$ successive application of $\mathsf{delete}_{\cdot,\cdot}$ to π_3 $]\!]$

$\Longrightarrow \pi_5 :=$  $\in \mathcal{P}$ $[\![\pi_5 = \mathsf{double}_{\cdot,3}(\pi_4)]\!]$

$\implies \pi_6 :=$  $\in \mathcal{P} \quad [\![\, \pi_6 = \mathsf{split}^3_{\cdot,2}(\pi_5,$ $) \,]\!]$

$\implies$  $\in \mathcal{P} \quad [\![\, \pi_6 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,3})(\pi_5) \,]\!]$

This finishes the proof for Claim 9 because the proof for partitions of the remaining types is analogously done. In Lemma 4.2.41 we have shown that $\mathsf{Gen}($$) = \mathsf{biNC}$ and this implies $\mathsf{biNC} \subseteq \mathcal{P}$ which contradicts our assumption $\mathcal{P} \in \mathsf{ipred}(\mathsf{biNC})$. Therefore, we have shown that Claim 7 is true.

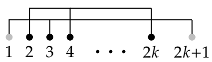
From Definition 4.2.13 we obtain

$$\mathsf{I}_\circ\mathsf{A}_\bullet \cap \mathsf{NC}_{\{\circ,\bullet\}} = \mathsf{I}_\circ\mathsf{NC}_\bullet. \tag{VIII}$$

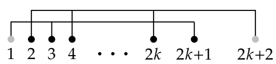
The proof is similar to equation (I) and is therefore omitted.

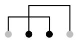
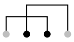
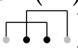
Claim 10. $\mathsf{ipred}(\mathsf{I}_\circ\mathsf{A}_\bullet) = \{\mathsf{I}_\circ\mathsf{NC}_\bullet\}$

We already know that $\mathsf{ipred}(\mathsf{I}_\circ\mathsf{A}_\bullet) \supseteq \mathsf{I}_\circ\mathsf{NC}_\bullet$. We prove the other direction by proof of contradiction. Hence, we assume there exists another two-colored universal class of partition denoted by $\mathcal{P}$ such that $\mathcal{P} \in \mathsf{ipred}(\mathsf{I}_\circ\mathsf{A}_\bullet)$ and $\emptyset \neq \mathcal{P} \neq \mathsf{I}_\circ\mathsf{NC}_\bullet$. Therefore, we have $\mathcal{P} \setminus \mathsf{I}_\circ\mathsf{NC}_\bullet \neq \emptyset$. Hence, there exists a partition $\pi \in \mathcal{P}$ such that $\pi \notin \mathsf{I}_\circ\mathsf{NC}_\bullet$. From equation (VIII) we can see that $\pi \notin \mathsf{NC}_{\{\circ,\bullet\}}$, since $\pi \in \mathcal{P} \subseteq \mathsf{I}_\circ\mathsf{A}_\bullet$. The fact $\pi \notin \mathsf{NC}_{\{\circ,\bullet\}}$ implies that π needs to have a crossing. Therefore, the set $\mathsf{Cross}(\pi)$ is not empty. Hence, there exist $((p_1,p_2),(q_1,q_2)) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $((p_1,p_2),(q_1,q_2)) \in \mathsf{Cross}(\pi)$. Because $\pi \in \mathsf{I}_\circ\mathsf{A}_\bullet$ and Definition 4.2.13 **(g)** we have $\varepsilon_{p_2} = \varepsilon_{q_1} = \bullet$. This implies the existence of two blocks b_p and b_q such that $b_p,\, b_q \in \pi$ and $\{p_1,p_2\} \subseteq \mathsf{set}(b_p)$ and $\{q_1,q_2\} \subseteq \mathsf{set}(b_q)$. Once again we apply our so-called "unification procedure", described in the proof for Claim 7. After application of this unification procedure to the partition $\pi \in \mathcal{P}$ we obtain that there needs to exist a partition, once again denoted by π, which as an element of $\mathcal{P}$ and needs to be of type of the following two. Either there exists $k \in \mathbb{N} \setminus \{1\}$ such that

 $\in \mathcal{P}$ (IX)

or there exists $k \in \mathbb{N}$ such that

 $\in \mathcal{P}$. (X)

Like done before, we want to regard  as a partition of type from equation (X) for $k = 1$. If $\pi \in \mathcal{P}$ is of type from equation (IX) or equation (X), then this implies $\in \mathcal{P}$. The proof is similar to Claim 4 in the proof of Theorem 4.1.17. We can analogously show, if $\pi \in \mathcal{P}$ is of type from equation (IX) or equation (X), then this implies  $\in \mathcal{P}$. In Lemma 4.2.37 we have shown that $\mathsf{Gen}($$) = \mathsf{I}_\circ\mathsf{A}_\bullet$ and this implies $\mathsf{I}_\circ\mathsf{A}_\bullet \subseteq \mathcal{P}$, which contradicts our assumption $\mathcal{P} \in \mathsf{ipred}(\mathsf{I}_\circ\mathsf{A}_\bullet)$. Therefore, we have shown that Claim 10 is true.

Claim 11. $\mathsf{ipred}(\mathsf{A}_\circ\mathsf{I}_\bullet) = \{\mathsf{NC}_\circ\mathsf{I}_\bullet\}$

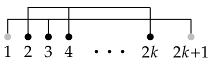
The proof of this claim is similar to the proof of Claim 10 and is therefore omitted.

Claim 12. $\mathsf{ipred}(\mathsf{NC}_\circ\mathsf{A}_\bullet) = \{\mathsf{pureNC}, \mathsf{I}_\circ\mathsf{A}_\bullet\}$

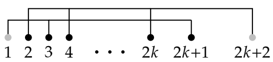
We have $\mathsf{ipred}(\mathsf{NC}_\circ\mathsf{A}_\bullet) \supseteq \{\mathsf{pureNC}, \mathsf{I}_\circ\mathsf{A}_\bullet\}$. We prove the other direction by contradiction. We assume therefor the existence of $\mathcal{P}$ as a two-colored universal class of partitions such that $\emptyset \neq \mathcal{P} \in \mathsf{ipred}(\mathsf{NC}_\circ\mathsf{A}_\bullet)$, $\mathcal{P} \neq \mathsf{pureNC}$ and $\mathcal{P} \neq \mathsf{I}_\circ\mathsf{A}_\bullet$. Hence we obtain that $\exists \pi_1 \in \mathsf{NC}_\circ\mathsf{A}_\bullet \setminus \mathsf{pureNC}$ and $\exists \pi_2 \in \mathsf{NC}_\circ\mathsf{A}_\bullet \setminus \mathsf{I}_\circ\mathsf{A}_\bullet$. We can show that

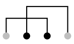
$$\mathsf{NC}_\circ\mathsf{A}_\bullet \cap \mathsf{NC}_{\{\circ,\bullet\}} = \mathsf{pureNC}. \tag{XI}$$

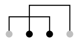
According to equation (XI) we have $\pi_1 \notin \mathsf{NC}_{\{\circ,\bullet\}}$ because $\pi_1 \notin \mathsf{pureNC}$ and $\pi_1 \in \mathcal{P} \subseteq \mathsf{NC}_\circ\mathsf{A}_\bullet$. The fact $\pi_1 \notin \mathsf{NC}_{\{\circ,\bullet\}}$ implies that π_1 needs to have a crossing. Therefore, the set $\mathsf{Cross}(\pi)$ is not empty. Hence there exist $\big((p_1, p_2), (q_1, q_2)\big) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $\big((p_1, p_2), (q_1, q_2)\big) \in \mathsf{Cross}(\pi)$. Because $\pi_1 \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ and by Definition 4.2.13 **(i)**, we have that $\varepsilon_{p_2} = \varepsilon_{q_1} = \bullet$. This implies the existence of two blocks b_p and b_q such that $b_p,\ b_q \in \pi$ and $\{p_1, p_2\} \subseteq \mathsf{set}(b_p)$ and $\{q_1, q_2\} \subseteq \mathsf{set}(b_q)$. Once again we apply our so-called "unification procedure", described in the proof for Claim 7. After application of this unification procedure to the partition $\pi_1 \in \mathcal{P}$, we obtain the existence of a partition $\pi \in \mathcal{P}$ which needs to be one of the following two types. Either there exists $k \in \mathbb{N} \setminus \{1\}$ such that

$\pi =$  $\in \mathcal{P}$ (XII)

or there exists $k \in \mathbb{N}$ such that

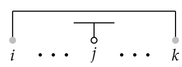
$\pi =$  $\in \mathcal{P}$. (XIII)

Like done before we want to regard  as a partition of type from equation (XIII) for $k = 1$. If $\pi \in \mathcal{P}$ is of type from equation (XII) or equation (XIII), then this implies  $\in \mathcal{P}$. The proof is similar to Claim 4 in the proof of Theorem 4.1.17. We can analogously show, if $\pi \in \mathcal{P}$ is of type from equation (XII) or equation (XIII), then this implies

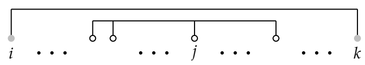
 $\in \mathcal{P}$. (XIV)

Now, let us consider $\pi_2 \in \mathsf{NC}_\circ\mathsf{A}_\bullet \setminus \mathsf{I}_\circ\mathsf{A}_\bullet$. The negation of equation (4.2.20a) leads to $\pi_2 \notin \mathsf{I}_\circ\mathsf{A}_\bullet \iff$ there exist distinct blocks $b,\ \beta \in \pi$, there exist indices $i,\ j,\ k \in \mathsf{set}(\mathsf{type}(b)) \cup \mathsf{set}(\mathsf{type}(\beta))$ such that

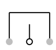
$$\Big(i,\ k \in \mathsf{set}\big(\mathsf{type}(b)\big),\ \varepsilon_j = \circ\Big) \wedge \big((j, \varepsilon_j) \notin \mathsf{set}(\beta)\big)$$

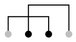
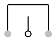
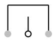
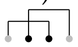
$\widehat{=}$  occurs in π_2

Thus, we can conclude that in the partition π_2 there is a block β, with two legs i and k and a leg j between i and k which does not belong to block β. The color of leg j is $\circ$. The fact that $\pi_2 \in \mathsf{NC}_\circ\mathsf{A}_\bullet$ can give us possibilities for the relative position of the block with the leg j with respect to the block β. Both blocks can not lead to a crossing because this would violate equation (4.2.22a), which needs to be satisfied because $\pi_2 \in \mathsf{NC}_\circ\mathsf{A}_\bullet$. The only allowed possibility according to equation (4.2.22a) is the following scenario

$\pi_2 \ \widehat{=}$  ,

where the block with the leg j only consists of legs with the color $\circ$. By the unification procedure applied to π_2, described in the proof of Claim 7, we obtain that

 $\in \mathcal{P}$. (XV)

Equation (XIV) and (XV) yield $\{$, $\} \subset \mathcal{P}$. In Lemma 4.2.39 we have shown that $\mathsf{Gen}\big(\{$, $\}\big) = \mathsf{NC}_\circ\mathsf{A}_\bullet$ and this implies $\mathsf{NC}_\circ\mathsf{A}_\bullet \subset \mathcal{P}$ which contradicts our assumption $\mathcal{P} \in \mathsf{ipred}(\mathsf{NC}_\circ\mathsf{A}_\bullet)$. Therefore, we have shown that Claim 12 is true.

Claim 13. $\mathsf{ipred}\big(\mathsf{A}_\circ\mathsf{NC}_\bullet\big) = \big\{\mathsf{pureNC}, \mathsf{A}_\circ\mathsf{I}_\bullet\big\}$

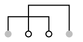
The proof of this claim is similar to the proof of Claim 12 and is therefore omitted.

Claim 14. $\mathsf{ipred}(\mathsf{pureC}) = \{\mathsf{NC}_\circ\mathsf{A}_\bullet, \mathsf{A}_\circ\mathsf{NC}_\bullet\}$

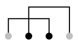
We already know that $\mathsf{ipred}(\mathsf{pureC}) \supseteq \{\mathsf{NC}_\circ\mathsf{A}_\bullet, \mathsf{A}_\circ\mathsf{NC}_\bullet\}$. We prove the other direction by contradiction. We assume therefor the existence of $\mathcal{P}$ as a two-colored universal class of partitions $\mathcal{P}$ such that $\emptyset \neq \mathcal{P} \in \mathsf{ipred}(\mathsf{pureC})$, $\mathcal{P} \neq \mathsf{NC}_\circ\mathsf{A}_\bullet$ and $\mathcal{P} \neq \mathsf{A}_\circ\mathsf{NC}_\bullet$. Hence we obtain that $\exists \pi_1 \in \mathsf{pureC} \setminus \mathsf{NC}_\circ\mathsf{A}_\bullet$ and $\exists \pi_2 \in \mathsf{pureC} \setminus \mathsf{A}_\circ\mathsf{NC}_\bullet$. Looking at Definition 4.2.13 **(i)** for a partition being noncrossing-crossing, we can see that it only puts restrictions on legs with color $\circ$. Thus, if a partition is not noncrossing-crossing, it is necessary that a leg with color $\circ$ does not satisfy equation (4.2.22a). For the partition $\pi_1 \notin \mathsf{NC}_\circ\mathsf{A}_\bullet$ we also need to take into account that π_1 must be pure crossing. Therefore $\exists \big((p_1, p_2), (q_1, q_2)\big) \in \mathsf{Cross}(\pi)$ such that

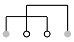
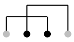
$$\varepsilon_{p_2} = \varepsilon_{q_1} = \circ.$$

Applying the unification procedure to π_1 and several other analogous steps as done in the proof of Claim 12 we obtain

 $\in \mathcal{P}.$

A similar reasoning holds for the implication

$\pi_2 \in \mathcal{P} \implies$  $\in \mathcal{P}.$

In Lemma 4.2.42 we have shown that $\mathsf{Gen}\big(\{$  ,  $\}\big) = \mathsf{pureC}$ and this implies $\mathsf{pureC} \subseteq \mathcal{P}$, which contradicts our assumption $\mathcal{P} \in \mathsf{ipred}(\mathsf{pureC})$. Therefore, we have shown that Claim 14 is true.

Claim 15. $\mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}}) = \{\mathsf{NC}_{\{\circ,\bullet\}}, \mathsf{biNC}, \mathsf{pureC}\}$

We already know that $\mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}}) \supseteq \{\mathsf{NC}_{\{\circ,\bullet\}}, \mathsf{biNC}, \mathsf{pureC}\}$. We prove the other direction by contradiction. We assume therefor the existence of $\mathcal{P}$ as a two-colored universal class of partitions such that $\emptyset \neq \mathcal{P} \in \mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}})$, $\mathcal{P} \neq \mathsf{NC}_{\{\circ,\bullet\}}$, $\mathcal{P} \neq \mathsf{biNC}$ and $\mathcal{P} \neq \mathsf{pureC}$. We have the existence of

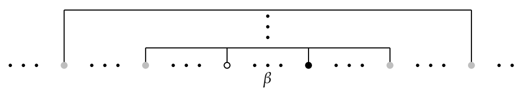
$$\pi_1 \in \mathsf{Part}_{\{\circ,\bullet\}} \setminus \mathsf{NC}_{\{\circ,\bullet\}},$$

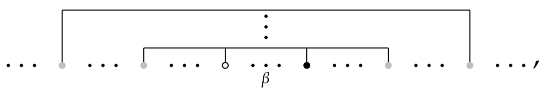
$$\pi_2 \in \mathsf{Part}_{\{\circ,\bullet\}} \setminus \mathsf{biNC} \text{ and}$$

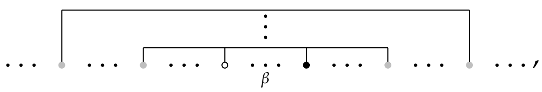
$$\pi_3 \in \mathsf{Part}_{\{\circ,\bullet\}} \setminus \mathsf{pureC}.$$

Since $\pi_1 \in \mathsf{Part}_{\{\circ,\bullet\}} \setminus \mathsf{NC}_{\{\circ,\bullet\}}$ the partition π_1 needs to have at least one crossing. At this point we do not make any further distinctions for π_1. But we make the following case considerations for π_2 and π_3

(a) $\pi_2 \in \mathsf{pureC}$ and $\pi_3 \in \mathsf{biNC}$,

(b) $\pi_2 \;\hat{=}$  and $\pi_3 \in \mathsf{biNC}$

(c) $\pi_2 \in \mathsf{pureC}$ and $\pi_3 \;\hat{=}$  ,

(d) π_2 and $\pi_3 \;\hat{=}$  ,

(e) π_2 or π_3 have an arbitrary crossing.

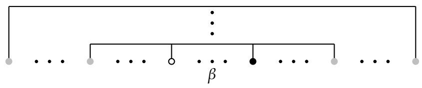
According to Definition 4.2.13 **(k)** and **(l)** the above case consideration displays all cases which can occur such that $\pi_2 \notin \mathsf{biNC}$ and $\pi_3 \notin \mathsf{pureC}$.
AD **(a)**: Since $\pi_2 \in \mathsf{pureC}$ and $\pi_2 \notin \mathsf{biNC}$, the partition π_2 needs to have a crossing. Thus, there exists $\big((p_1, p_2), (q_1, q_2)\big) \in \mathsf{Cross}(\pi_2)$. The colors of the legs p_2 and q_1 need to have the same color, thus there are two possibilities; either $\varepsilon_{p_2} = \varepsilon_{q_1} = \circ$ or $\varepsilon_{p_2} = \varepsilon_{q_1} = \bullet$. We claim the existence of $\pi_2 \in \mathcal{P}$ with the above properties implies

$$\in \mathcal{P} \text{ or } \in \mathcal{P}.$$

The proof can be taken from the proof of Claim 12. Furthermore we have

$$\mathcal{P} \ni \pi_3 \in \big((\mathsf{Part}_{\{\circ,\bullet\}} \setminus \mathsf{pureC}) \cap \mathsf{biNC}\big) \implies \in \mathcal{P}. \tag{XVI}$$

The proof can be taken from the proof of Claim 7. In Lemma 4.2.43 we have shown that $\mathsf{Gen}(\{ , \}) = \mathsf{Gen}(\{ , \}) = \mathsf{Part}_{\{\circ,\bullet\}}$. This implies that $\mathcal{P} = \mathsf{Part}_{\{\circ,\bullet\}}$ which contradicts our assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}})$. Thus, in case of **(a)** we have shown that Claim 15 is true.
AD **(b)**: Like in the case of **(a)** we can conclude from equation (XVI) that $\in \mathcal{P}$. In the proof of Claim 6 we have shown that



$$\pi_2 \,\hat{=}\, \ldots \in \mathcal{P} \implies \in \mathcal{P}. \tag{XVII}$$

Hence, we obtain that $\{ , \} \subseteq \mathcal{P}$. But in Lemma 4.2.43 we have shown that $\mathsf{Gen}(\{ , \}) \subseteq= \mathsf{Part}_{\{\circ,\bullet\}}$. This implies that $\mathcal{P} = \mathsf{Part}_{\{\circ,\bullet\}}$ which contradicts our assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}})$. Thus, in case of **(b)** we have shown that Claim 15 is true.
AD **(c)**: Like in the case of **(a)** we can conclude that

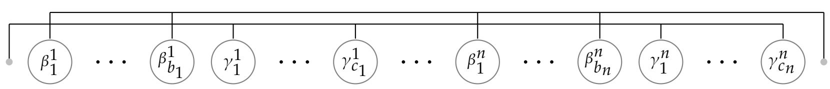
$$\mathcal{P} \ni \pi_2 \in \big((\mathsf{Part}_{\{\circ,\bullet\}} \setminus \mathsf{biNC}) \cap \mathsf{pureC}\big) \implies \in \mathcal{P} \vee \in \mathcal{P}.$$

Like in equation (XVII) applied to $\pi_3 \in \mathcal{P}$ we can conclude that $\in \mathcal{P}$. Thus, we have shown that $\{ , \} \subseteq \mathcal{P}$ or $\{ , \} \subseteq \mathcal{P}$. This implies that $\mathcal{P} = \mathsf{Part}_{\{\circ,\bullet\}}$ which contradicts our assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}})$. Thus, in case of **(c)** we have shown that Claim 15 is true.
AD **(d)**: Since the partition $\mathcal{P} \ni \pi_1 \notin \mathsf{NC}_{\{\circ,\bullet\}}$, we have $\mathsf{Cross}(\pi_1) \neq \emptyset$, hence π_1 must have a crossing. By the unification procedure we obtain the existence of a "minimal two-block crossing partition" $\mathsf{m2bC}(\alpha, \beta, \gamma, \delta) \in \mathcal{P}$, defined in equation (VIII) in the proof Lemma 4.2.43 **(c)**. This partition $\pi := \mathsf{m2bC}(\alpha, \beta, \gamma, \delta)$ must be of the type from equation (VI) or equation (VII) of the proof of Lemma 4.2.43 **(c)**. We first claim the following statement

$$\forall \alpha,\, \beta,\, \gamma,\, \delta \in \mathbb{A}(\{\circ, \bullet\})\colon\ \mathsf{m2bC}(\alpha, \beta, \gamma, \delta) \in \mathcal{P} \implies \in \mathcal{P}. \tag{XVIII}$$

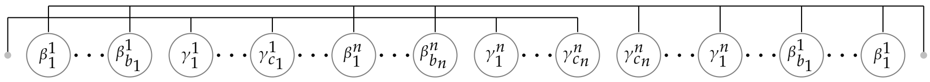
For the proof let us assume $\pi \in \mathcal{P}$ is of type from equation (VI) of the proof of Lemma 4.2.43 **(c)** for some $n \in \mathbb{N}$. The proof for the other type is similar. We use the notation from equation (VI) of the proof of Lemma 4.2.43 **(c)**. We calculate for any $\beta,\ \gamma \in \mathbb{A}(\{\circ, \bullet\})$

$\pi :=$



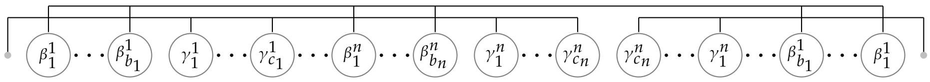
$\in \mathcal{P}$

$\implies \mathcal{P} \ni \pi_1 :=$

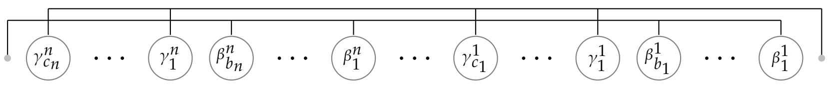


⟦ we insert the sequence of legs in reverse order from the beginning of π: starting with $(\gamma^n_{(c_n+1)-i})_{i\in[c_n]}$, then $(\beta^n_{(b_n+1)-i})_{i\in[b_n]}$ until $(\gamma^1_{(c_1+1)-i})_{i\in[c_n]}$, then $(\beta^1_{(b_1+1)-i})_{i\in[b_1]}$ are reached ⟧

$\implies \mathcal{P} \ni \pi_2 :=$



⟦ split the last two legs of π_1 by

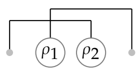


$= \text{mirror}(\pi) \in \mathcal{P}$ ⟧

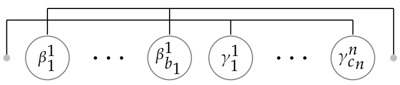
$\implies$  $\in \mathcal{P}$ ⟦ apply $\text{pr}_2 \circ \text{UMem}$ to neighboring legs with color $\gamma^n_{c_n}$ in π_2, then successively delete ⟧

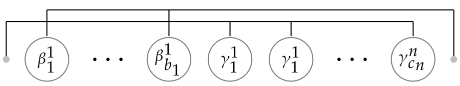
Next, we want to proof

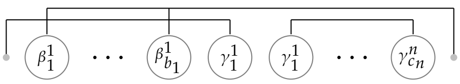
$$\forall \alpha, \beta, \gamma, \delta \in \mathbb{A}(\{\circ, \bullet\}),\ \exists (\rho_1, \rho_2) \in \{\circ, \bullet\}^{\times 2}:$$

$$\text{m2bC}(\alpha, \beta, \gamma, \delta) \in \mathcal{P} \implies$$  $\in \mathcal{P}$. (XIX)

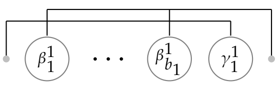
For the proof assume that $\pi \in \mathcal{P}$ is of type from equation (VI) for some $n \in \mathbb{N}$. The proof for the other type from equation (VII) of the proof of Lemma 4.2.43 **(c)** is similar. We calculate

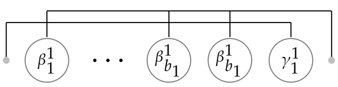
$\pi =$  $\in \mathcal{P}$

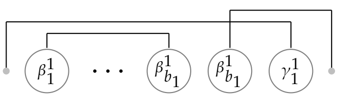
$\implies \pi_1 :=$  $\in \mathcal{P}$ ⟦ $\pi_1 = \text{double}(\pi)$ ⟧

$\implies \pi_2 :=$  $\in \mathcal{P}$

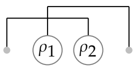
⟦ use eq. (XVIII) to split π_1 by two-block interval partition ⟧

$\implies \pi_3 :=$  $\in \mathcal{P}$ ⟦ $\pi_3 = (\text{pr}_2 \circ \text{UMem}^1_{\cdot,3})(\pi_2)$ ⟧

$\implies \pi_4 :=$  $\in \mathcal{P}$ ⟦ $\pi_4 = \text{double}(\pi_3)$ ⟧

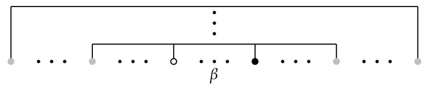
$\implies \pi_5 :=$  $\in \mathcal{P}$

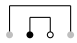
$[\![$ use eq. (XVIII) to split π_4 by two-block interval partition $]\!]$

$\Longrightarrow \pi_6 :=$ 

$\in \mathcal{P}$ $[\![\ \pi_6 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,3})(\mathsf{mirror}(\pi_5)),\ \rho_1 := \beta^1_{b_1},\ \rho_2 := \gamma^1_1\ ]\!]$

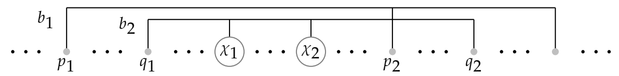
In the proof of Claim 6 we have shown that

$\pi_2 \mathrel{\widehat{=}}$ 

$\in \mathcal{P} \Longrightarrow$  $\in \mathcal{P}$ (XX)

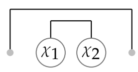
From equation (XIX), equation (XX) and Lemma 4.2.43 we can conclude $\mathcal{P} = \mathsf{Part}_{\{\circ,\bullet\}}$, which contradicts our assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}})$. Thus, in case of **(d)** we have shown that Claim 15 is true.

Ad **(e)**: In this case we have the existence of a partition $\pi \in \mathcal{P}$ such that $\mathsf{Cross}(\pi) \neq \emptyset \wedge \pi \notin \mathsf{pureC} \wedge \pi \notin \mathsf{biNC}$. If $\pi \in \mathcal{P}$ satisfies these prerequisites, then we have the following occurrence in the partition π, which we schematically express in a diagram

$\pi \mathrel{\widehat{=}}$ 

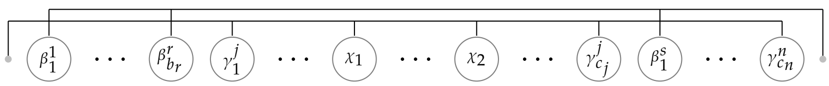
$\in \mathcal{P}.$ (XXI)

Hence, in $\pi \in \mathcal{P}$ there exist two blocks b_1 and b_2, which lead to a crossing $((p_1,p_2),(q_1,q_2)) \in \mathsf{Cross}(\pi)$. This crossing violates the defining properties for biNC and pureC. Hence, there need to exist two legs with the colors χ_1 and χ_2 such that $(\chi_1,\chi_2) \in \mathbb{A}(\{\circ,\bullet\})$. Now, we claim

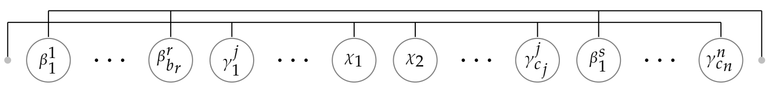
$\exists(\chi_1,\chi_2) \in \mathbb{A}(\{\circ,\bullet\})\colon \mathcal{P} \ni \pi$ from eq. (XXI) $\Longrightarrow$ 

$\in \mathcal{P}.$ (XXII)

For the proof consider the following argumentation. If we apply the "unification procedure" to π until b_1 and b_2 are the first two blocks of π, then we obtain by $\{b_1,b_2\} \in \mathcal{P}$. By deleting all block neighboring legs by delete, there needs to exist a reduced, two-block, crossing partition $\mathsf{m2bC}(\alpha,\beta,\gamma,\delta) \in \mathcal{P}$ for certain $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{A}(\{\circ,\bullet\})$. We denote this two-block partition again by π and for the proof of equation (XXII) we assume it is of type from equation (VI) of the proof of Lemma 4.2.43 **(c)** for some $n \in \mathbb{N}$. There is a similar proof if it is of type from equation (VII) of the proof of Lemma 4.2.43 **(c)**. The legs with the color $\chi_1 \neq \chi_2 \in \{\circ,\bullet\}$, which exist by equation (XXI), have to appear at a certain position in the partition π. Let us assume they appear in one of the sequences $(\gamma^j_i)_{i\in[c_j]}$ for some $j \in [n]$. If these legs show up in one of the sequences $(\beta^j_i)_{i\in[b_j]}$ for some $j \in [n]$, the proof is analogous to the following one. For convenience of drawing the diagrams for the partitions let us assume $1 < j < n$. We set $r := j-1$ and $s := j+1$, then we calculate

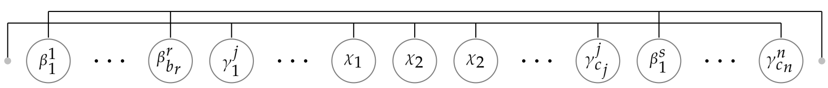
$\pi =$ 

$\in \mathcal{P}$

$\Longrightarrow \pi_1 :=$ 

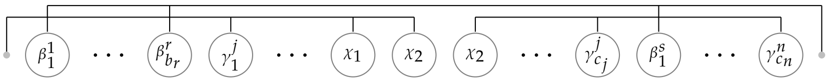
$\in \mathcal{P}$

$[\![$ π is a reduced partition and $(\chi_1,\chi_2) \in \mathbb{A}(\{\circ,\bullet\})$, hence we can assume that there exist neighboring legs, which differ in their color, we denote the color of these neighboring legs again by $(\chi_1,\chi_2) \in \mathbb{A}(\{\circ,\bullet\})$ $]\!]$

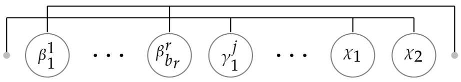
$\Longrightarrow \pi_2 :=$ 

$\in \mathcal{P}$

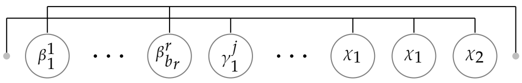
$[\![\ \pi_2 = \mathsf{double}(\pi_1)\ ]\!]$

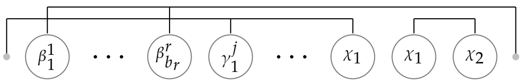
$\Longrightarrow \pi_3 :=$  $\in \mathcal{P}$

$[\![$ use eq. (XVIII) to split π_2 by two-block interval partition $]\!]$

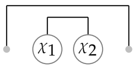
$\Longrightarrow \pi_4 :=$  $\in \mathcal{P}$

$[\![\ \pi_4 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,3})(\pi_3)\ ]\!]$

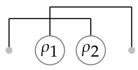
$\Longrightarrow \pi_5 :=$  $\in \mathcal{P} \quad [\![\ \pi_5 = \mathsf{double}(\pi_4)\ ]\!]$

$\Longrightarrow \pi_6 :=$  $\in \mathcal{P}$

$[\![$ use eq. (XVIII) to split π_5 by two-block interval partition $]\!]$

$\Longrightarrow \pi_7 :=$  $\in \mathcal{P} \quad [\![\ \pi_7 = (\mathsf{pr}_2 \circ \mathsf{UMem}^1_{\cdot,3})\big(\mathsf{mirror}(\pi_6)\big)\ ]\!]$

This finishes the proof of equation (XXII). Furthermore we have

$\mathcal{P} \ni \pi$ from eq. (XXI) $\Longrightarrow$  $\in \mathcal{P}.$ (XXIII)

Once again we can apply the above described unification procedure to π to obtain the existence of a reduced, two-block, crossing partition $\mathsf{m2bC}(\alpha,\beta,\gamma,\delta) \in \mathcal{P}$ for certain $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{A}(\{\circ,\bullet\})$. The implication of equation (XIX) shows equation (XXIII). From equation (XXII), equation (XXIII) and Lemma 4.2.43 we can conclude $\mathcal{P} = \mathsf{Part}_{\{\circ,\bullet\}}$ which contradicts our assumption that $\mathcal{P} \in \mathsf{ipred}(\mathsf{Part}_{\{\circ,\bullet\}})$. Thus, in case of **(e)** we have shown that Claim 15 is true. □

Now, that we have classified all the two-colored universal classes of partitions we can display them in a so-called Hasse diagram. This can be seen in Figure 4.1 at the end of this section, where we have drawn the immediate predecessor relations as lines and the convention is that if $x \in \mathsf{ipred}(y)$, then x is drawn below y.

4.2.45 Definition (Lattice, complete lattice [Jac85, Def. 8.2]). A *lattice* is a partially ordered set P in which any two elements have a least upper bound and a greatest lower bound. A partially ordered set is called *complete lattice* if every subset $A \subseteq P$ has a least upper bound and a greatest lower bound.

We recall the following Theorem.

4.2.46 Theorem ([Jac85, Thm. 8.1]). A partially ordered set P with greatest element $\mathbb{1}_P$ such that every non-empty subset $P' \subseteq P$ has a greatest lower bound is a complete lattice, i. e., if every subset $P' \subseteq P$ has a sup and inf.

4.2.47 Remark. Assume we have a finite partially ordered set P, where $\pi \wedge \sigma$ exists for all $\pi,\ \sigma \in P$. Then, by induction we can show that for any subset $P' \subseteq P$ its meet $\bigwedge P'$ exists. Thus, by the above theorem we can conclude, that if P is a finite poset with greatest element $\mathbb{1}_P$ and if every two elements $\pi,\ \sigma \in P$ have a meet, then P is a complete lattice.

For a two-colored universal class of partitions $\mathcal{P}$, for each $n \in \mathbb{N}$ and $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ its subset $\mathcal{P}_\varepsilon$ is a poset by the reversed refinement order, i. e., we write $\pi \leq \sigma$ if and only if each block of π is completely contained in one of the blocks of σ. This is the definition given in [NS06, Def. 9.14]. Therefore, the following statement makes sense.

4.2.48 Lemma. Let $\mathcal{P}$ be a two-colored universal class of partitions. If $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ for some $n \in \mathbb{N}$, then $\mathcal{P}_\varepsilon$ is a complete lattice.

Proof: This is only a sketch of proof. Let $\mathcal{P}_\varepsilon$ be such a two-colored universal class of partitions. From Definition 3.4.9 **(a)** we have $\mathbb{1}_\varepsilon \in \mathcal{P}_\varepsilon$. Hence $\mathcal{P}_\varepsilon$ has a greatest element. It remains to show that any two partitions $\pi, \sigma \in \mathcal{P}_\varepsilon$ have a meet $\pi \wedge \sigma$. From the proof of [NS06, Prop. 9.17] we obtain a candidate for the meet of $\pi = \{V_1, \ldots, V_r\}$ and $\sigma = \{W_1, \ldots, W_k\}$, namely

$$\{\, V_i \cap W_j \mid i \in [r],\ j \in [k],\ V_i \cap W_j \neq \emptyset \,\}. \tag{I}$$

We claim the above expression defines a partition in $\mathcal{P}_\varepsilon$ and is the supremum of π and σ. Abbreviate the expression of equation (I) by $\pi \wedge \sigma$, although we have not yet shown that the expression of equation (I) has the desired properties. Assume that $\pi \wedge \sigma$ exists in $\mathcal{P}_\varepsilon$, then we need to show that $\pi \wedge \sigma$ is the greatest lower bound of π and σ. Of course, it is a lower bound of π and σ in the reversed refinement order. Assume that $\rho \in \mathcal{P}_\varepsilon$ is a lower bound of π and σ, for each block b of ρ there exists a block $V_i \in \pi$ and a block $W_j \in \sigma$ such that $b \subseteq V_i$ and $b \subseteq W_j$. Therefore, $b \subseteq V_i \cap W_j$ which implies $\rho \leq \pi \wedge \sigma$. It remains to show the existence of $\pi \wedge \sigma$ in the regarding subset $\mathcal{P}_\varepsilon$ of a two-colored universal class of partitions $\mathcal{P}$. Since we have already classified all possible classes $\mathcal{P}$, we will show existence of the partition $\pi \wedge \sigma$ for each subset $\mathcal{P}_\varepsilon$ with arbitrary but fixed $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [m]^{\times n}$ for any $n \in \mathbb{N}$.

Let $\mathcal{P} = 1\mathsf{B}_{\{\circ,\bullet\}}$, then $\pi \wedge \sigma \in \mathcal{P}_\varepsilon$, since $\mathcal{P}_\varepsilon$ only consists of one element and $\pi \wedge \pi = \pi \in \mathcal{P}_\varepsilon$.

Let $\mathcal{P} = \mathsf{I}_{\{\circ,\bullet\}}$ and let $\pi, \sigma \in \mathcal{P}_\varepsilon$. Assume that $\pi \wedge \sigma \notin \mathcal{P}_\varepsilon$. According to Definition 4.2.13,**(b)** the implication of equation (4.2.15a) does not hold for $\pi \wedge \sigma$. Thus, there need to exist blocks $b, b' \in \pi \wedge \sigma$ and legs $i, j, k \in \mathsf{set}(\pi \wedge \sigma))$ such that $b \neq b'$ and

$$i, k \in \mathsf{set}(\mathsf{pr}_1(b)) \text{ and } j \in \mathsf{set}(\mathsf{pr}_1(b')). \tag{II}$$

By definition of blocks in $\pi \wedge \sigma$, there need to exist blocks V of π and W of σ such that $b = V \cap W$. Similarly, we have blocks V', W' such that $b' = V' \cap W'$. We have $V \neq V'$ or $W \neq W'$. Assume $V \neq V'$, then we conclude from equation (II) that $i, k \in \mathsf{set}(\mathsf{pr}_1(V))$ and $j \in \mathsf{set}(\mathsf{pr}_1(W)$. This violates equation (4.2.15a) and therefore contradicts our assumption that $\pi \in \mathsf{I}_{\{\circ,\bullet\}}$. A similar statement holds for the case $W \neq W'$. By proof of contradiction we have shown that $\forall \pi, \sigma \in \mathcal{P}_\varepsilon \colon \pi \wedge \sigma \in \mathcal{P}_\varepsilon$.

Let $\mathcal{P} = \mathsf{I}_\circ\mathsf{NC}_\bullet$ and let $\pi, \sigma \in \mathcal{P}_\varepsilon$. Assume $\pi \wedge \sigma \notin \mathcal{P}_\varepsilon$. Thus according to Definition 4.2.13 **(c)** equation (4.2.16a) is not satisfied or the implication of equation (4.2.16b) is false for $\pi \wedge \sigma$. Assume that $\mathsf{pr}_1(\pi \wedge \sigma) \notin \mathsf{NC}$. This means that $\exists\,\big((p_1, p_2), (q_1, q_2)\big) \in \mathsf{Cross}(\pi \wedge \sigma)$. This implies the existence of two blocks $b, b' \in \pi \wedge \sigma$ with $b \neq b'$ such that $p_1, p_2 \in \mathsf{set}\, b$ and $q_1, q_2 \in \mathsf{set}\, b'$. Like in the above case for $\mathcal{P} = \mathsf{I}_{\{\circ,\bullet\}}$ this would imply that $\mathsf{pr}_1(\pi) \notin \mathsf{NC}$ or $\mathsf{pr}_1(\sigma) \notin \mathsf{NC}$ which is a contradiction to our assumption. We can look at the proof of Lemma 4.2.18 for the negation of the implication of equation (4.2.16b). We claim that the property of equation (II) of Lemma 4.2.18 is inherited by the partition π or σ, because of the definition of the blocks in $\pi \wedge \sigma$. But this would contradict the assumption that $\pi, \sigma \in \mathsf{I}_\circ\mathsf{NC}_\bullet$.

For convenience of the reader we do not further elaborate the proofs for the remaining classes. We just say that they all share the same proof of concept. We let $\pi, \sigma \in \mathcal{P}$ and assume

$\pi \wedge \sigma \notin \mathcal{P}$. Negations for the conditions that a partition is an element in a certain universal class of partition have been done in the proofs of Lemma 4.2.16 till Lemma 4.2.27. If we go through all the possible scenarios of how $\pi \wedge \sigma$ can not be an element of a certain universal class, we see by the definition of blocks in $\pi \wedge \sigma$ which are non-empty intersections of blocks from π and σ, that any violating scenario is inherited by π or σ which contradicts the assumption $\pi \in \mathcal{P}$ and $\sigma \in \mathcal{P}$. □

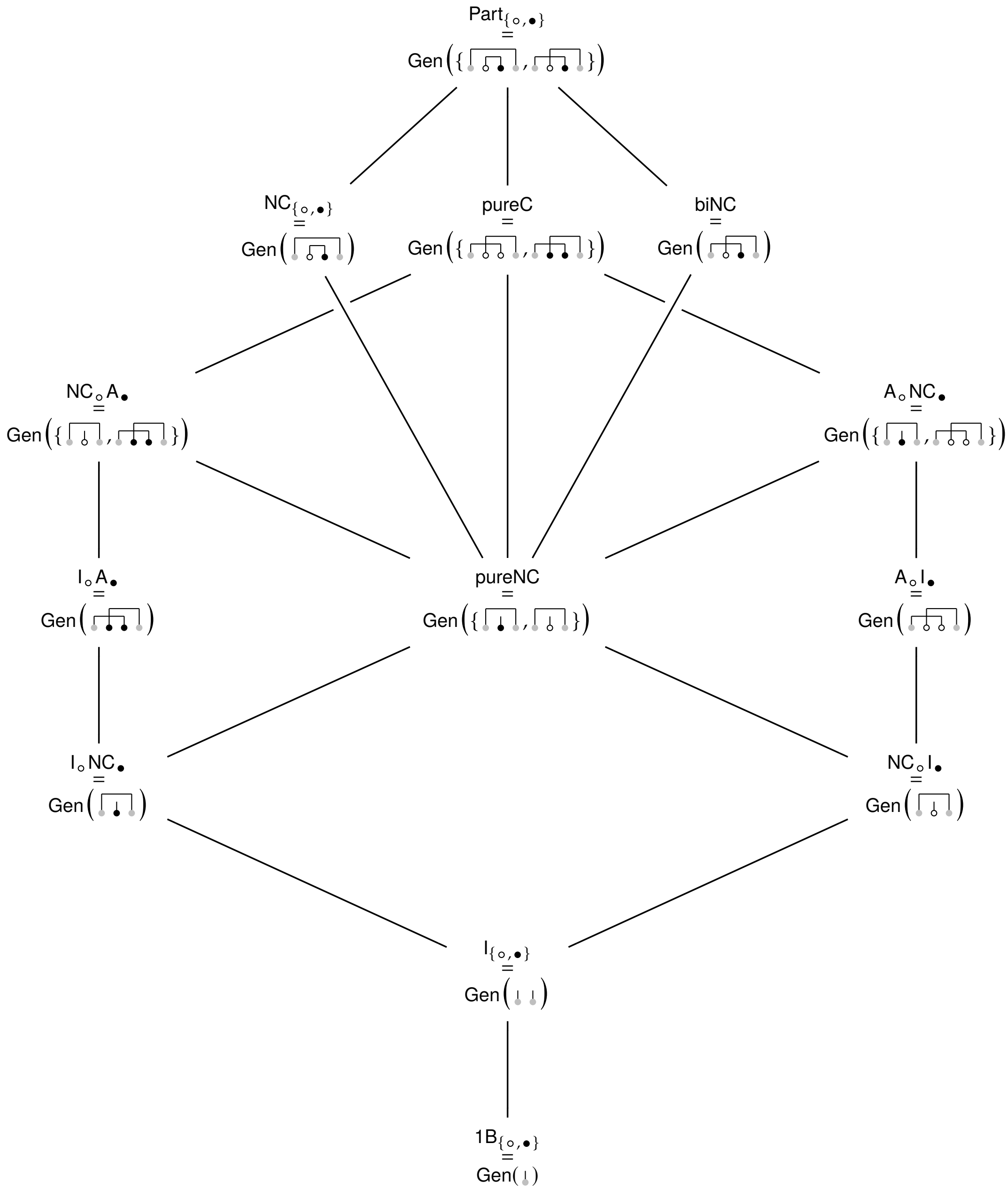


Figure 4.1: Hasse diagram of all two-colored universal classes of partitions.

Chapter 5

Universal class of partitions from positive and symmetric u.a.u.-products

In this chapter we use the classification results of the previous chapter to seek for a classification of positive and symmetric u.a.u.-products in the single-faced case and the two-faced case. For this, the main tool is to show that the partitions corresponding to nonzero highest coefficients for such products obey the properties of a universal class of partitions. In Section 5.1 we explicitly do this for the single-colored case and confirm Speicher's classification result from [Spe97] or in our axiomatic setting Ben Ghorbal's and Schürmann's result [BS02]. In Section 5.2 we provide a partial classification result for the two-faced case.

5.1 Classification of positive and symmetric single-faced u.a.u.-products

5.1.1 Lemma. Let $\odot$ be a u.a.u.-product in the category AlgP and let $k \in \mathbb{N}$. Let $\big(\mathcal{A}_i, \varphi_i\big)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$. By Convention 3.3.1 we set

$$\forall\, r \in [k]\colon \mathrm{j}_i \coloneqq \mathcal{T}\Big(\bigoplus_{i=1}^{k} \Delta_{i,r}\Big). \tag{5.1.1}$$

Put

$$V \coloneqq \bigoplus_{i=1}^{k} \mathcal{A}_i \tag{5.1.2}$$

and consider $(\mathrm{T}(V), \Delta, 0)$ as the single-faced dual semigroup with primitive comultiplication Δ from Example 2.2.8. Then,

$$\varphi_1 \odot \cdots \odot \varphi_k = \Big((\varphi_1 \circ \mathrm{j}_1) \circledast \cdots \circledast (\varphi_k \circ \mathrm{j}_k)\Big) \circ \mathsf{inc}_{\bigsqcup_{i=1}^{k} \mathcal{A}_i, \mathrm{T}(V)} \tag{5.1.3}$$

where $\circledast$ denotes the convolution with respect to the primitive comultiplication Δ and $\mathsf{inc}_{\bigsqcup_{i=1}^{k} \mathcal{A}_i, \mathrm{T}(V)}$ is the canonical inclusion of vector spaces.

Proof: The proof is similar to the proof of Theorem 2.4.12, where we just need to set the color index to one, i. e., $m = 1$. On the other hand, we need to extend the "type index" to $k \in \mathbb{N}$ algebras. The necessary steps can be directly adjusted and therefore we omit the proof and instead refer to the proof of Theorem 2.4.12. □

Similar to the result of Lemma 2.4.6 we want to define a "linearized" version of ⊙ without the prerequisite of a comonoid in the category $\mathsf{Alg_m}$. Let ⊙ be an u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $(\mathscr{A}_i, \varphi_i)_{i\in[k]} \in (\mathsf{Obj}(\mathsf{AlgP_m}))^{\times k}$. Then, we define

$$T_{k\boxdot}(\varphi_1, \dots, \varphi_k) \colon \begin{cases} \prod_{i=1}^{k} \mathscr{A}_i \longrightarrow \mathbb{C} \\ a \longmapsto \dfrac{\partial^n}{\partial t_1 \dots \partial t_n}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a)\Big)\Big|_{t_1=\ldots=t_k=0}. \end{cases} \tag{5.1.4}$$

Thanks to the universal coefficient theorem (Theorem 2.3.3) we can see that the derivatives exist and furthermore $T_{k\boxdot}(\varphi_1, \dots, \varphi_k) \in \mathrm{Lin}(\bigsqcup_{i=1}^k \mathscr{A}_i, \mathbb{C})$. This justifies the following definition

5.1.2 Definition. Let ⊙ be a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $(\mathscr{A}_i, \varphi_i)_{i\in[k]} \in (\mathsf{Obj}(\mathsf{AlgP_m}))^{\times k}$. Then, we set

$$T_{k\boxdot} \colon \begin{cases} \prod_{i=1}^{k} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C}) \longrightarrow \mathrm{Lin}\Big(\bigsqcup_{i=1}^{k} \mathscr{A}_i, \mathbb{C}\Big) \\ (\varphi_1, \dots, \varphi_k) \longmapsto T_{k\boxdot}(\varphi_1, \dots, \varphi_k). \end{cases} \tag{5.1.5}$$

We also use the notation $\varphi_1 \boxdot \cdots \boxdot \varphi_k \coloneqq T_{k\boxdot}(\varphi_1, \dots, \varphi_k)$.

5.1.3 Remark. The relation of $T_{k\boxdot}$ to the k-fold operation $T_{\boxplus k}$ on the dual space of a given comonoid in the category AlgP, is the following. If we consider equation (5.1.3), then we obtain for any basis element $a \in \bigsqcup_{i=1}^k \mathscr{A}_i$

$$\begin{aligned}
&(\varphi_1 \boxdot \cdots \boxdot \varphi_k)(a) \\
&= \frac{\partial^n}{\partial t_1 \dots \partial t_n}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ eq. (5.1.4) }]\!] \\
&= \frac{\partial}{\partial t_1 \dots \partial t_n}\Big(\Big(\big((t_1\varphi_1)\circ \mathrm{j}_1\big) \circledast \cdots \circledast \big((t_k\varphi_k)\circ \mathrm{j}_k\big)\Big)\big(\mathsf{inc}_{\bigsqcup_{i=1}^k \mathscr{A}_i, \mathrm{T}(V)}(a)\big)\Big)\Big|_{t_1=\ldots=t_k=0} \\
&\quad [\![\text{ eq. (5.1.3) }]\!] \\
&= \frac{\partial}{\partial t_1 \dots \partial t_n}\Big(\Big(\big(t_1(\varphi_1\circ \mathrm{j}_1)\big) \circledast \cdots \circledast \big(t_k(\varphi_k\circ \mathrm{j})_k\big)\Big)\big(\mathsf{inc}_{\bigsqcup_{i=1}^k \mathscr{A}_i, \mathrm{T}(V)}(a)\big)\Big)\Big|_{t_1=\ldots=t_k=0} \\
&= \big((\varphi_1\circ \mathrm{j}_1) \boxplus \cdots \boxplus (\varphi_k\circ \mathrm{j}_k)\big)\big(\mathsf{inc}_{\bigsqcup_{i=1}^k \mathscr{A}_i, \mathrm{T}(V)}(a)\big) \qquad (5.1.6) \\
&\quad [\![\text{ Lem. 2.4.6 applied to } \mathbb{N}_0\text{-graded comonoid } \mathrm{T}\big(\textstyle\bigoplus_{i=1}^k \mathscr{A}_i\big)]\!].
\end{aligned}$$

Roughly speaking, the $(k-1)$-fold operation $T_{k\boxdot}$ on $\prod_{i=1}^k \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ can be seen as a restriction of the $(k-1)$-fold operation $T_{k\boxplus}$ on $\mathrm{Lin}\big(\mathrm{T}(\bigoplus_{i=1}^k \mathscr{A}_i), \mathbb{C}\big)$.

5.1.4 Lemma. Assume ⊙ is a u.a.u.-product in the category AlgP. Then,

$\forall k \in \mathbb{N},\ \forall n \in \mathbb{N} \setminus [k-1],\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in [k]^{\times n},\ \forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k},\ \forall (a_i)_{i\in[n]} \in$

$\prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$:

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(\tilde{a}_1 \cdot \cdots \cdot \tilde{a}_{\tilde{n}}), \tag{5.1.7}$$

where $(\tilde{a}_i)_{i\in[\tilde{n}]} \in \prod_{i=1}^{n} \mathscr{A}_{\tilde{\varepsilon}_i}$ is the unique tuple such that

$$\exists\, \tilde{n} \in [n] \colon (\tilde{\varepsilon}_i)_{i\in[\tilde{n}]} = \operatorname{red}(\varepsilon) \in \mathbb{A}([k]). \tag{5.1.8}$$

The definition of the reduced tuple $\operatorname{red}(\varepsilon)$ is due to Convention 2.5.8.

Proof: This follows from Definition 5.1.2 and Convention 2.5.8. □

5.1.5 Lemma. Let $\odot$ be u.a.u.-product with the right-ordered monomials property in the category AlgP. With respect to the notation from Theorem 2.3.3 we have

(a) $\forall\, k \in \mathbb{N} \setminus \{1\}$, $\forall\, n \in \mathbb{N} \setminus [k-1]$, $\forall\, \varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k])$, $\forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathrm{Obj}(\mathsf{AlgP})\big)^{\times k}$, $\forall (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$:

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \underbrace{\alpha^{(\varepsilon)}_{\mathbb{1}_1,\ldots,\mathbb{1}_k}}_{\equiv \alpha^{(\varepsilon)}_{\max}} \prod_{i=1}^{k} \varphi_i\big(j(a_1,\ldots,a_n)(\mathbb{1}_i)\big), \tag{5.1.9}$$

where $\mathbb{1}_i$ denotes the unique monomial in $\mathcal{OM}(X^{(i)}_\varepsilon)$ of degree $|X^{(i)}_\varepsilon|$ (notation introduced in Remark 2.3.4 **(b)**) and $\alpha^{(\varepsilon)}_{\max}$ is the highest coefficient w.r.t. $\odot$, defined in Definition 2.5.3.

(b) $\forall\, k \in \mathbb{N} \setminus \{1\}$, $\forall\, n \in \mathbb{N} \setminus [k-1]$, $\forall\, \varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k])$, $\exists (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathrm{Obj}(\mathsf{AlgP})\big)^{\times k}$, $\exists (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$:

$$\forall\, i \in [k] \colon \varphi_i\big(j(a_1,\ldots,a_n)(\mathbb{1}_i)\big) = 1, \tag{5.1.10a}$$

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \alpha^{(\varepsilon)}_{\max}. \tag{5.1.10b}$$

Proof: Ad **(a)**: By the given assumptions we calculate

$$\begin{aligned}
&\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n)\\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ def. of } \boxdot \text{ in eq. (5.1.4) }]\!]\\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\sum_{\substack{\pi_1\\ \in \mathcal{OM}(X^{(1)}_\varepsilon)}} \cdots \sum_{\substack{\pi_k\\ \in \mathcal{OM}(X^{(k)}_\varepsilon)}} \alpha^{(\varepsilon)}_{\pi_1,\ldots,\pi_k}\\
&\prod_{M_1\in\pi_1} (t_1\varphi_1)\big(j(a_1,\ldots,a_n)(M_1)\big) \ldots \prod_{M_k\in\pi_k} (t_k\varphi_k)\big(j(a_1,\ldots,a_n)(M_k)\big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}\\
&\quad [\![\text{ [MS17, equation (4.5)] }]\!]\\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\alpha^{(\varepsilon)}_{\max}(t_1\varphi_1)\big(j(a_1,\ldots,a_n)(\mathbb{1}_1)\big) \ldots (t_k\varphi_k)\big(j(a_1,\ldots,a_n)(\mathbb{1}_k)\big)
\end{aligned}$$

$$+ \sum_{\substack{\pi_1 \\ \in \mathcal{OM}(X_\varepsilon^{(1)}), \\ \pi_1 \neq \mathbb{1}_1}} \cdots \sum_{\substack{\pi_k \\ \in \mathcal{OM}(X_\varepsilon^{(k)})}} \alpha^{(\varepsilon)}_{\pi_1,\ldots,\pi_k}$$

$$\prod_{M_1 \in \pi_1} (t_1\varphi_1)\big(j(a_1,\ldots,a_n)(M_1)\big) \ldots \prod_{M_k \in \pi_k} (t_k\varphi_k)\big(j(a_1,\ldots,a_n)(M_k)\big)$$

$$+ \cdots + \sum_{\substack{\pi_1 \\ \in \mathcal{OM}(X_\varepsilon^{(1)})}} \cdots \sum_{\substack{\pi_k \\ \in \mathcal{OM}(X_\varepsilon^{(i)}), \\ \pi_i \neq \mathbb{1}_i}} \cdots \sum_{\substack{\pi_k \\ \in \mathcal{OM}(X_\varepsilon^{(k)})}} \alpha^{(\varepsilon)}_{\pi_1,\ldots,\pi_i,\ldots,\pi_k}$$

$$\prod_{M_1 \in \pi_1} (t_1\varphi_1)\big(j(a_1,\ldots,a_n)(M_1)\big) \ldots \prod_{M_i \in \pi_i} (t_i\varphi_i)\big(j(a_1,\ldots,a_n)(M_i)\big)$$

$$+ \cdots + \sum_{\substack{\pi_1 \\ \in \mathcal{OM}(X_\varepsilon^{(1)})}} \cdots \sum_{\substack{\pi_k \\ \in \mathcal{OM}(X_\varepsilon^{(k)}), \\ \pi_k \neq \mathbb{1}_k}} \alpha^{(\varepsilon)}_{\pi_1,\ldots,\pi_k}$$

$$\prod_{M_1 \in \pi_1} (t_1\varphi_1)\big(j(a_1,\ldots,a_n)(M_1)\big) \ldots \prod_{M_k \in \pi_k} (t_k\varphi_k)\big(j(a_1,\ldots,a_n)(M_k)\big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$$= \alpha^{(\varepsilon)}_{\max}\varphi_1\big(j(a_1,\ldots,a_n)(\mathbb{1}_1)\big) \ldots \varphi_k\big(j(a_1,\ldots,a_n)(\mathbb{1}_k)\big)$$

$$[\![\ \forall i \in [k]\colon \pi_i \neq \mathbb{1}_i \in \mathcal{OM}(X_\varepsilon^{(i)}) \implies \deg\Big(\prod_{M_i \in \pi_i} (t_i\varphi_i)\big(j(a_1,\ldots,a_n)(M_i)\big)\Big) \geq 2\ ]\!].$$

Ad **(b)**: Let V_j be a one-dimensional vector space for each $j \in [k]$. Let $a_j \in V_j$ be a basis vector for each $j \in [k]$. For each $j \in [k]$ define a linear functional on basis elements by

$$\varphi_j\colon \begin{cases} V_j \longrightarrow \mathbb{C} \\ a_j \longmapsto 1. \end{cases} \tag{I}$$

Now, let $\varepsilon = (\varepsilon_i)_{i\in[n]} \in [k]^{\times n}$, such that $\alpha^{(\varepsilon)}_{\max} \neq 0$. We can see that $\forall j \in [k]\colon (\mathrm{T}(V_j), \mathcal{T}(\varphi_j)) \in$ Obj(AlgP). Let $\iota_j\colon V_j \hookrightarrow \bigsqcup_{j=1}^k \mathrm{T}(V_j)$ be the canonical inclusion map for each $j \in [k]$. If we use $\bigsqcup_{j=1}^k \mathrm{T}(V_j) \simeq \mathrm{T}\big(\bigoplus_{i=1}^k V_j\big)$, then we can calculate

$$\begin{aligned}
&\big(\mathcal{T}(\varphi_1) \boxdot \cdots \boxdot \mathcal{T}(\varphi_k)\big)\big(\iota_{\varepsilon_1}(a_{\varepsilon_1}) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_{\varepsilon_n})\big) \\
&= \alpha^{(\varepsilon)}_{\max} \prod_{i=1}^k \mathcal{T}(\varphi_i)\big(j(a_{\varepsilon_1},\ldots,a_{\varepsilon_n})(\mathbb{1}_i)\big) \quad [\![\text{ Lem. 5.1.5 }\textbf{(a)}\ ]\!] \\
&= \alpha^{(\varepsilon)}_{\max} \quad [\![\text{ UMP for } \mathrm{T}\big(\textstyle\bigoplus_{i=1}^k V_j\big) \text{ and def. of } \varphi_j \text{ in eq. (I) }]\!].
\end{aligned}$$

□

5.1.6 Proposition. Let $\odot$ be a u.a.u.-product with the right-ordered monomials property in the category AlgP. Let $k \in \mathbb{N} \setminus \{1\}$, $n \in \mathbb{N} \setminus [k-1]$. Set

$$E_{n,k} := \{\, (\varepsilon_i)_{i\in[n]} \in [k]^{\times n} \mid \exists \ell \in \{0,\ldots,n-2\} \subseteq \mathbb{N}\colon \varepsilon_{\ell+1} = 1,\ \varepsilon_{\ell+2} = 2 \,\}. \tag{5.1.11}$$

Then, holds

$\forall k \in \mathbb{N} \setminus \{1\}, \forall n \in \mathbb{N} \setminus [k-1],\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in E_{n,k},\ \forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k}$, $\forall (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$:

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)\big(a_1 \cdot \dots \cdot \underbrace{a_{\ell+1}}_{\in \mathscr{A}_1} \cdot \underbrace{a_{\ell+2}}_{\in \mathscr{A}_2} \cdot \dots \cdot a_n\big)$$
$$= \Big(\big(\varphi_1 \boxdot \varphi_2\big) \boxdot \varphi_3 \boxdot \cdots \boxdot \varphi_k\Big)\big(a_1 \cdot \dots \cdot \underbrace{a_{\ell+1} \cdot a_{\ell+2}}_{\in \mathscr{A}_1 \sqcup \mathscr{A}_2} \cdot \dots \cdot a_n\big). \quad (5.1.12)$$

Proof: According to Lemma 5.1.4 it suffices to show the assertion for alternating tuples $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k])$. Furthermore, let $\varepsilon = (\varepsilon_i)_{i\in[n]} \in E_{n,k}$. Each n-tuple $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$ induces an element $a_1 \cdot \dots \cdot a_n \in \bigsqcup_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$. But by equation (1.1.17) we have a canonical algebra isomorphism $\bigsqcup_{i=1}^{n} \mathscr{A}_i \cong (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup (\bigsqcup_{i=3}^{n} \mathscr{A}_i)$. By application of this isomorphism we define

$$a_1 \cdot \dots \cdot a_n =: \tilde{a}_1 \cdot \dots \cdot \tilde{a}_{\tilde{n}} \in (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup (\bigsqcup_{i=3}^{n} \mathscr{A}_i),$$

where $\tilde{n} < n$, since we have assumed that $\varepsilon \in E_{n,k}$. We stress that we suppress any canonical homomorphic insertions by this notation. Moreover, for each $i \in [\tilde{n}]$ we either have $\tilde{a}_i \in \mathscr{A}_j$ for some $j \in [n] \setminus \{1,2\}$ or $\tilde{a}_i \in \mathscr{A}_1 \sqcup \mathscr{A}_2$. Therefore we can define for all $i \in [\tilde{n}]$

$$\tilde{\varepsilon}_i := \begin{cases} j, \text{ if } \tilde{a}_i \in \mathscr{A}_j \\ 1 \sqcup 2, \text{ if } \tilde{a}_i \in \mathscr{A}_1 \sqcup \mathscr{A}_2. \end{cases}$$

We set

$$\tilde{\varepsilon} := (\tilde{\varepsilon}_i)_{i\in[\tilde{n}]} \in \mathbb{A}(\{1 \sqcup 2\} \cup ([k] \setminus \{1,2\}))$$

and by definition for all $i \in [\tilde{n}]$ we have that $\tilde{a}_i \in \mathscr{A}_{\tilde{\varepsilon}_i}$. If we set $\varphi_{1\sqcup 2} := \varphi_1 \boxdot \varphi_2$, then we calculate for the right hand side of equation (5.1.12)

$$\Big(\big(\varphi_1 \boxdot \varphi_2\big) \boxdot \varphi_3 \boxdot \cdots \boxdot \varphi_k\Big)\big(a_1 \cdot \dots \cdot (a_{\ell+1} \cdot a_{\ell+2}) \cdot \dots \cdot a_n\big)$$
$$= \big(\varphi_{1\sqcup 2} \boxdot \varphi_3 \boxdot \cdots \boxdot \varphi_n\big)(\tilde{a}_1 \cdot \dots \cdot \tilde{a}_{\tilde{n}})$$
$$= \alpha_{\max}^{(\tilde{\varepsilon})} \varphi_{1\sqcup 2}\big(j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big) \dots \varphi_k\big(j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)$$

⟦ application of Lemma 5.1.5 **(a)** to $\tilde{\varepsilon}$ instead of ε and $\tilde{a}_i$ instead of a_i ⟧

$$= \alpha_{\max}^{(\tilde{\varepsilon})} (\varphi_1 \boxdot \varphi_2)\big(j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big) \dots \varphi_k\big(j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big) \quad [\![\text{def. of } \varphi_{1\sqcup 2}]\!]$$
$$= \alpha_{\max}^{(\tilde{\varepsilon})} (\varphi_1 \boxdot \varphi_2)\big(j(\bar{a}_1, \dots, \bar{a}_{\bar{n}})(\mathbb{1}_{1\sqcup 2})\big) \dots \varphi_k\big(j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)$$

⟦ $\exists \bar{n} \in \mathbb{N}, \exists \bar{\varepsilon} = (\bar{\varepsilon}_i)_{i\in[\bar{n}]} \in \mathbb{A}([2]), \exists (\bar{a}_i)_{i\in[\bar{n}]} \in \prod_{i=1}^{\bar{n}} \mathscr{A}_{\varepsilon_i}$: $j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2}) = \bar{a}_1 \cdot \dots \cdot \bar{a}_{\bar{n}} \in \mathscr{A}_1 \sqcup \mathscr{A}_2$ ⟧

$$= \alpha_{\max}^{(\tilde{\varepsilon})} \alpha_{\max}^{(\bar{\varepsilon})} \varphi_1\big(j(\bar{a}_1, \dots, \bar{a}_{\bar{n}})(\mathbb{1}_1)\big) \varphi_2\big(j(\bar{a}_1, \dots, \bar{a}_{\bar{n}})(\mathbb{1}_2)\big) \dots \varphi_k\big(j(\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)$$

⟦ application of Lemma 5.1.5 **(a)** to $\bar{\varepsilon}$ instead of ε and $\bar{a}_i$ instead of a_i ⟧

$$= \alpha^{(\tilde{\varepsilon})}_{\max} \alpha^{(\bar{\varepsilon})}_{\max} \prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \quad [\![\text{ by def. of } (\tilde{a}_i)_{i\in[\tilde{n}]} \text{ and } (\bar{a}_i)_{i\in[\bar{n}]}]\!]. \tag{I}$$

Before calculating the left hand side of equation (5.1.12), we note that the expression $\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n)$ is polynomial in $t_1, \ldots, t_k \in \mathbb{R}$ for any value of $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathcal{A}_{\varepsilon_i}$. This is a consequence of equation (2.3.12). Hence, for any partial derivative of this expression we may apply Schwarz's theorem. We do this without explicitly mentioning. We calculate

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)\big(a_1 \cdot \cdots \cdot (a_{\ell+1} \cdot a_{\ell+2}) \cdot \cdots \cdot a_n\big)$$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)\big(a_1 \cdot \cdots \cdot (a_{\ell+1} \cdot a_{\ell+2}) \cdot \cdots \cdot a_n\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

$[\![$ def. of $\boxdot$ in eq. (5.1.4) $]\!]$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1) \odot (t_2\varphi_2)\big) \odot (t_3\varphi_3) \odot \cdots \odot (t_k\varphi_k)\Big)\big(a_1 \cdot \cdots \cdot (a_{\ell+1} \cdot a_{\ell+2}) \cdot \cdots \cdot a_n\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

$[\![$ $\odot$ is associative $]\!]$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\alpha^{(\tilde{\varepsilon})}_{\max}\big((t_1\varphi_1) \odot (t_2\varphi_2)\big)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big) \ldots (t_k\varphi_k)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)$$
$$+ \underbrace{\sum_{\substack{\pi_{1\sqcup 2} \\ \in \mathcal{OM}(X^{(1\sqcup 2)}_{\tilde{\varepsilon}})}} \cdots \sum_{\substack{\pi_k \\ \in \mathcal{OM}(X^{(k)}_{\tilde{\varepsilon}})}}}_{\exists \ell \in \{1\sqcup 2, 3, \ldots, k\}:\ \pi_\ell \neq \mathbb{1}_\ell} \alpha^{(\tilde{\varepsilon})}_{\pi_{1\sqcup 2}, \ldots, \pi_k} \prod_{M_{1\sqcup 2}\in\pi_{1\sqcup 2}}\big((t_1\varphi_1) \odot (t_2\varphi_2)\big)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(M_{1\sqcup 2})\big)$$
$$\ldots \prod_{M_k\in\pi_k}(t_k\varphi_k)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(M_k)\big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\alpha^{(\tilde{\varepsilon})}_{\max}\big((t_1\varphi_1) \odot (t_2\varphi_2)\big)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big) \ldots (t_k\varphi_k)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)$$
$$+ \sum_{\substack{\pi_{1\sqcup 2} \\ \in \mathcal{OM}(X^{(1\sqcup 2)}_{\tilde{\varepsilon}}), \\ \pi_{1\sqcup 2} \neq \mathbb{1}_{1\sqcup 2}}} \alpha^{(\tilde{\varepsilon})}_{\pi_{1\sqcup 2}, \mathbb{1}_3, \ldots, \mathbb{1}_k} \prod_{M_{1\sqcup 2}\in\pi_{1\sqcup 2}}\big((t_1\varphi_1) \odot (t_2\varphi_2)\big)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(M_{1\sqcup 2})\big)$$
$$\cdot \prod_{i\in\{3,\ldots,k\}}(t_i\varphi_i)\big(j(\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}})(\mathbb{1}_i)\big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$[\![$ $\forall i \in [k] \setminus \{1,2\},\ :\ \pi_i \neq \mathbb{1}_i \in \mathcal{OM}(X^{(i)}_{\varepsilon})$
$\Longrightarrow \deg\Big(\prod_{M_i\in\pi_i}(t_i\varphi_i)\big(j(a_1, \ldots, a_n)(M_i)\big)\Big) \geq 2,$
linearity of partial derivative $]\!]$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\alpha^{(\tilde{\varepsilon})}_{\max}\big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big)\ldots(t_k\varphi_k)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)$$

$$+ \sum_{\substack{\pi_{1\sqcup 2} \\ \in \mathcal{OM}(X^{(1\sqcup 2)}_{\tilde{\varepsilon}}), \\ \pi_{1\sqcup 2}\neq \mathbb{1}_{1\sqcup 2}}} \alpha^{(\tilde{\varepsilon})}_{\pi_{1\sqcup 2},\mathbb{1}_3,\ldots,\mathbb{1}_k}$$

$$\prod_{\substack{M_{1\sqcup 2}\in\pi_{1\sqcup 2} \\ M_{1\sqcup 2}\neq M'_{1\sqcup 2}}} \big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M_{1\sqcup 2})\big) \cdot \big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M'_{1\sqcup 2})\big)$$

$$\cdot \prod_{i\in\{3,\ldots,k\}} (t_i\varphi_i)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_i)\big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$$\left[\!\left[\begin{array}{l} \forall \pi_{1\sqcup 2}\neq \mathbb{1}_{1\sqcup 2}\in\mathcal{OM}(X^{(1\sqcup 2)}_{\tilde{\varepsilon}}),\ \exists!\ M'_{1\sqcup 2}\in\pi_{1\sqcup 2}: \\ j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M'_{1\sqcup 2}) = \cdots a_{\ell+1}\cdot a_{\ell+2}\cdots \in \mathscr{A}_1\sqcup\mathscr{A}_2 \end{array}\right]\!\right]$$

$$= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\alpha^{(\tilde{\varepsilon})}_{\max}\big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big)\ldots(t_k\varphi_k)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_k)\big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0}$$

$$\left[\!\left[\begin{array}{l} \text{by Thm. 2.3.3} \\ \deg\Big(\big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M_{1\sqcup 2})\big) \\ \qquad \cdot\big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M'_{1\sqcup 2})\big)\Big) \geq 2, \\ \text{because} \\ \big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M'_{1\sqcup 2})\big) \\ \sim (t_1\varphi_1)(\cdots \underbrace{a_\ell}_{\in\mathscr{A}_1} \cdots)(t_2\varphi_2)(\cdots \underbrace{a_{\ell+1}}_{\in\mathscr{A}_2}\cdots) + \sum_{\pi_1}\sum_{\pi_2}\prod_{\ldots}\prod_{\ldots}\cdots, \\ \big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(M_{1\sqcup 2})\big) \sim (t_1\varphi_1)(\mathbb{1}_1)(t_2\varphi_2)(\mathbb{1}_2) + \sum_{\pi_1}\sum_{\pi_2}\prod_{\ldots}\prod_{\ldots}\cdots \\ \text{and partial derivative is linear} \end{array}\right]\!\right]$$

$$= \alpha^{(\tilde{\varepsilon})}_{\max}\frac{\partial^2}{\partial t_1\,\partial t_2}\Big(\big((t_1\varphi_1)\odot(t_2\varphi_2)\big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big)\Big)\Big|_{t_1=t_2=0}\prod_{i=3}^{k}\varphi_i\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_i)\big)$$

$$[\![\ \text{linearity of partial derivative}\ ]\!]$$

$$= \alpha^{(\tilde{\varepsilon})}_{\max}\Big(\varphi_1 \boxdot \varphi_2\Big)\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})\big)\prod_{i=3}^{k}\varphi_i\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_i)\big)$$

$$[\![\ \text{def. of } \boxdot \text{ in eq. (5.1.4)}\ ]\!]$$

$$= \alpha^{(\tilde{\varepsilon})}_{\max}(\varphi_1 \boxdot \varphi_2)\big(j(\bar{a}_1,\ldots,\bar{a}_{\bar{n}})(\mathbb{1}_{1\sqcup 2})\big)\prod_{i=3}^{k}\varphi_i\big(j(\tilde{a}_1,\ldots,\tilde{a}_{\tilde{n}})(\mathbb{1}_i)\big)$$

$$\left[\!\left[\begin{array}{l}\exists \bar{n}\in\mathbb{N},\ \exists\bar{\varepsilon}=(\bar{\varepsilon}_i)_{i\in[\bar{n}]}\in\mathbb{A}([2]),\ \exists(\bar{a}_i)_{i\in[\bar{n}]}\in\prod_{i=1}^{\bar{n}}\mathscr{A}_{\varepsilon_i}:\\ j(\tilde{a}_1,\dots,\tilde{a}_{\tilde{n}})(\mathbb{1}_{1\sqcup 2})=\bar{a}_1\cdot\dots\cdot\bar{a}_{\bar{n}}\in\mathscr{A}_1\sqcup\mathscr{A}_2\end{array}\right]\!\right]$$

$$=\alpha_{\max}^{(\tilde{\varepsilon})}\alpha_{\max}^{(\bar{\varepsilon})}\varphi_1\big(j(\bar{a}_1,\dots,\bar{a}_{\bar{n}})(\mathbb{1}_1)\big)\varphi_2\big(j(\bar{a}_1,\dots,\bar{a}_{\bar{n}})(\mathbb{1}_2)\big)\prod_{i=3}^{k}\varphi_i\big(j(\tilde{a}_1,\dots,\tilde{a}_{\tilde{n}})(\mathbb{1}_i)\big)$$

$[\![$ application of Lemma 5.1.5 **(a)** to $\bar{\varepsilon}$ instead of ε and $\bar{a}_i$ instead of a_i $]\!]$

$$=\alpha_{\max}^{(\tilde{\varepsilon})}\alpha_{\max}^{(\bar{\varepsilon})}\prod_{i=1}^{k}\varphi_i\big(j(a_1,\dots,a_n)(\mathbb{1}_i)\big)\quad[\![\text{ by def. of }(\tilde{a}_i)_{i\in[\tilde{n}]}\text{ and }(\bar{a}_i)_{i\in[\bar{n}]}\ ]\!].\tag{II}$$

Since equations (I) and (II) are equal to each other, we have shown that equation (5.1.12) holds. □

5.1.7 Example. We want to give an example that if $\varepsilon\notin E_{n,k}$, where the set $E_{n,k}$ is defined in equation (5.1.11), then it can happen that

$$\big(\varphi_1\boxdot\cdots\boxdot\varphi_k\big)\big(a_1\cdot\dots\cdot a_n\big)\neq\Big(\big(\varphi_1\boxdot\varphi_2\big)\boxdot\varphi_3\boxdot\cdots\boxdot\varphi_k\Big)\big(a_1\cdot\dots\cdot a_n\big).\tag{5.1.13}$$

This shows that the operation $\boxdot$ is in general not associative. We already know from Theorem 4.1.17 that the set of noncrossing partitions $\mathcal{P}:=\mathsf{NC}$ is a universal class of partitions which induces a u.a.u.-product denoted by $\odot_{\mathsf{NC}}$ according to Theorem 3.3.9. In Theorem 3.3.9 **(e)** we have shown that $\odot_{\mathcal{P}}$ satisfies the right-ordered monomials property. Therefore, the following calculation provides an example if $\varepsilon\notin E_{n,k}$, then equation (5.1.12) does not hold. Assume we are given $(\mathscr{A}_i,\varphi_i)_{i\in[3]}\in\big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times 3}$ and linear functionals $\varphi_i\neq 0$. Let $\varepsilon=(\varepsilon_i)_{i\in[4]}=(1,3,2,3)\notin E_{4,3}$ and a tuple $(a_i)_{i\in[4]}\in\prod_{i=1}^{4}\mathscr{A}_{\varepsilon_i}$. Then, we can calculate for $t_{12},\ t_3\in\mathbb{R}\setminus\{0\}$

$$\Bigg(\Big(t_{12}\underbrace{\big(\varphi_1\boxdot\varphi_2\big)}_{=:\varphi_{12}}\Big)\odot(t_3\varphi_3)\Bigg)\big(\underbrace{a_1}_{\in\mathscr{A}_1\sqcup\mathscr{A}_2}\cdot\underbrace{a_2}_{\in\mathscr{A}_3}\cdot\underbrace{a_3}_{\in\mathscr{A}_1\sqcup\mathscr{A}_2}\cdot\underbrace{a_4}_{\in\mathscr{A}_3}\big)$$

$$=\Big(\big((t_{12}\varphi_{12}\circ \mathrm{j}_{12})\,\widetilde{\circledcirc_{\mathcal{P}}}\,(\varphi_2\circ \mathrm{j}_3)\big)\circ(\mathbf{i}_{12}\sqcup\mathbf{i}_3)\Big)(a_1\cdot a_2\cdot a_3\cdot a_4)$$

$[\![$ def. of $\odot_{\mathcal{P}}$ in eq. (3.3.13) $]\!]$

$$=\Big(\big((t_{12}\varphi_{12}\circ \mathrm{j}_{12})\circledcirc_{\mathcal{P}}(t_3\varphi_3\circ \mathrm{j}_3)\big)\Big)(a_1\otimes a_2\otimes a_3\otimes a_4)\quad[\![\text{ def. }\widetilde{\circledcirc_{\mathcal{P}}}\text{ in eq. (3.3.10) }]\!]$$

$$=\Big(\sum_{\pi\in\mathcal{P}_4}\prod_{b\in\pi}\big(\log_{\mathcal{P}}(t_{12}\varphi_{12}\circ \mathrm{j}_{12})+\log_{\mathcal{P}}(t_3\varphi_3\circ \mathrm{j}_3)\big)\Big)(a_1\otimes a_2\otimes a_3\otimes a_4)$$

$[\![$ def. of $\circledcirc_{\mathcal{P}}$ in eq. (3.2.7) $]\!]$

$$=t_{12}\varphi_{12}(a_1\cdot a_3)\big((t_3)^2\varphi_3(a_2)\varphi_3(a_4)\big)+\big((t_{12})^2\varphi_{12}(a_1)\varphi_{12}(a_3)\big)t_3\varphi_3(a_2a_4)$$
$$-\big((t_{12})^2\varphi_{12}(a_1)\varphi_{12}(a_3)\big)\big((t_3)^2\varphi_3(a_2)\varphi_3(a_4)\big)\quad[\![\text{ Lem. 3.2.12 }]\!]$$

$$\Longrightarrow\ \Big(\big(\varphi_1\boxdot\varphi_2\big)\boxdot\varphi_3\Big)(a_1\cdot a_2\cdot a_3\cdot a_4)=0\quad[\![\text{ def. of }\boxdot\text{ in eq. (5.1.5) }]\!].\tag{5.1.14}$$

On the other hand, we can obtain for $t_1,\ t_2,\ t_3\in\mathbb{R}\setminus\{0\}$

$$\big((t_1\varphi_1)\odot_{\mathcal{P}}(t_2\varphi_2)\odot_{\mathcal{P}}(t_3\varphi_3)\big)(a_1\cdot a_2\cdot a_3\cdot a_4)$$

$$= \Big(\underbrace{\big((t_1\varphi_1) \odot_{\mathcal{P}} (t_2\varphi_2)\big)}_{=:\varphi_{12}} \odot_{\mathcal{P}} (t_3\varphi_3)\Big)(a_1 \otimes a_2 \otimes a_3 \otimes a_4) \quad [\![\odot_{\mathcal{P}} \text{ is associative}]\!]$$

$$= \varphi_{12}(a_1 \cdot a_3) \cdot \big((t_3)^2\varphi_3(a_2)\varphi_3(a_4)\big) + \big(\varphi_{12}(a_1)\varphi_{12}(a_3)\big)t_3\varphi_3(a_2a_4)$$

$$- \varphi_{12}(a_1)\varphi_{12}(a_3)\big((t_3)^2\varphi_3(a_2)\varphi_3(a_4)\big) \quad [\![\text{similar calculation as above}]\!]$$

$$= t_1\varphi_1(a_1)t_2\varphi_2(a_3)\big((t_3)^2\varphi_3(a_2)\varphi_3(a_4)\big) + (t_1\varphi_1(a_1))(t_2\varphi_2(a_3))(t_3\varphi_3(a_2a_4))$$

$$- t_1\varphi_1(a_1)t_2\varphi_2(a_3)\big((t_3)^2\varphi_3(a_2)\varphi_3(a_4)\big) \quad [\![\odot_{\mathcal{P}} \text{ is unital}]\!]$$

$$\Longrightarrow \big(\varphi_1 \boxdot \varphi_2 \boxdot \varphi_3\big)(a_1 \cdot a_2 \cdot a_3 \cdot a_4) = \varphi_1(a_1)\varphi_2(a_3)\varphi_3(a_2a_4) \tag{5.1.15}$$

$[\![$ def. of $\boxdot$ in eq. (5.1.5) $]\!]$.

The two above calculations show equation (5.1.13) for an appropriate choice of linear functionals $\varphi_1, \varphi_2, \varphi_3$. For instance, we can choose $\mathscr{A}_i = \mathrm{T}(V_i)$ for some vector spaces V_i for each $i \in [3]$. Assume the tuple $(a_i)_{i\in[4]} \in \prod_{i=1}^4 V_{\varepsilon_i}$ are basis vectors. Then, we define linear functionals $f_1\colon V_1 \longrightarrow \mathbb{C}$, $a_1 \longmapsto 1$,$f_2\colon V_2 \longrightarrow \mathbb{C}$, $a_3 \longmapsto 1$ and $f_3\colon V_3 \longrightarrow \mathbb{C}, a_2 \longmapsto 1, a_4 \longmapsto 1$ and on other basis elements all maps are zero. Then, we can set $\forall\, i \in [3]\colon \varphi_i := \mathcal{T}(f_i)$.

Let $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus [k-1]$ and $\varepsilon =: (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k])$. We want to derive two other tuples from this tuple ε. We set

$$S := \big\{\, i \in [n] \mid (\varepsilon_i = 1) \vee (\varepsilon_i = 2) \,\big\}. \tag{5.1.16}$$

We define the following tuple $\gamma = (\gamma_i)_{i\in[n]} \in [k-1]^{\times n}$ by

$$\forall\, i \in [n]\colon \gamma_i := \begin{cases} 1 & \text{for } i \in S \\ \varepsilon_i - 1 & \text{else.} \end{cases} \tag{5.1.17}$$

Then, we set

$$\tilde{\varepsilon} := (\tilde{\varepsilon}_i)_{i\in[n]} := \mathrm{red}(\gamma) \in \mathbb{A}([k-1]). \tag{5.1.18}$$

For example, if $\varepsilon = (1,2,3,2,4,1) \in \mathbb{A}([4])$, then according to the above $\gamma = (1,1,2,1,3,1)$ and therefore $\tilde{\varepsilon} = (1,2,1,3,1) \in \mathbb{A}([3])$. We want to define a second tuple originating from ε by

$$\bar{\varepsilon} := (\bar{\varepsilon}_i)_i := \mathrm{red}\big((\varepsilon_i)_{i\in S}\big) \in \mathbb{A}([2]). \tag{5.1.19}$$

Notice that we make use of Convention 2.5.5 **(a)**. As an example, for $\varepsilon = (1,2,3,2,4,1) \in \mathbb{A}([4])$ we have $\bar{\varepsilon} = (1,2,1) \in \mathbb{A}([2])$. Using these notations we want to formulate a Corollary to Proposition 5.1.6.

5.1.8 Corollary (to Proposition 5.1.6). Let $k \in \mathbb{N}\setminus\{1\}$, $n \in \mathbb{N}\setminus[k-1]$ and assume $\varepsilon = (\varepsilon_i)_{i\in[n]} \in \big(E_{n,k} \cap \mathbb{A}([k])\big)$, where the set $E_{n,k}$ has been defined in equation (5.1.11).
Then, we have

$$\alpha_{\max}^{(\varepsilon)} = \alpha_{\max}^{(\tilde{\varepsilon})} \cdot \alpha_{\max}^{(\bar{\varepsilon})}. \tag{5.1.20}$$

Proof: From Lemma 5.1.5 **(b)** we obtain the existence of algebras $\mathscr{A}_i$, linear functionals φ_i and elements $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_i}$ such that $(\varphi_1 \boxdot \cdots \boxdot \varphi_k)(a_1 \cdots\cdot a_n) = \alpha_{\max}^{(\varepsilon)}$ and $\prod_{i=1}^k \varphi_i\big(j(a_1,\dots,a_n)(\mathbb{1}_i) = 1$,. We can calculate

$$\alpha_{\max}^{(\varepsilon)} = (\varphi_1 \boxdot \cdots \boxdot \varphi_k)(a_1 \cdot \cdots \cdot a_n)$$

$$= \Big(\underbrace{(\varphi_1 \boxdot \varphi_2)}_{=:\tilde{\varphi}_1} \boxdot \underbrace{\varphi_3}_{=:\tilde{\varphi}_2} \boxdot \cdots \boxdot \underbrace{\varphi_{k-1}}_{=:\tilde{\varphi}_{k-1}} \Big) \big(a_1 \cdot \cdots \cdot \underbrace{(a_{\ell+1} \cdot a_{\ell+2})}_{\in \mathscr{A}_1 \sqcup \mathscr{A}_2} \cdot \cdots \cdot a_n\big)$$

⟦ equation (5.1.12) ⟧

$$= \alpha^{(\tilde{\varepsilon})}_{\max} \prod_{i=1}^{k} \tilde{\varphi}_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \quad [\![\text{ eq. (5.1.9) }]\!]$$

$$= \alpha^{(\tilde{\varepsilon})}_{\max} (\varphi_1 \boxdot \varphi_2)\Big(\overrightarrow{\prod_{i \in S}} a_i\Big) \quad [\![\text{ eq. (5.1.10a), def. of } S \text{ in eq. (5.1.16), Conv. 2.5.5 } \textbf{(b)}]\!]$$

$$= \alpha^{(\tilde{\varepsilon})}_{\max} \alpha^{(\bar{\varepsilon})}_{\max} \varphi_1\big(j(a_1, \ldots, a_n)(\mathbb{1}_1)\big) \varphi_2\big(j(a_1, \ldots, a_n)(\mathbb{1}_2)\big) \quad [\![\text{ eq. (5.1.9) }]\!]$$

$$= \alpha^{(\tilde{\varepsilon})}_{\max} \alpha^{(\bar{\varepsilon})}_{\max} \quad [\![\text{ eq. (5.1.10a) }]\!]. \qquad \square$$

Now, we have some hope to make the transition from a u.a.u.-product $\odot$ to a universal class of partitions since we have found a possibility to "reduce" the length of a highest coefficient. For this, we will introduce a more diagrammatic notation in the following.

5.1.9 Definition (Induced single-colored partition). We set

$$\forall k \in \mathbb{N},\ \forall n \in \mathbb{N} \setminus [k-1]\colon T_{n,k} := \{\, \varepsilon \in [k]^{\times n} \mid |\mathsf{set}\, \varepsilon| = k \,\}, \tag{5.1.21}$$

$$T := \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N} \setminus [k-1]} T_{n,k}. \tag{5.1.22}$$

We shall define a prescription which maps an n-tuple $\varepsilon \in T_{n,k}$ to a partition $\pi \in \mathsf{Part}_{n,k}$ with k blocks. Let $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus [k-1]$ and $\varepsilon \in T_{n,k}$. We set

$$\forall i \in [k]\colon \beta_{\varepsilon,i} := \{\, \ell \in \mathbb{N} \mid \varepsilon_\ell = i \,\}. \tag{5.1.23}$$

And by this we set

$$\mathsf{indPart}(\varepsilon) := \bigcup_{i \in [k]} \{\, \mathsf{block}(\beta_{\varepsilon,i}) \,\} \in \mathsf{Part}_{n,k}\,. \tag{5.1.24}$$

Thus, we define

$$\mathsf{indPart}\colon \begin{cases} T \longrightarrow \mathsf{Part} \\ \varepsilon \longmapsto \mathsf{indPart}(\varepsilon). \end{cases} \tag{5.1.25}$$

5.1.10 Remark. The above defined map is not injective but surjective. Moreover, for each $k \in \mathbb{N}$ and $n \in \mathbb{N} \setminus [k-1]$ we have

$$(\mathbb{A}([k]) \cap T_{n,k}) = \mathsf{indPart}^{-1}\big(\mathrm{red}(\mathsf{Part}_{n,k})\big). \tag{5.1.26}$$

Or in other words, we can say that $\mathsf{indPart}(\varepsilon)$ is reduced as a (single-colored) partition (Definition 4.1.10 **(a)**) if and only if ε is reduced as a tuple (Convention 2.5.8 for $m = 1$).

5.1.11 Lemma. If $\odot$ is a symmetric u.a.u.-product in the category AlgP, then

$$\forall \sigma \in S_k\colon \varphi_1 \boxdot \cdots \boxdot \varphi_k = \varphi_{\sigma(1)} \boxdot \cdots \boxdot \varphi_{\sigma(k)} \circ \mathsf{can}, \tag{5.1.27}$$

where S_k denotes the symmetric group on the set $[k]$ and $\mathsf{can}\colon \bigsqcup_{i=1}^{k} \longrightarrow \mathcal{A}_{\sigma(1)} \sqcup \cdots \sqcup \mathcal{A}_{\sigma(k)}$ the canonical isomorphism of the free product of rearranged algebras $\mathcal{A}_i$ in the sense of equation (1.1.17).

Proof: We only sketch the proof. It suffices to show the statement for a transposition $\sigma \in S_k$ since each permutation can be made up of a finite composition of transpositions. The statement then follows by induction. Assume that $\sigma \in S_k$ is a transposition. Due to Lemma 2.4.7 we have

$$[\,\cdot\,,\,\cdot\,]_{\boxplus} = 0.$$

Then, according to Proposition 2.4.5 **(e)** we have for each $\ell \in [k-1] \subseteq \mathbb{N}$

$$\varphi_1 \boxplus \cdots \boxplus \varphi_{\ell-1} \boxplus \varphi_\ell \boxplus \varphi_{\ell+1} \boxplus \varphi_{\ell+2} \boxplus \cdots \boxplus \varphi_k = \varphi_1 \boxplus \cdots \boxplus \varphi_{\ell-1} \boxplus \varphi_{\ell+1} \boxplus \varphi_\ell \boxplus \varphi_{\ell+2} \boxplus \cdots \boxplus \varphi_k.$$

Now, the statement of equation (5.1.27) in case that σ is a transposition follows again by induction and equation (5.1.6). □

The above Lemma tells us that in the case $[\,\cdot\,,\,\cdot\,]_{\boxplus} = 0$ and whenever we want to calculate $\varphi_1 \boxdot \cdots \boxdot \varphi_n$ then we can arbitrarily rearrange the order of φ_i in this expression (up to a canonical isomorphism).

5.1.12 Lemma. Let $\odot$ be a symmetric u.a.u.-product with right-ordered monomials property in the category AlgP. Let $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus [k-1]$, $\pi \in \mathsf{Part}_{n,k}$. Choose $\varepsilon = (\varepsilon_i)_{i \in [n]} \in T_{n,k}$ such that $\mathsf{indPart}(\varepsilon) = \pi$. Define the set

$$\mathcal{N}_\varepsilon := \{\, \varepsilon' = (\varepsilon'_i)_{i\in[n]} \in \mathbb{N} \mid \exists f \in S_k \colon \forall i \in [n] \colon \varepsilon'_i = f(\varepsilon_i) \,\}. \tag{5.1.28}$$

Then, we have

$$\mathcal{N}_\varepsilon = (\mathsf{indPart})^{-1}(\pi), \tag{5.1.29}$$

$$\forall \varepsilon' \in (\mathsf{indPart})^{-1}(\pi) \colon \alpha_{\max}^{(\mathrm{red}\,\varepsilon')} = \alpha_{\max}^{(\mathrm{red}\,\varepsilon)}. \tag{5.1.30}$$

Proof: By definition of the map $\mathsf{indPart}$ in equation (5.1.25) the assertion of equation (5.1.29) immediately follows. For the proof of equation (5.1.30) it suffices to assume $\varepsilon = \mathrm{red}(\varepsilon) \in T$. If $\varepsilon \in T$ is not reduced, then the same calculation applies to $\mathrm{red}(\varepsilon)$ due to Lemma 5.1.4. We can calculate for some $f \in S_k$

$$\begin{aligned}
\alpha_{\max}^{(\varepsilon)} &= \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) \quad [\![\text{ Lem. 5.1.5 \textbf{(b)} }]\!] \\
&= \big(\varphi'_{f(1)} \boxdot \varphi'_{f(2)} \boxdot \cdots \boxdot \varphi'_{f(k)}\big)\underbrace{(a'_1 \cdot \cdots \cdot a'_n)}_{\in \mathcal{A}_{\varepsilon'}} \\
&\qquad [\![\ \forall i \in [k] \colon \big(\mathcal{A}'_{f(i)}, \varphi'_{f(i)}\big) := \big(\mathcal{A}_i, \varphi_i\big),\ \forall i \in [n] \colon a'_i := a_i \in \mathcal{A}'_{\varepsilon'_i} \text{ for } \varepsilon' \in \mathcal{N}_\varepsilon\ ]\!] \\
&= \big(\varphi'_1 \boxdot \cdots \boxdot \varphi'_k\big)(a'_1 \cdot \cdots \cdot a'_n) \quad [\![\ \odot \text{ is symmetric, Lem 2.4.7 and Lem. 5.1.11 }]\!] \\
&= \alpha_{\max}^{(\varepsilon')} \quad [\![\text{ Lem. 5.1.5 \textbf{(a)}, def. of } \varepsilon' \in \mathcal{N}_\varepsilon \text{ in eq. (5.1.28) }]\!]
\end{aligned}$$

This shows equation (5.1.30), since equation (5.1.29) holds. □

We want to introduce a more handy diagrammatic expression for $\alpha_{\max}^{(\varepsilon)}$ for some $\varepsilon \in T$ which can help us to see when we are allowed to apply the assertion of Corollary 5.1.8.

5.1.13 Convention. Set

$$\forall \pi \in \mathsf{Part}\colon \alpha_\pi := \alpha_{\max}^{(\mathrm{red}\,\varepsilon')} \text{ for some } \varepsilon' \in (\mathsf{indPart})^{-1}(\pi). \tag{5.1.31}$$

This convention is well-defined since $\mathsf{indPart}\colon T \longrightarrow \mathsf{Part}$ is surjective and because of equation (5.1.30). Due to Remark 5.1.10 this convention implies $\forall \pi \in \mathsf{Part}\colon \alpha_\pi = \alpha_{\mathrm{red}\,\pi}$.

5.1.14 Corollary (to Proposition 5.1.6). Let $k \in \mathbb{N}\setminus\{1\}$, $n \in \mathbb{N}\setminus[k-1]$. For all $\varepsilon = (\varepsilon_i)_{i\in[n]} \in T_{n,k}$ with $\pi := \mathsf{indPart}(\varepsilon) \in \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}$ holds

$$\alpha_\pi = \alpha_{(\mathsf{pr}_1 \circ \mathsf{UMem}_{n,k}^{\ell+1})(\pi)} \cdot \alpha_{(\mathsf{pr}_2 \circ \mathsf{UMem}_{n,k}^{\ell+1})(\pi)}. \tag{5.1.32}$$

In other words, for all $(\tilde{\varepsilon}, \bar{\varepsilon}) \in (T_{n,k-1} \times T_{\cdot,2})$ with $(\tilde{\pi}, \bar{\pi}) := (\mathsf{indPart}(\tilde{\varepsilon}), \mathsf{indPart}(\bar{\varepsilon})) \in \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big)$ holds

$$\alpha_{\tilde{\pi}} \cdot \alpha_{\bar{\pi}} = \alpha_{\mathsf{split}_{n,k-1}^{\ell+1}(\tilde{\pi},\bar{\pi})}. \tag{5.1.33}$$

Proof: Both equations are equivalent since split and UMem are inverse to each other because of Lemma 3.1.7. Therefore, it suffices to prove equation (5.1.32). Let $\varepsilon = (\varepsilon_i)_{i\in[n]} \in T$ such that $\pi := \mathsf{indPart}(\varepsilon) \in \mathsf{Part}_{n,k}^{(\ell+1)\vee(\ell+2)}$. At this point we may not apply Corollary 5.1.8 since $\varepsilon_{\ell+1} \neq 1$ or $\varepsilon_{\ell+2} \neq 2$. We want to define a bijection $f\colon [k] \longrightarrow [k]$ according to the following cases

- **(a)** $(\varepsilon_{\ell+1} = 1) \wedge (\varepsilon_{\ell+2} = 2)$ then set $f = \mathsf{id}$.
- **(b)** $(\varepsilon_{\ell+1} = 2) \wedge (\varepsilon_{\ell+2} = 1)$ then we define f by $1 \longmapsto 2, 1 \longmapsto 2$ and $f\restriction_{[k]\setminus\{1,2\}} = \mathsf{id}$.
- **(c)** $(\varepsilon_{\ell+1} = 2) \wedge (\varepsilon_{\ell+2} \neq 1)$ then we define f by $\varepsilon_{\ell+2} \longmapsto 1, 1 \longmapsto \varepsilon_{\ell+2}$, and $f\restriction_{[k]\setminus\{1,\varepsilon_{\ell+2}\}} = \mathsf{id}$. Then, we apply case **(b)** and we obtain a composition of maps.
- **(d)** $(\varepsilon_{\ell+1} \neq 2) \wedge (\varepsilon_{\ell+2} = 1)$ then we define f by $\varepsilon_{\ell+1} \longmapsto 1, 1 \longmapsto \varepsilon_{\ell+1}$, and $f\restriction_{[k]\setminus\{1,\varepsilon_{\ell+1}\}} = \mathsf{id}$. Then, we apply case **(b)** and we obtain a composition of maps.
- **(e)** $(\varepsilon_{\ell+1} \notin \{1,2\}) \wedge (\varepsilon_{\ell+2} \notin \{1,2\})$, then we define f by $\varepsilon_{\ell+1} \longmapsto 1$, $\varepsilon_{\ell+2} \longmapsto 2, 1 \longmapsto \varepsilon_{\ell+1}, 2 \longmapsto \varepsilon_{\ell+2}$ and $f\restriction_{[k]\setminus\{1,2,\varepsilon_{\ell+1},\varepsilon_{\ell+2}\}} = \mathsf{id}$.

For the bijection $f \in S_k$ we define a tuple $\varepsilon' = (\varepsilon'_i)_{i\in[n]} \in T$ by $\forall i \in [n]\colon \varepsilon'_i := f(\varepsilon_i)$. The tuple ε' has the property $\varepsilon'_{\ell+1} = 1$ and $\varepsilon'_{\ell+2} = 2$. Hence, we can calculate

$$\begin{aligned}
\alpha_{\mathsf{indPart}(\varepsilon)} &= \alpha_{\max}^{(\mathrm{red}\,\varepsilon)} \quad [\![\text{ Conv. 5.1.13 }]\!] \\
&= \alpha_{\max}^{(\mathrm{red}\,\varepsilon')} \quad [\![\text{ Lem. 5.1.12 }]\!] \\
&= \alpha_{\max}^{(\widetilde{\mathrm{red}\,\varepsilon'})} \cdot \alpha_{\max}^{(\overline{\mathrm{red}\,\varepsilon'})} \quad [\![\text{ Cor. 5.1.8 }]\!] \\
&= \alpha_{(\mathsf{pr}_1 \circ \mathsf{UMem}_{n,k}^{\ell+1})(\pi)} \cdot \alpha_{(\mathsf{pr}_2 \circ \mathsf{UMem}_{n,k}^{\ell+1})(\pi)} \\
&\qquad [\![\text{ def. of } \tilde{\cdot} \text{ and } \bar{\cdot} \text{ in eq. (5.1.18) resp. (5.1.19), Conv. 5.1.13 }]\!].
\end{aligned}$$

□

We want to further elaborate nonzero highest coefficients for a given symmetric u.a.u.-product with the right-ordered monomials property. We want to use them in order to assign a set of partitions to such a universal product $\odot$, thus we set

$$\mathfrak{P}_\odot := \{\, \pi \in \mathsf{Part} \mid \alpha_\pi \neq 0 \,\}. \tag{5.1.34}$$

5.1.15 Theorem. Let $\odot$ be a symmetric u.a.u.-product with the right-ordered monomials property in the category AlgP. We set

$$\forall n \in \mathbb{N} \setminus \{1\},\ \forall \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N} \colon (\mathfrak{P}_\odot)_n^{(\ell+1)\wedge(\ell+2)} := \mathfrak{P}_\odot \cap \mathsf{Part}_{n,k}^{(\ell+1)\wedge(\ell+2)}, \tag{5.1.35}$$

$$\forall n \in \mathbb{N} \colon (\mathfrak{P}_\odot)_n := \mathfrak{P}_\odot \cap \mathsf{Part}_n\,. \tag{5.1.36}$$

Then,

$$\alpha_{\shortmid} = 1, \tag{5.1.37}$$

$$\forall n \in \mathbb{N} \setminus \{1\},\ \forall \ell \in \{0, \dots, n-2\} \subseteq \mathbb{N},\ \forall \pi \in (\mathfrak{P}_\odot)_n^{(\ell+1)\wedge(\ell+2)} \colon \alpha_\pi = \alpha_{\mathsf{delete}_{n,\ell+2}(\pi)}, \tag{5.1.38}$$

$$\forall n \in \mathbb{N},\ \forall \ell \in \{0, \dots, n-1\} \subseteq \mathbb{N},\ \forall \pi \in (\mathfrak{P}_\odot)_n \colon \alpha_\pi = \alpha_{\mathsf{double}_{n,\ell+1}(\pi)}. \tag{5.1.39}$$

Moreover, the induced set of partitions $\mathfrak{P}_\odot$ is a (single-colored) universal class of partitions.

Proof: For the proof of equation (5.1.37) we observe the following. Since $\odot$ is assumed to be unital, we have for any algebras $\mathscr{A}_1$, $\mathscr{A}_2 \in \mathsf{Obj}(\mathsf{Alg})$ and linear functionals $0 \neq \varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ for $i \in \{1,2\}$ that there needs to exists an element $a \in \mathscr{A}_1$ such that

$$(\varphi_1 \odot \varphi_2)(\iota_1(a)) = \varphi_1(a) \neq 0.$$

If we apply equation (2.3.12) to $(\varphi_1 \odot \varphi_2)(\iota_1(a))$, we obtain

$$(\varphi_1 \odot \varphi_2)(\iota_1(a)) = \underbrace{1}_{=\alpha_{\max}^{(1)}} \cdot \varphi_1(a).$$

For the proof of equation (5.1.38) we observe the following. Let $\pi \in (\mathfrak{P}_\odot)_{n,k}^{(\ell+1)\wedge(\ell+2)}$ for some $k \in \mathbb{N} \setminus \{1\}$, $n \in \mathbb{N} \setminus [k-1]$, $\ell \in \{0, \dots, n-2\}$. There exists $\varepsilon \in T_{n,k}$ such that $\mathsf{indPart}(\varepsilon) = \pi$. Let us fix such a tuple ε. We use the assertion of Lemma 5.1.5 **(b)** and obtain the existence of algebras and linear functionals such that $(\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k}$, $(\tilde a_i)_{i\in[\tilde n]} \in \prod_{i=1}^{\tilde n} \mathscr{A}_{(\mathrm{red}\,\varepsilon)_i}$ and

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \dots \cdot a_n) = \underbrace{\alpha_\pi}_{=\alpha_{\max}^{(\varepsilon)}} \underbrace{\prod_{i=1}^{k} \varphi_i\big(j(\tilde a_1, \dots, \tilde a_{\tilde n})(\mathbb{1}_i)\big)}_{=1} \neq 0, \tag{I}$$

where the existence of the tuple $(\tilde a_i)_{i\in[\tilde n]}$ is due to Lemma 5.1.4. We calculate

$$\begin{aligned}
&\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \dots \cdot a_{\ell+1} \cdot a_{\ell+2} \cdot \dots \cdot a_n)\\
&\quad = \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \dots \cdot (a_{\ell+1}a_{\ell+2}) \ \cdot \dots \cdot a_n) \quad [\![\, \varepsilon_{\ell+1} = \varepsilon_{\ell+2},\ a_{\ell+i} \in \mathscr{A}_{\varepsilon_i} \text{ for } i \in [2] \,]\!]\\
&\quad = \alpha_{\max}^{(\mathrm{red}\,\tilde\varepsilon)} \prod_{i=1}^{k} \varphi_i\big(j(\tilde a_1, \dots, \tilde a_{\tilde n})(\mathbb{1}_i)\big)\\
&\qquad\quad [\![\, \tilde\varepsilon := (\varepsilon_1, \dots, \varepsilon_{\ell+1}, \varepsilon_{\ell+3}, \dots, \varepsilon_n) \in [k]^{\times(n-1)},\ \mathrm{red}\,\varepsilon = \mathrm{red}\,\tilde\varepsilon,\ \text{Lem. 5.1.5 } \textbf{(a)}, \text{Lem. 5.1.4} \,]\!].
\end{aligned}$$

By this calculation and by equation (I), for the tuple $\varepsilon \in T_{n,k}$ we have found a tuple $\tilde{\varepsilon} \in T_{n-1,k}$ such that $\alpha_{\mathsf{indPart}(\varepsilon)} = \alpha_{\mathsf{indPart}(\tilde{\varepsilon})}$ and $\mathsf{delete}_{n,\ell+2}(\mathsf{indPart}(\varepsilon)) = \mathsf{indPart}(\tilde{\varepsilon})$.

For the proof of equation (5.1.39) we consider the following. We denote by $\mathscr{A}_i^{\mathbb{1}}$ the unitization of the algebra $\mathscr{A}_i$. We have homomorphic embeddings $\iota_i\colon \mathscr{A}_i \hookrightarrow \mathscr{A}_i^{\mathbb{1}}$. Furthermore, denote by $(\varphi_i)^{\mathbb{1}}\colon \mathscr{A}_i^{\mathbb{1}} \longrightarrow \mathbb{C}$ the unique linear extension of φ_i with $(\varphi_i)^{\mathbb{1}}(\mathbb{1}) = 1$, i. e., $(\varphi_i)^{\mathbb{1}}$ is the unital extension of φ such that

$$(\varphi_i)^{\mathbb{1}} \circ \iota_i = \varphi_i. \tag{II}$$

By the fact that $\iota_i\colon \mathscr{A}_i \hookrightarrow (\mathscr{A}_i)^{\mathbb{1}}$ is a homomorphism of algebras and equation (2.1.19) we have

$$\Big(\bigodot_{i=1}^{k}(\varphi_i)^{\mathbb{1}}\Big) \circ \Big(\coprod_{i=1}^{k} \iota_i\Big) = \bigodot_{i=1}^{k}\underbrace{((\varphi_i)^{\mathbb{1}} \circ \iota_i)}_{=\varphi_i}. \tag{III}$$

Now, let us assume that $\varepsilon = (\varepsilon_i)_{i\in[n]} \in T_{n,k}$ such that $\pi := \mathsf{indPart}(\varepsilon) \in \mathfrak{P}_{\odot}$, i. e., $\alpha_\pi \neq 0$. We need to show that $\alpha_\pi = \alpha_{\mathsf{double}_{n,\ell+1}(\pi)} \neq 0$. Let us further assume that $\varepsilon = \operatorname{red}\varepsilon$. If the tuple ε is not reduced, then due to Lemma 5.1.4 the proof in this case is similar to the following one. We use the assertion of Lemma 5.1.5 **(b)** and obtain the existence of algebras and linear functionals such that $(\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k}$, $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$ and

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \underbrace{\alpha_\pi}_{=\alpha_{\max}^{(\varepsilon)}} \underbrace{\prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)}_{=1} \neq 0. \tag{IV}$$

Let $\mathbb{1}_{\varepsilon_{\ell+1}}$ denote the unit element of the unital algebra $(\mathscr{A}_{\varepsilon_{\ell+1}})^{\mathbb{1}}$, then we can calculate

$$\begin{aligned}
&\alpha_{\max}^{(\varepsilon)} \prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)\\
&= \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) \quad [\![\text{ eq. (IV) }]\!]\\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ eq. (5.1.4) }]\!]\\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1)^{\mathbb{1}} \odot \cdots \odot (t_k\varphi_k)^{\mathbb{1}}\big)\\
&\qquad\qquad \big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_{\ell+1}}(a_{\ell+1}) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ eq. (III) }]\!]\\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1)^{\mathbb{1}} \odot \cdots \odot (t_k\varphi_k)^{\mathbb{1}}\big)\\
&\qquad\qquad \big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_{\ell+1}}(a_{\ell+1}) \cdot \iota_{\varepsilon_{\ell+1}}(\mathbb{1}_{\varepsilon_{\ell+1}}) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0}\\
&\qquad [\![\, (\mathscr{A}_{\varepsilon_{\ell+1}})^{\mathbb{1}} \text{ is unital }]\!]\\
&= \big((\varphi_1)^{\mathbb{1}} \boxdot \cdots \boxdot (\varphi_k)^{\mathbb{1}}\big)\big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_{\ell+1}}(a_{\ell+1}) \cdot \iota_{\varepsilon_{\ell+1}}(\mathbb{1}_{\varepsilon_{\ell+1}}) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big)\\
&\qquad [\![\text{ eq. (5.1.4) }]\!]
\end{aligned}$$

$$= \alpha_{\max}^{(\mathrm{red}\,\tilde{\varepsilon})} \prod_{i=1}^{k} (\varphi_i)^{\mathbb{1}} \big(j\big(\iota_{\varepsilon_1}(a_1), \ldots, \iota_{\varepsilon_n}(a_n)\big)(\mathbb{1}_i)\big)$$

$$[\![\ \tilde{\varepsilon} := (\varepsilon_1, \ldots, \varepsilon_\ell, \varepsilon_{\ell+1}, \varepsilon_{\ell+1}, \varepsilon_{\ell+2}, \ldots, \varepsilon_n) \in T_{n+1,k},\ \text{Lem. 5.1.4, Lem. 5.1.5 \textbf{(a)}}\ ]\!]$$

$$= \alpha_{\max}^{(\mathrm{red}\,\tilde{\varepsilon})} \prod_{i=1}^{k} \varphi_i \big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \quad [\![\ \text{eq. (II)}\ ]\!].$$

From the above calculation we obtain that $\alpha_{\mathsf{indPart}(\tilde{\varepsilon})} = \alpha_\pi$ and $\mathsf{double}_{n,\ell+1}(\pi) = \mathsf{indPart}(\tilde{\varepsilon})$.

For the last part of the assertion we need to check the axioms from Definition 3.1.9.

- That $\shortmid \in \mathfrak{P}_\odot$ is equation (5.1.37).
- That $\mathfrak{P}_\odot$ is closed under the delete-operation follows from equation (5.1.38).
- That $\mathfrak{P}_\odot$ is closed under the double-operation follows from equation (5.1.39).
- That $\mathfrak{P}_\odot$ is closed under the UMem-operation follows from equation (5.1.32).
- That $\mathfrak{P}_\odot$ is closed under the split-operation follows from equation (5.1.33).

□

5.1.16 Proposition. Let $\odot$ be a symmetric u.a.u.-product with the right-ordered monomials property in the category AlgP. Let $k \in \mathbb{N} \setminus \{1\}$. If $\pi \in \mathfrak{P}_\odot$ with $|\pi| = k$. Then, there exists a $(k-1)$-tuple of two-block partitions $(\delta_i)_{i\in[k-1]} \in \big((\mathfrak{P}_\odot)_{\cdot,2}\big)^{\times(k-1)}$ such that

$$\alpha_\pi = \prod_{i=1}^{k-1} \alpha_{\delta_i}. \tag{5.1.40}$$

Proof: We prove this assertion by induction over $k \in \mathbb{N} \setminus \{1\}$. For the induction base $k = 2$, the statement follows, if we set $\delta_1 := \pi$ and using $\pi \in \mathfrak{P}_\odot \implies \alpha_\pi \neq 0$.

For the induction step $k \to k+1$, we assume that the assertion holds for $k \in \mathbb{N} \setminus \{1\}$ with $\pi \in (\mathfrak{P}_\odot)_{n,k}$ for some $n \in \mathbb{N} \setminus [k-1]$. We calculate

$$\alpha_\pi = \alpha_{\tilde{\pi}} \cdot \alpha_{\bar{\pi}} \quad \left[\!\!\left[\begin{array}{l} \text{apply eq. (5.1.32) to "smallest" neighbored legs} \\ \text{of the first and second block of } \pi, \\ \text{this a similar procedure as described in Def. 4.1.2} \end{array} \right]\!\!\right]$$

$$= \alpha_{\tilde{\pi}} \cdot \alpha_{\delta_k} \quad [\![\ \delta_k := \bar{\pi} \in (\mathfrak{P}_\odot)_{\cdot,2} \text{ since } \pi \in \mathfrak{P}_\odot \implies \alpha_\pi \neq 0\ ]\!]$$

$$= \Big(\prod_{i=1}^{k-1} \alpha_{\delta_i}\Big) \cdot \alpha_{\delta_k} \quad [\![\ \text{induction hypothesis applied to } \tilde{\pi} \in \mathfrak{P}_\odot,\ |\pi| = k\ ]\!].$$

The existence of the tuple $(\delta_i)_{i\in[k]}$ of two-block partitions in $\mathfrak{P}_\odot$ proves the statement of equation (5.1.40). □

All of our theory concerning partitions is developed to determine admissible choices of highest coefficients $\alpha_{\max}^{(\varepsilon)}$ which lead to a u.a.u.-product. For the classification of universal classes of partitions $\mathcal{P}$ we have seen that they possess "generators". These generators are certain two-block partitions which can generate any partition $\pi \in \mathcal{P}$. The same also holds for nonzero highest coefficients which belong to two-block partitions as the following lemma shows. The following theorem is a stronger version of Theorem 2.5.13.

5.1.17 Theorem. Assume that $\odot$, $\tilde{\odot}$ are symmetric u.a.u.-products with the right-ordered monomials property in the category AlgP. If

$$\forall \pi \in \mathrm{red}\big((\mathfrak{P}_{\odot})_{\cdot,2}\big) \colon \alpha_{\pi} = \tilde{\alpha}_{\pi}, \tag{5.1.41}$$

then $\odot = \tilde{\odot}$ as bifunctors. Or in other words, $\odot$ is uniquely determined by the highest coefficients α_{π}, where π is a reduced two-block partition of $\mathfrak{P}_{\odot}$

Proof: Because of Theorem 2.5.13, a u.a.u.-product with the right-ordered monomials property is uniquely determined by its highest coefficients $\alpha^{(\varepsilon)}_{\max}$ with $\varepsilon \in \mathbb{A}([k]) \cap T_{n,k}$ for some $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus [k-1]$. Equation (5.1.26), Convention 5.1.13 and equation (5.1.34) give us the following equivalent characterization of the initial assumption: If

$$\forall k \in \mathbb{N},\ \forall \pi \in \mathrm{red}(\mathsf{Part}_{\cdot,k}) \colon \alpha_{\pi} = \tilde{\alpha}_{\pi},$$

then $\odot = \tilde{\odot}$ as bifunctors.

Let us fix $k = 1$ and assume $\pi \in \mathsf{Part}_{\cdot,1}$. By equation (5.1.37) and equation (5.1.39) we have $\pi \in \mathfrak{P}_{\odot}$ and $\alpha_{\pi} = 1 = \tilde{\alpha}_{\pi}$.

Now, let us assume $k \geq 2$ and $\pi \in (\mathfrak{P}_{\odot})_{\cdot,k}$. Due to Proposition 5.1.16 each coefficient α_{π} can be expressed as a product of coefficients α_{δ_i}, where $\delta_i \in (\mathfrak{P}_{\odot})_{\cdot,2}$. Now, the assertion follows. □

We still do not know which values a nonzero highest coefficient can have. At this point two different u.a.u.-products, which are represented by the same set of nonzero highest coefficients, can induce the same universal class of partitions although their nonzero highest coefficients might not be equal. We prospect that the choice of the value of the coefficient $\alpha_{|\,|}$ determines all the remaining coefficients α_{π} for $\pi \in \mathsf{Part}_{\cdot,2}$. In particular, we notice that $\alpha_{|} = 1$ holds for any u.a.u.-product which follows from unitality as we did in the proof of Theorem 5.1.15. We continue to determine possible nonzero values for the highest coefficients of a u.a.u.-product with the right-ordered monomials property.

5.1.18 Proposition. Let $\odot$ be a symmetric u.a.u.-product with the right-ordered monomials property in the category AlgP. Then,

$$\forall \pi \in (\mathfrak{P}_{\odot})_{\cdot,2} \colon \alpha_{\pi} = \alpha_{|\,|}. \tag{5.1.42}$$

Proof: We claim

$$\alpha_{|\,|} \neq 0,\ \alpha_{\text{┌╷┐}} \neq 0 \implies \alpha_{\text{┌╷┐}} = \alpha_{|\,|}. \tag{I}$$

For the proof of this claim we calculate

$$\begin{aligned} \left(\alpha_{\text{┌╷┐}}\right)^2 &= \alpha_{\text{┌╷┬┐}} \cdot \alpha_{\text{┌╷┐}} && [\![\text{ eq. (5.1.31) }]\!] \\ &= \alpha_{\text{┌╷╷┐}} && [\![\text{ eq. (5.1.33) }]\!] \\ &= \alpha_{\text{┌┌┐┐}} \cdot \alpha_{|\,|} && [\![\text{ eq. (5.1.32) }]\!] \\ &= \alpha_{\text{┌╷┐}} \cdot \alpha_{|\,|} && [\![\text{ eq. (5.1.38) }]\!] \end{aligned}$$

From the above we obtain that

$$\left(\alpha_{\text{┌╷╷┐}}\right)^2 = \alpha_{\text{┌╷╷┐}} \cdot \alpha_{\text{| |}}.$$

Now, we claim

$$\alpha_{\text{| |}} \neq 0,\ \alpha_{\text{┌┬┐┐}} \neq 0 \implies \forall \pi \in \mathrm{red}(\mathsf{Part}_{\cdot,2} \setminus \mathsf{NC}_{\cdot,2})\colon \alpha_\pi = \alpha_{\text{┌╷╷┐}} = \alpha_{\text{| |}}. \tag{II}$$

We calculate

$$\begin{aligned}\left(\alpha_{\text{┌┬┐┐}}\right)^2 &= \alpha_{\text{┌┬┬┬┐}} \cdot \alpha_{\text{┌┬┐┐}} && [\![\text{ eq. (5.1.39) }]\!]\\ &= \alpha_{\text{┌┬┬┐┐}} && [\![\text{ eq. (5.1.33) }]\!]\\ &= \alpha_{\text{┌┬┬┐┐┐}} \cdot \alpha_{\text{┌┌┐┐}} && [\![\text{ eq. (5.1.32) }]\!]\\ &= \alpha_{\text{┌┬┐┐}} \cdot \alpha_{\text{┌╷╷┐}} && [\![\text{ eq. (5.1.38) }]\!]\end{aligned}$$

Thus, we obtain

$$\left(\alpha_{\text{┌┬┐┐}}\right)^2 = \alpha_{\text{┌┬┐┐}} \cdot \alpha_{\text{┌╷╷┐}}.$$

From equation (I) follows

$$\alpha_{\text{┌┬┐┐}} \neq 0 \implies \alpha_{\text{┌┬┐┐}} = \alpha_{\text{| |}}. \tag{III}$$

Let $C := \mathrm{red}(\mathsf{Part}_{\cdot,2} \setminus \mathsf{NC}_{\cdot,2})$ be the set of all reduced, two-block partitions $\pi \in \mathsf{Part}$ which have a crossing. Denote by C_1 the set of all partitions $\pi \in C$ which are of type

$$\exists k \in \mathbb{N}\colon \pi = \underset{1\ 2\ 3\ 4\ \cdots\ 2k\ \ 2k+1\ \ \ 2k+2}{\text{┌┬┬┬ ⋯ ┬┐ ┐}}$$

and by C_2 the set of all partitions $\pi \in C$ which are of type

$$\exists k \in \mathbb{N} \setminus \{1\}\colon \pi = \underset{1\ 2\ 3\ 4\ \cdots\ 2k\ \ 2k+1}{\text{┌┬┬┬ ⋯ ┬┐}}. \tag{IV}$$

Then, $C = C_1 \mathbin{\dot\cup} C_2$. Therefore, we need to consider two cases. For brevity we will only consider the case $\pi \in C_2$ since the other case is done analogously. We have to show

$$\alpha_{\text{┌┬┐┐}} \neq 0 \implies \alpha_{\underset{1\ 2\ 3\ 4\ \cdots\ 2k\ \ 2k+1}{\text{┌┬┬┬ ⋯ ┬┐}}} = \alpha_{\text{| |}} \tag{V}$$

for any $k \in \mathbb{N} \setminus \{1\}$. For any partition $\pi \in C_2$ there exists a unique $k \in \mathbb{N} \setminus \{1\}$ such that we can say that π is of type from equation (IV). Thus, we prove equation (V) for $\pi \in C_2$ by induction over $k \in \mathbb{N} \setminus \{1\}$. For the induction base we calculate

$$\begin{aligned}\left(\alpha_{\text{┌┬┐┐}}\right)^2 &= \alpha_{\text{┌┬┐┐╷╷}} \cdot \alpha_{\text{┌┬┐┐}} && [\![\text{ eq. (5.1.31) }]\!]\\ &= \alpha_{\text{┌┬┐┌┐┐}} && [\![\text{ eq. (5.1.33) }]\!]\\ &= \alpha_{\text{┌┬┬┬┐}} \cdot \alpha_{\text{┌┐ ┌┐}} && [\![\text{ eq. (5.1.32) }]\!]\end{aligned}$$

$$= \alpha_{\sqcap\!\sqcap\!\sqcap} \cdot \alpha_{\vert\,\vert} \qquad [\![\text{ eq. (5.1.38) }]\!].$$

From equation (III) we can conclude the induction base for $\alpha_{\sqcap\!\sqcap} \neq 0$. Hence, the induction base holds.

For the induction step $k \to k+1$ we calculate

$$\alpha_{\underset{1\,2\,3\,4\,\cdots\,2k\quad 2k+1}{\sqcap\sqcap\sqcap\cdots\sqcap}} \cdot \alpha_{\sqcap\!\sqcap\!\sqcap} = \alpha_{\underset{1\,2\,3\,4\,5\,6\,7\,\cdots\,2(k+1)+1\quad 2(k+1)+2}{\sqcap\sqcap\sqcap\cdots\sqcap}} \cdot \alpha_{\sqcap\!\sqcap\cdots\sqcap} \qquad [\![\text{ eq. (5.1.39) }]\!]$$

$$= \alpha_{\sqcap\sqcap\sqcap\sqcap\cdots\sqcap} \qquad [\![\text{ eq. (5.1.33) }]\!]$$

$$= \alpha_{\sqcap\sqcap\sqcap\sqcap\cdots\sqcap} \cdot \alpha_{\sqcap\cdots\sqcap\ \sqcap\cdots\sqcap} \qquad [\![\text{ eq. (5.1.32) }]\!]$$

$$= \alpha_{\underset{1\,2\,3\,4\,5\,6\,\cdots\,2(k+1)\quad 2(k+1)+1}{\sqcap\sqcap\sqcap\cdots\sqcap}} \cdot \alpha_{\vert\,\vert} \qquad [\![\text{ eq. (5.1.38) }]\!].$$

If we use the induction hypothesis for $k \in \mathbb{N} \setminus \{1\}$ and the induction base for expressions on the left hand side of the above equation, then the induction step follows.

Now, we finally prove equation (5.1.42). By Convention 5.1.13 and equation (5.1.26) it suffices to show the assertion for $\pi \in \mathsf{Part}$ with $\pi = \operatorname{red}\pi$. In Theorem 5.1.15 we have shown that a symmetric u.a.u.-product with the right-ordered monomials property $\odot$ induces a universal class of partitions $\mathfrak{P}_\odot$. By the classification result provided in Theorem 4.1.17 we have

$$1\mathsf{B} \subsetneq \underset{= \mathrm{Gen}(\vert\,\vert)}{\mathsf{I}} \subsetneq \underset{= \mathrm{Gen}(\sqcap)}{\mathsf{NC}} \subsetneq \underset{= \mathrm{Gen}(\sqcap\!\sqcap)}{\mathsf{Part}}, \tag{VI}$$

$$\operatorname{red}(\mathsf{Part}_{\cdot,2}) = \operatorname{red}\big(\mathsf{I}_{\cdot,2} \uplus \mathsf{NC}_{\cdot,2} \setminus \mathsf{I}_{\cdot,2} \uplus \mathsf{Part}_{\cdot,2} \setminus \mathsf{NC}_{\cdot,2}\big). \tag{VII}$$

By equation (VI) we need to make the following case consideration for $\mathfrak{P}_\odot$:

- $\mathfrak{P}_\odot = 1\mathsf{B}$: Then, we have $(\mathfrak{P}_\odot)_{\cdot,2} = \emptyset$ and in particular $\alpha_{\vert\,\vert} = 0$. The statement now follows from the definition of $\mathfrak{P}_\odot$ in equation (5.1.34).
- $\mathfrak{P}_\odot = \mathsf{I}$: Then, $\operatorname{red}((\mathfrak{P}_\odot)_{\cdot,2}) = \{\vert\,\vert\}$, i. e., $\alpha_{\vert\,\vert} \neq 0$ and the statement follows from equation (5.1.34).
- $\mathfrak{P}_\odot = \mathsf{NC}$: Then from equation (VII) we have $\operatorname{red}((\mathfrak{P}_\odot)_{\cdot,2}) = \operatorname{red}(\mathsf{I}_{\cdot,2}) \uplus \operatorname{red}(\mathsf{NC}_{\cdot,2} \setminus \mathsf{I}_{\cdot,2}) = \{\vert\,\vert\} \uplus \{\sqcap\}$, i.e. $\alpha_{\vert\,\vert} \neq 0$ and $\alpha_{\sqcap} \neq 0$. The assertion of equation (5.1.42) for reduced partitions now follows from equation (I).
- $\mathfrak{P}_\odot = \mathsf{Part}$: This means that $\operatorname{red}((\mathfrak{P}_\odot)_{\cdot,2}) = \operatorname{red}(\mathsf{I}_{\cdot,2}) \uplus \operatorname{red}(\mathsf{NC}_{\cdot,2} \setminus \mathsf{I}_{\cdot,2}) \uplus \operatorname{red}(\mathsf{Part}_{\cdot,2} \setminus \mathsf{NC}_{\cdot,2}) = \{\vert\,\vert\} \uplus \{\sqcap\} \uplus \operatorname{red}(\mathsf{Part}_{\cdot,2} \setminus \mathsf{NC}_{\cdot,2})$, hence $\alpha_{\vert\,\vert} \neq 0$ and $\alpha_{\sqcap\!\sqcap} \neq 0$. The assertion of equation (5.1.42) for reduced partitions now follows in particular from equation (II).

□

5.1.19 Remark. Note the similarities between the calculations done in the proof of Theorem 4.1.17 and the above calculations for the coefficients α_π in Proposition 5.1.18. We can use the diagrammatic calculations from the proof of Theorem 4.1.17 if we combine this with the fact from Convention 5.1.13 and Corollary 5.1.14.

Before we formulate our main theorem in this section, we need one more formula for the calculation of the universal product $\odot_{\mathcal{P}}$ if $\mathcal{P}$ is any universal class of partitions. $\mathcal{P}_n$ is a poset w. r. t. reversed refinement denoted by $\leq$. Therefore, the following assertion is well posed.

5.1.20 Lemma. Let $\mathcal{P}$ be a universal class of partitions. Then,

$\forall n \in \mathbb{N},\ \forall k \in [n],\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in T_{n,k},\ \forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_i}$:

$$(\varphi_1 \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} \varphi_k)(a_1 \cdot \cdots \cdot a_n)$$
$$= \begin{cases} \displaystyle\sum_{\pi \leq \mathsf{indPart}(\varepsilon)} \prod_{b\in\pi} \Big(\sum_{i=1}^{k} (\log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i))(\overrightarrow{\bigotimes_{i\in \mathsf{set}\, b}} a_i)\Big) & \text{for } \mathsf{indPart}(\varepsilon) \in \mathcal{P} \\ \displaystyle\sum_{j=k+1}^{n} \sum_{\substack{\pi\in\mathcal{P}_{n,j}:\\ \forall b\in\pi,\\ \exists b'\in \mathsf{indPart}(\varepsilon):\\ \mathsf{set}\, b\subseteq \mathsf{set}\, b'}} \prod_{b\in\sigma} \Big(\sum_{i=1}^{k} (\log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i))(\overrightarrow{\bigotimes_{i\in \mathsf{set}\, b}} a_i)\Big) & \text{for } \mathsf{indPart}(\varepsilon) \notin \mathcal{P}. \end{cases} \tag{5.1.43}$$

Proof: For the following calculations we define canonical insertion maps $\forall j \in [k]$

$$\iota_j \colon \mathscr{A}_j \hookrightarrow \bigsqcup_{i=1}^{k} \mathscr{A}_i,$$

$$\mathsf{inc}_j \colon \mathscr{A}_j \hookrightarrow \mathrm{T}(\bigoplus_{i=1}^{k} \mathscr{A}_i).$$

Assume that $\mathsf{indPart}(\varepsilon) \in \mathcal{P}$, then we can calculate

$$(\varphi_1 \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} \varphi_k)\big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big)$$
$$= \Big(\mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big) \circ \mathsf{can}\Big)\big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big)$$

[[$\odot_{\mathcal{P}}$ is associative, can neglect putting brackets, Lem. 3.3.7]]

$$= \Big(\mathsf{lift}\big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big) \circ \mathsf{pr}\Big)\big(\mathsf{inc}_{\varepsilon_1}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\varepsilon_n}(a_n)\big)$$

[[$\mathsf{can}\colon \bigsqcup_{i=1}^{k} \mathscr{A}_i \longrightarrow \mathrm{T}(\bigoplus_{i=1}^{k} \mathscr{A}_i / I_{1,\dots,k})$]]

$$= \big((\varphi_1 \circ \mathrm{j}_1) \circledcirc_{\mathcal{P}} \cdots \circledcirc_{\mathcal{P}} (\varphi_k \circ \mathrm{j}_k)\big)\big(\mathsf{inc}_{\varepsilon_1}(a_1) \cdot \cdots \cdot \mathsf{inc}_{\varepsilon_n}(a_n)\big) \quad [\![\text{ eq. (3.3.9) }]\!]$$
$$= \Big(\exp_{\mathcal{P}}\big(\sum_{\pi\in\mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)\Big)(a_1 \otimes \cdots \otimes a_n) \quad [\![\text{ Cor. 3.2.10 }]\!]$$
$$= \sum_{\pi\in\mathcal{P}_n} \prod_{b\in\pi} \big(\sum_{\pi\in\mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(a_b) \quad [\![\text{ eq. (3.2.4) }]\!]$$
$$= \sum_{\substack{\pi\in\mathcal{P}_n\\ \pi\leq \mathsf{indPart}(\varepsilon)}} \prod_{b\in\pi} \big(\sum_{\pi\in\mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(a_b) + \underbrace{\sum_{\substack{\pi\in\mathcal{P}_n\\ \pi\not\leq \mathsf{indPart}(\varepsilon)}} \prod_{b\in\pi} \big(\sum_{\pi\in\mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\big)(a_b)}_{=0}$$

[[$\mathsf{indPart}(\varepsilon) \in \mathcal{P}$]].

The last sum vanishes since any partition $\sigma \in \mathcal{P}_n$ for which $\sigma \not\leq \mathsf{indPart}(\varepsilon)$ holds, has the property that there exists a block $b \in \sigma$, which is not properly contained in any block of $\mathsf{indPart}(\varepsilon)$. Hence, such a block has the property that there exist $i, j \in [n]$ with $i \neq j$, $i, j \in \mathsf{set}\, b$ such that $\mathcal{A}_{\varepsilon_i} \neq \mathcal{A}_{\varepsilon_j}$. The result now follows in particular from Lemma 3.2.12 and equation (3.3.3).

For the case $\mathsf{indPart}(\varepsilon) \notin \mathcal{P}$ we calculate

$$
\begin{aligned}
&(\varphi_1 \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} \varphi_k)\big(\iota_{\varepsilon_1}(a_1) \cdot \cdots \cdot \iota_{\varepsilon_n}(a_n)\big) \\
&\quad = \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b) \quad [\![\text{ same steps as above }]\!] \\
&\quad = \underbrace{\sum_{j=1}^{k-1} \sum_{\pi \in \mathcal{P}_{n,j}} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b)}_{=0} + \sum_{j=k}^{n} \sum_{\pi \in \mathcal{P}_{n,j}} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b) \\
&\quad = \sum_{j=k}^{n} \sum_{\pi \in \mathcal{P}_{n,j}} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b) \quad [\![\text{ Lem. 3.2.12, eq. (3.3.3) }]\!] \\
&\quad = \sum_{j=k}^{n} \sum_{\substack{\pi \in \mathcal{P}_{n,j} \\ \forall b \in \pi, \\ \exists b' \in \mathsf{indPart}(\varepsilon): \\ \mathsf{set}\, b \subseteq \mathsf{set}\, b'}} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b) + \underbrace{\sum_{j=k}^{n} \sum_{\substack{\pi \in \mathcal{P}_{n,j} \\ \exists b \in \pi, \\ \forall b' \in \mathsf{indPart}(\varepsilon): \\ \mathsf{set}\, b \not\subseteq \mathsf{set}\, b'}} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b)}_{=0} \\
&\qquad [\![\text{ similar reasoning for the last step as in foregoing calculation }]\!] \\
&\quad = \sum_{j=k+1}^{n} \sum_{\substack{\pi \in \mathcal{P}_{n,j} \\ \forall b \in \pi, \\ \exists b' \in \mathsf{indPart}(\varepsilon): \\ \mathsf{set}\, b \subseteq \mathsf{set}\, b'}} \prod_{b \in \pi} \Big(\sum_{\pi \in \mathcal{P}_n} \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)(a_b) \quad [\![\mathsf{indPart}(\varepsilon) \notin \mathcal{P}]\!]
\end{aligned}
$$

□

5.1.21 Lemma. *Let $\mathcal{P}$ be a universal class of partitions. Consider $\odot_{\mathcal{P}}$ and let us denote its associated operation $\boxdot$ defined in equation (5.1.5) by $\boxdot_{\mathcal{P}}$. Then,*

$\forall n \in \mathbb{N}$, $\forall k \in [n]$, $\forall \varepsilon = (\varepsilon_i)_{i \in [n]} \in \mathbb{A}([k])$, $\forall (\mathcal{A}_i, \varphi_i)_{i \in [k]} \in \big(\mathsf{Obj}(\mathsf{AlgP})\big)^{\times k}$, $\forall (a_i)_{i \in [n]} \in \prod_{i=1}^{n} \mathcal{A}_{\varepsilon_i}$:

$$
(\varphi_1 \boxdot_{\mathcal{P}} \cdots \boxdot_{\mathcal{P}} \varphi_k)(a_1 \cdot \cdots \cdot a_n) = \begin{cases} \displaystyle\prod_{b \in \mathsf{indPart}(\varepsilon)} \Big(\sum_{i=1}^{k} (\varphi_i \circ \mathrm{j}_i)(a_b) \Big) & \text{for } \mathsf{indPart}(\varepsilon) \in \mathcal{P} \\ 0 & \text{for } \mathsf{indPart}(\varepsilon) \notin \mathcal{P}. \end{cases} \tag{5.1.44}
$$

Proof: According to the definition of the k-fold operation $T_{\boxdot_{\mathcal{P}} k}$ in equation (5.1.4) and (5.1.5) we need to determine an expression for $(t_1\varphi_1) \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} (t_k\varphi_k)$, where we can neglect orders of

t_i^2. Let us assume that $\mathsf{indPart}(\varepsilon) \in \mathcal{P}$, then we calculate

$$\begin{aligned}
&\big((t_1\varphi_1) \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n) \\
&= \sum_{\pi \leq \mathsf{indPart}(\varepsilon)} \prod_{b\in\pi} \Big(\sum_{i=1}^{k} \Big(\log_{\mathcal{P}}\big((t_i\varphi_i) \circ \mathrm{j}_i\big)\Big)(a_b)\Big) \quad [\![\text{ Lem. 5.1.20 }]\!] \\
&= \prod_{b\in\mathsf{indPart}(\varepsilon)} \Big(\sum_{i=1}^{k} \Big(\log_{\mathcal{P}}\big((t_i\varphi_i) \circ \mathrm{j}_i\big)\Big)(a_b)\Big) + \text{ orders of } t_i^2 \quad [\![\leq \text{ is reversed refinement }]\!] \\
&= \prod_{b\in\mathsf{indPart}(\varepsilon)} \Big(\sum_{i=1}^{k} \big((t_i\varphi_i) \circ \mathrm{j}_i\big)(a_b)\Big) + \text{ orders of } t_i^2 \quad [\![\text{ recursion of } \log_{\mathcal{P}}(\,\cdot\,) \text{ in eq. (3.2.5) }]\!].
\end{aligned}$$

Hence, we have determined the part with linear orders of t_i and the result now follows if we take the partial derivatives at $t_1 = \ldots = t_k = 0$.

For the case $\mathsf{indPart}(\varepsilon) \notin \mathcal{P}$ we can see from equation (5.1.43) that there is no term which can contribute to linear orders of t_i and thus the result follows. □

5.1.22 Remark. In the setting of Lemma 5.1.20 we notice the following. If $\mathsf{indPart}(\varepsilon) \in \mathcal{P}$, then

$$\begin{aligned}
&(\varphi_1 \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} \varphi_k)(a_1 \cdot \cdots \cdot a_n) \\
&= \sum_{\pi \leq \mathsf{indPart}(\varepsilon)} \prod_{b\in\pi} \Big(\sum_{i=1}^{k} (\log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i))\big(\overrightarrow{\bigotimes_{i\in \mathsf{set}\, b}} a_i\big)\Big) \\
&= \sum_{\pi \leq \{b_1,\ldots,b_k\}} \prod_{b\in\pi} \Big(\sum_{i=1}^{k} (\log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i))\big(\overrightarrow{\bigotimes_{i\in \mathsf{set}\, b}} a_i\big)\Big) \\
&= \Big(\sum_{\pi_1 \leq \mathsf{set}\, b_1} \prod_{\tilde{b}_1\in\pi_1} (\log_{\mathcal{P}}(\varphi_1 \circ \mathrm{j}_1))\big(\overrightarrow{\bigotimes_{i\in \mathsf{set}\, \tilde{b}_1}} a_i\big)\Big) \cdots \Big(\sum_{\pi_k \leq b_k} \prod_{\tilde{b}_k\in\pi_k} (\log_{\mathcal{P}}(\varphi_k \circ \mathrm{j}_k))\big(\overrightarrow{\bigotimes_{i\in \mathsf{set}\, \tilde{b}_k}} a_i\big)\Big) \\
&= \prod_{i=1}^{k} \Big((\varphi_i \circ \mathrm{j}_i)(a_{b_i})\Big)
\end{aligned} \tag{5.1.45}$$

We can compare this result to equation (5.1.44) and see that

$$(\varphi_1 \boxdot_{\mathcal{P}} \cdots \boxdot_{\mathcal{P}} \varphi_k)(a_1 \cdot \cdots \cdot a_n) = (\varphi_1 \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} \varphi_k)(a_1 \cdot \cdots \cdot a_n). \tag{5.1.46}$$

The above result shortens the computation of the right hand side whenever $\mathsf{indPart}(\varepsilon) \in \mathcal{P}$.

5.1.23 Lemma. Let $\mathcal{P}$ be a universal class of partitions. According to Theorem 3.3.9 **(e)** $\odot_{\mathcal{P}}$ satisfies the right-ordered monomials property. Therefore, we may speak of highest coefficients for $\odot_{\mathcal{P}}$. Then, all the nonzero highest coefficients w.r.t $\odot_{\mathcal{P}}$ are equal to 1 and the induced universal class of partitions w.r.t $\odot_{\mathcal{P}}$ is again $\mathcal{P}$. In other words, if we denote the highest coefficients belonging to $\odot_{\mathcal{P}}$ by α_π, then

$$\forall \pi \in \mathsf{Part}\colon \alpha_\pi \neq 0 \implies \alpha_\pi = 1, \tag{5.1.47a}$$

$$\mathfrak{P}_{\odot_{\mathcal{P}}} = \mathcal{P}. \tag{5.1.47b}$$

PROOF: Let us show that $\mathcal{P} \subseteq \mathfrak{P}_{\odot_\mathcal{P}}$ and that equation (5.1.47a) holds. We assume $\pi \in \mathcal{P}_{n,k}$ for some $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then, there exists at least one tuple $\varepsilon \in [k]^{\times n}$ such that $\mathsf{indPart}(\varepsilon) = \pi \in \mathcal{P}$. Let us fix this tuple $\varepsilon \in [k]^{\times n}$. The universal product $\odot_\mathcal{P}$ satisfies the right-ordered monomials property by Theorem 3.3.9 **(e)**. Therefore, we may apply Lemma 5.1.5 for evaluation of the expression $(\varphi_1 \boxdot_\mathcal{P} \cdots \boxdot_\mathcal{P} \varphi_k)(a_1 \cdot \cdots \cdot a_n)$ for $a_i \in \mathcal{A}_{\varepsilon_i}$, which basically says that it is determined by a product of the highest coefficient $\alpha_{\max}^{(\mathrm{red}\,\varepsilon)}$ and a product of evaluations of the linear functionals. On the other hand, we can also use Lemma 5.1.21 for calculation of $(\varphi_1 \boxdot_\mathcal{P} \cdots \boxdot_\mathcal{P} \varphi_k)(a_1 \cdot \cdots \cdot a_n)$. Comparing both equations against each other, for the highest coefficient we obtain $\alpha_{\max}^{(\mathrm{red}\,\varepsilon)} = 1$. Using Convention 5.1.13 we can see $\alpha_{\mathsf{indPart}(\varepsilon)} = 1 \neq 0$. From equation (5.1.34) we can see that $\mathsf{indPart}(\varepsilon) = \pi \in \mathfrak{P}_{\odot_\mathcal{P}}$. In a similar way we can prove equation (5.1.47a).

Now, let us show that $\mathfrak{P}_{\odot_\mathcal{P}} \subseteq \mathcal{P}$. We will show this by contraposition. Therefore let us assume that $\pi \notin \mathcal{P}$. If $\pi \in \mathsf{Part}_{n,k}$ for some $n, k \in \mathbb{N}$, then there exists at least one tuple $\varepsilon \in [k]^{\times n}$ such that $\mathsf{indPart}(\varepsilon) = \pi \notin \mathcal{P}$. The universal product $\odot_\mathcal{P}$ satisfies the right-ordered monomials property by Theorem 3.3.9 **(e)**. Therefore, we may apply Lemma 5.1.5 and if we compare this against Lemma 5.1.21, then we obtain for the highest coefficient $\alpha_{\max}^{(\mathrm{red}\,\varepsilon)} = 0$. Using Convention 5.1.13 we can see $\alpha_{\mathsf{indPart}(\varepsilon)} = 0$. From equation (5.1.34) we can see that $\mathsf{indPart}(\varepsilon) = \pi \notin \mathfrak{P}_{\odot_\mathcal{P}}$. □

5.1.24 Theorem. Let U denote the set of all symmetric u.a.u.-products with the right-ordered monomials property in the category AlgP such that their corresponding highest coefficients satisfy $\alpha_{|\,|} = 0$ or $\alpha_{|\,|} = 1$. Let P denote the set of all single-colored universal classes of partitions. Then, the map

$$f\colon \begin{cases} U \longrightarrow P, \\ \odot \longmapsto \mathfrak{P}_\odot. \end{cases} \tag{5.1.48}$$

is bijective.

PROOF: For injectivity we consider the following. From Theorem 4.1.17 we can determine the structure of P. Thus, we can make the following case consideration. Let $\odot, \tilde{\odot} \in U$ and assume $f(\odot) = \mathsf{1B} = f(\tilde{\odot})$. This means $\mathfrak{P}_\odot = \mathsf{1B} = \mathfrak{P}_{\tilde{\odot}}$. From Proposition 4.1.5 we can conclude $(\mathfrak{P}_\odot)_{\cdot,2} = \emptyset = (\mathfrak{P}_{\tilde{\odot}})_{\cdot,2}$. By Theorem 5.1.17 we obtain $\odot = \tilde{\odot}$.

Now, let $\odot, \tilde{\odot} \in U$ and assume $f(\odot) = f(\tilde{\odot}) \neq \mathsf{1B}$. Then, we can calculate

$$\begin{aligned}
&f(\odot) = f(\tilde{\odot}) \\
&\quad\iff \mathfrak{P}_\odot = \mathfrak{P}_{\tilde{\odot}} \quad [\![\text{ def. of } f\ ]\!] \\
&\quad\iff (\mathfrak{P}_\odot)_{\cdot,2} = (\mathfrak{P}_{\tilde{\odot}})_{\cdot,2} \quad [\![\text{ Proposition 4.1.5 }]\!] \\
&\quad\iff \{\,\pi \in \mathsf{Part}_{\cdot,2} \mid \alpha_\pi \neq 0\,\} = \{\,\pi \in \mathsf{Part}_{\cdot,2} \mid \tilde{\alpha}_\pi \neq 0\,\} \quad [\![\text{ def. of } \mathfrak{P}_\odot \text{ in eq. (5.1.34) }]\!] \\
&\quad\implies \{\,\pi \in \mathsf{Part}_{\cdot,2} \mid \alpha_\pi = \alpha_{|\,|}\,\} = \{\,\pi \in \mathsf{Part}_{\cdot,2} \mid \tilde{\alpha}_\pi = \tilde{\alpha}_{|\,|}\,\} \quad [\![\text{ Prop. 5.1.18 }]\!] \\
&\quad\iff \forall \pi \in \mathsf{Part}_{\cdot,2}\colon \alpha_\pi = \tilde{\alpha}_\pi \quad [\![\ \alpha_{|\,|} = 1 = \tilde{\alpha}_{|\,|}\ ]\!] \\
&\quad\iff \odot = \tilde{\odot} \quad [\![\text{ Thm. 5.1.17 }]\!].
\end{aligned}$$

This shows injectivity.

For surjectivity: We need to show that a universal product $\odot \in U$ with the property that $\mathfrak{P}_\odot = \mathcal{P}$ for any $\mathcal{P} \in P$ exists. We claim that $\odot_\mathcal{P}$ does the job. We first need to show that $\odot_\mathcal{P} \in U$.

From Theorem 3.3.9 **(d)** we know that $\odot_{\mathcal{P}}$ is a symmetric u.a.u.-product with the right-ordered monomials property. For the value of the coefficient $\alpha_{|\,|}$ we can obtain from Lemma 5.1.23 that $\alpha_{|\,|} = 0$ if $\mathcal{P}_{\cdot,2} = \emptyset$ and $\alpha_{|\,|} = 1$ if $\mathcal{P}_{\cdot,2} \neq \emptyset$. This shows that $\odot_{\mathcal{P}} \in U$ for any $\mathcal{P} \in P$. From Lemma 5.1.23 we obtain $f(\odot_{\mathcal{P}}) = \mathcal{P}$ for any $\mathcal{P} \in P$. □

5.1.25 Corollary. Let $f: U \longrightarrow P$ denote the map from equation (5.1.48). Let $\tilde{U}$ denotes the set of all positive and symmetric u.a.u.-products. Then, $\tilde{U} \subseteq U$ and the map $f\restriction_{\tilde{U}}: \tilde{U} \longrightarrow P \setminus \mathsf{1B}$ is bijective.

Proof: In [Voß13, Satz 1.7.7] or [SV14, Prop. 2.1] it is shown only by the use of positivity that any positive u.a.u.-product satisfies $\alpha_{|\,|} = 1$. We have a similar result in the m-faced case (apply Proposition 5.2.22 for $m = 1$). Then, positive u.a.u.-products have the right-ordered monomials property and thus $\tilde{U} \subseteq U$. Now, the assertion follows from Theorem 5.1.24. □

5.1.26 Remark. It is known that the boolean, free and the tensor product are elements of $\tilde{U}$. For instance in [Sch95] we can find their GNS-representations. By the above one-to-one correspondence we can conclude that the boolean product equals $\odot_{\mathsf{I}}$, the free product equals $\odot_{\mathsf{NC}}$ and the tensor product equals $\odot_{\mathsf{Part}}$ and we reproduce the result of [Spe97] or [BS02].

5.2 Partial classification of positive and symmetric two-faced u.a.u.-products

In this section we want to move on from the single-faced case to allow m-faced algebras for some $m \in \mathbb{N}$. Most of the statements in this section have a counterpart in the single-faced case. Thus, mostly for the proofs in this section we can take the regarding proofs from the single-faced case, where we need to make the following replacements

- AlgP replaced by $\mathsf{AlgP_m}$,
- $\mathbb{A}([k])$ replaced by $\mathbb{A}([k] \times [m])$ and
- $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathcal{A}_{\varepsilon_i}$ replaced by $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathcal{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$. (5.2.1)

Like we did in the single-faced case, we show for the two-faced case that nonzero highest coefficients of a positive and symmetric u.a.u.-product induces a two-colored universal class of partition.

5.2.1 Lemma. Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $\big(\mathcal{A}_i, (\mathcal{A}_i^{(\ell)})_{\ell\in[m]}, \varphi_i\big)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$. Define a similar "colored map-like Kronecker delta" as in Convention 3.4.26 with a minor modification

$$\forall \ell \in [m],\ \forall r \in [k]\colon \tilde{\Delta}_{i,r}^{(\ell)} := \begin{cases} 0\colon \mathcal{A}_i^{(j)} \longrightarrow \mathcal{A}_i^{(j)} & \text{for } i \neq r \\ \mathsf{id}_{\mathcal{A}_r^{(\ell)}}\colon \mathcal{A}_r^{(\ell)} \longrightarrow \mathcal{A}_r^{(\ell)} & \text{for } i = r. \end{cases} \tag{5.2.2}$$

We use the following abbreviation

$$\forall \ell \in [m],\ \forall r \in [k]\colon \mathrm{j}_r^{(\ell)} := \mathcal{T}\Big(\bigoplus_{i=1}^{k} \tilde{\Delta}_{i,r}^{(\ell)}\Big). \tag{5.2.3}$$

Put

$$\forall \ell \in [m]\colon V_\ell := \bigoplus_{i=1}^{k} \mathscr{A}_i^{(\ell)} \tag{5.2.4}$$

and consider $(\bigsqcup_{\ell=1}^m \mathrm{T}(V_\ell), \coprod_{\ell=1}^m \Delta_\ell, 0)$ as the m-faced dual semigroup with primitive comultiplication $\coprod_{\ell=1}^m \Delta_\ell$ using Convention 2.4.10. Then,

$$\begin{aligned} &\varphi_1 \odot \cdots \odot \varphi_k \\ &= \left(\left(\varphi_1 \circ \left(\coprod_{\ell\in[m]} \mathrm{j}_1^{(\ell)}\right)\right) \circledast \cdots \circledast \left(\varphi_k \circ \left(\coprod_{\ell\in[m]} \mathrm{j}_k^{(\ell)}\right)\right)\right) \circ \mathsf{inc}_{\bigsqcup_{i=1}^k \mathscr{A}_i, \bigsqcup_{\ell=1}^m \mathrm{T}(V_\ell)}, \end{aligned} \tag{5.2.5}$$

where $\circledast$ denotes the convolution with respect to the primitive comultiplication $\coprod_{\ell=1}^m \Delta_\ell$ and $\mathsf{inc}_{\mathscr{A}_1 \sqcup \mathscr{A}_2, \bigsqcup_{\ell=1}^m \mathrm{T}(V_\ell)}$ is a canonical inclusion of vector spaces.

Proof: The proof is similar to the proof of Theorem 2.4.12 where we just need to adjust the type index k to an arbitrary value in $\mathbb{N}$. The necessary steps can be easily adjusted and therefore we omit the proof and instead refer to the proof of Theorem 2.4.12. □

The definition of $T_{k\boxdot}$ in equation (5.1.4) has already been done for the case $m \in \mathbb{N}$, thus there is no need for adjustment. The result of equation (5.1.6) also holds for $m \in \mathbb{N} \setminus \{1\}$, except that the $(k-1)$-fold operation $\boxplus$ is to be seen on the dual semigroup discussed in Lemma 5.2.1. Similar to the proof of equation (5.1.6) we can obtain

$$\begin{aligned} &\varphi_1 \odot \cdots \odot \varphi_k \\ &= \left(\left(\varphi_1 \circ \left(\coprod_{\ell\in[m]} \mathrm{j}_1^{(\ell)}\right)\right) \boxplus \cdots \boxplus \left(\varphi_k \circ \left(\coprod_{\ell\in[m]} \mathrm{j}_k^{(\ell)}\right)\right)\right) \circ \mathsf{inc}_{\bigsqcup_{i=1}^k \mathscr{A}_i, \bigsqcup_{\ell=1}^m \mathrm{T}(V_\ell)} \end{aligned} \tag{5.2.6}$$

for the multi-faced case. The importance of expressing the $(k-1)$-fold operation $T_{\boxdot k}$ by $T_{\boxplus k}$ is given by the following implication

$$\begin{aligned} &\odot \text{ is a u.a.u.-product in the category } \mathsf{AlgP_m} \text{ and } [\,\cdot\,,\,\cdot\,]_\boxplus = 0 \\ &\implies \forall \sigma \in S_k\colon \varphi_1 \boxdot \cdots \boxdot \varphi_k = \varphi_{\sigma(1)} \boxdot \cdots \boxdot \varphi_{\sigma(k)} \circ \mathsf{can}, \end{aligned} \tag{5.2.7}$$

where S_k denotes the symmetric group on the set $[k]$ and $\mathsf{can}\colon \bigsqcup_{i=1}^k \longrightarrow \mathscr{A}_{\sigma(1)} \sqcup \cdots \sqcup \mathscr{A}_{\sigma(k)}$ the canonical isomorphism of the free product of rearranged algebras $\mathscr{A}_i$ in the sense of equation (1.1.17). Since the proof for the m-faced case is done similarly to the single-faced case, we omit it. Because the next lemma is similar to Lemma 5.1.4, we state it without proof.

5.2.2 Lemma. Assume $\odot$ is a u.a.u.-product with the right-ordered monomials property in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. We set

$\forall k \in \mathbb{N},\ \forall n \in \mathbb{N} \setminus [k-1],\ \forall \varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big) \in ([k] \times [m])^{\times n},\ \forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$:

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) := \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(\tilde{a}_1 \cdot \cdots \cdot \tilde{a}_{\tilde{n}}), \tag{5.2.8}$$

where $(\tilde{a}_i)_{i\in[\tilde{n}]} \in \prod_{i=1}^{n} \mathscr{A}_{\tilde{\varepsilon}_{i,1}}^{(\tilde{\varepsilon}_{i,2})}$ is the unique tuple such that

$$\exists\, \tilde{n} \in [n]\colon \big((\tilde{\varepsilon}_{i,1}, \tilde{\varepsilon}_{i,2})\big)_{i\in[\tilde{n}]} = \mathrm{red}(\varepsilon) \in \mathbb{A}\big([k]\times[m]\big). \tag{5.2.9}$$

The definition of the reduced tuple red(ε) is due to Convention 2.5.8.

5.2.3 Lemma. Let $\odot$ be a u.a.u.-product with the right-ordered monomials property in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. With respect to the notation from Theorem 2.3.3 we have

(a) $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{N}\setminus[k-1]$, $\forall \varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in \mathbb{A}([k]\times[m])$, $\forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$, $\forall (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$:

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \underbrace{\alpha^{(\varepsilon)}_{\mathbb{1}_1,\ldots,\mathbb{1}_k}}_{\equiv \alpha^{(\varepsilon)}_{\max}} \prod_{i=1}^{k} \varphi_i\big(j(a_1,\ldots,a_n)(\mathbb{1}_i)\big), \tag{5.2.10}$$

where $\mathbb{1}_i$ denotes the unique monomial in $\mathcal{OM}(X_\varepsilon^{(i)})$ of degree $|X_\varepsilon^{(i)}|$ (notation introduced in Remark 2.3.4 **(b)**) and $\alpha^{(\varepsilon)}_{\max}$ is the highest coefficient w.r.t. $\odot$, defined in Definition 2.5.3.

(b) $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{N}\setminus[k-1]$, $\forall \varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in \mathbb{A}([k]\times[m])$, $\exists \big(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i\big)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$, $\exists (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$:

$$\forall i \in [k]\colon \varphi_i\big(j(a_1,\ldots,a_n)(\mathbb{1}_i)\big) = 1, \tag{5.2.11a}$$

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \alpha^{(\varepsilon)}_{\max}. \tag{5.2.11b}$$

Furthermore, each algebra $\mathscr{A}_i$ can be chosen to be a $*$-algebra such that for all $j \in [m]$ we have that $\mathscr{A}_i^{(j)}$ is a $*$-subalgebra. Additionally, each linear functional $\varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ can be chosen to be strongly positive.

Proof: If we put for $\varepsilon = (\varepsilon_i)_{i\in[n]} = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in \mathbb{A}([k]\times[m])$

$$\forall j \in [k]\colon X_\varepsilon^{(j)} = \{\, x_\ell \in X_n \colon \varepsilon_{\ell,1} = j \,\}, \tag{I}$$

where $X_n := \{x_1,\ldots,x_n\}$ is the set formed by $n \in \mathbb{N}$ indeterminates $x_1,\ldots,x_n$ then the universal coefficient theorem (Theorem 2.3.3) in the multi-faced case looks formally the same as in the single-faced case.

Ad **(a)**: This follows from the definition in equation (I) and the replacements of (5.2.1). Then, we can formally perform the same steps as in the proof of Lemma 5.1.5 **(a)**, since the proof of Lemma 5.1.5 **(a)** basically relies on the universal coefficient theorem.

Ad **(b)**: We can slightly modify the proof of Lemma 5.1.5 **(b)** for the case $m \in \mathbb{N}$. We mention that $(\mathrm{T}(V_i), \mathcal{T}(\varphi_i)) \in \mathsf{Obj}(\mathsf{AlgP_m})$ gives the desired property, if we make the following modifications for the definitions made in the proof of Lemma 5.1.5 **(b)**

- $V_i := \bigoplus_{j=1}^{m} V_i^{(j)}$ for some 1-dimensional vector spaces $V_i^{(j)}$,
- $a_i^{(j)} \in V_i^{(j)}$ denotes a basis vector,
- $\varphi_i\restriction_{V_i^{(j)}}\colon V_i^{(j)} \ni a_i^{(j)} \longmapsto 1 \in \mathbb{C}$, then $\varphi_i \in \mathrm{Lin}(V_i, \mathbb{C})$.

We can turn each algebra $\mathscr{A}_i$ into a $*$-algebra if we assume that basis elements of $V_i^{(j)}$ are self-adjoint. Then, for all $j \in [m]$ we have that $\mathscr{A}_i^{(j)}$ is a $*$-subalgebra. Furthermore, the above definition for the linear functional $(\varphi_i)_{i\in[k]} \in \prod_{i\in[k]} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ leads to the property that each φ_i is strongly positive. □

The next statement extends the statement of Proposition 5.1.6 to the case $m \in \mathbb{N}$. Because of its importance we formulate it here once again.

5.2.4 Proposition. Let $m \in \mathbb{N}$. Let $\odot$ be a u.a.u.-product with the right-ordered monomials property in the category $\mathsf{AlgP_m}$. Let $k \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N} \setminus [k-1]$. Set

$$E_{n,k}^{(m)} := \left\{ (\varepsilon_i)_{i\in[n]} \in ([k]\times[m])^{\times n} \;\middle|\; \begin{array}{l} \exists \ell \in \{0,\dots,n-2\} \subseteq \mathbb{N},\ \exists i \in [m]: \\ (\varepsilon_{\ell+1,1} = 1,\ \varepsilon_{\ell+2,1} = 2) \\ \wedge (\varepsilon_{\ell+1,2} = \varepsilon_{\ell+2,2} = i) \end{array} \right\}. \tag{5.2.12}$$

Then, holds

$\forall k \in \mathbb{N}\setminus\{1\}, \forall n \in \mathbb{N}\setminus[k-1],\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in E_{n,k}^{(m)},\ \forall (\mathscr{A}_i, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$,
$\forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_{i,1}}^{\varepsilon_{i,2}}$:

$$\begin{aligned}\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)\big(a_1 \cdot \cdots \cdot \underbrace{a_{\ell+1}}_{\in\mathscr{A}_1^{(i)}} \cdot \underbrace{a_{\ell+2}}_{\in\mathscr{A}_2^{(i)}} \cdot \cdots \cdot a_n\big) \\ = \Big(\big(\varphi_1 \boxdot \varphi_2\big) \boxdot \varphi_3 \boxdot \cdots \boxdot \varphi_k\Big)\big(a_1 \cdot \cdots \cdot \underbrace{a_{\ell+1}\cdot a_{\ell+2}}_{\substack{\in(\mathscr{A}_1\sqcup\mathscr{A}_2)^{(i)} \\ =\mathscr{A}_1^{(i)}\sqcup\mathscr{A}_2^{(i)}}} \cdot \cdots \cdot a_n\big).\end{aligned} \tag{5.2.13}$$

Proof: The proof is virtually the same as the proof of Proposition 5.1.6, if we use the replacements from (5.2.1) and the universal coefficient theorem for the multi-faced case (Theorem 2.3.3). We emphasize that on the right hand side of equation (5.2.13) the product $a_{\ell+1}\cdot a_{\ell+2}$ is to be considered in the algebra $(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(i)}$. This yields that it will be treated as one indeterminate in the universal coefficient formula (equation (2.3.12)). □

5.2.5 Example. We want to give an example that if $\varepsilon \notin E_{n,k}^{(m)}$, where the set $E_{n,k}^{(m)}$ is defined in equation (5.2.12), then it can happen that

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)\big(a_1 \cdot \cdots \cdot a_n\big) \neq \Big(\big(\varphi_1 \boxdot \varphi_2\big) \boxdot \varphi_3 \boxdot \cdots \boxdot \varphi_k\Big)\big(a_1 \cdot \cdots \cdot a_n\big). \tag{5.2.14}$$

We already know from Lemma 4.2.26 that the set of binoncrossing partitions $\mathcal{P} := \mathsf{biNC}$ is a two-colored universal class of partitions which induces a u.a.u.-product denoted by $\odot_{\mathsf{biNC}}$ according to Theorem 3.4.32. Assume we are given $(\mathscr{A}_i, \varphi_i)_{i\in[3]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times 3}$ for $m = 2$. Let $\varepsilon = (\varepsilon_i)_{i\in[4]} = ((1,2),(3,2),(2,2),(3,2)) \notin E_{4,3}^{(2)}$ and a tuple $(a_i)_{i\in[4]} \in \prod_{i=1}^4 \mathscr{A}_{\varepsilon_i}^{(2)}$. For our given ε-tuple we can find a bijection between the set of binoncrossing partitions $\mathsf{biNC}_\varepsilon$ and the set of noncrossing partitions NC_4. In our ε-tuple only legs with color $\bullet$ are involved (Convention 4.2.7 **(b)**) and the assertion in particular follows from equation (4.2.24e), which essentially says that $\forall \pi \in \mathsf{biNC}_\varepsilon\ \mathsf{type}(\pi) \in \mathsf{NC}$ holds. Therefore, the steps of calculation formally hold as in example 5.1.7 and the calculation in example 5.1.7 leads to an counterexample in this case.

Let $m, k \in \mathbb{N} \setminus \{1\}$, $n \in \mathbb{N} \setminus [k-1]$ and $\varepsilon =: ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in \mathbb{A}([k] \times [m])$. We want to derive two other tuples from this tuple ε. We set

$$S := \left\{ i \in [n] \mid \big(\mathsf{type}(\varepsilon_i) = 1\big) \vee \big(\mathsf{type}(\varepsilon_i) = 2\big) \right\}. \tag{5.2.15}$$

We define the following tuple $\gamma = ((\gamma_{i,1}, \gamma_{i,2}))_{i\in[n]} \in ([k-1] \times [m])^{\times n}$ by

$$\forall\, i \in [n]\colon (\gamma_{i,1}, \gamma_{i,2}) := \begin{cases} \big(1, \mathsf{col}(\varepsilon_i)\big) & \text{for } i \in S \\ \big(\mathsf{type}(\varepsilon_i) - 1, \mathsf{col}(\varepsilon_i)\big) & \text{else.} \end{cases} \tag{5.2.16}$$

Then, we set

$$\tilde{\varepsilon} := \big((\tilde{\varepsilon}_{i,1}, \tilde{\varepsilon}_{i,2})\big)_{i\in[n]} := \mathrm{red}(\gamma) \in \mathbb{A}([k-1]). \tag{5.2.17}$$

For example, if $\varepsilon = \big((1,1),(2,1),(3,1),(2,2),(4,1),(1,2)\big) \in \mathbb{A}([4] \times [2])$, then according to the above $\gamma = \big((1,1),(1,1),(2,1),(1,2),(3,1),(1,2)\big)$ and therefore $\tilde{\varepsilon} = \big((1,1),(2,1),(1,2),(3,1),(1,2)\big) \in \mathbb{A}([3] \times [2])$. We want to define a second tuple originating from ε by

$$\bar{\varepsilon} := \big((\bar{\varepsilon}_{i,1}, \bar{\varepsilon}_{i,2})\big)_i := \mathrm{red}\Big(\big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in S}\Big) \in \mathbb{A}([2]). \tag{5.2.18}$$

Notice that we make use of Convention 2.5.5 **(a)**. As an example, for $\varepsilon = \big((1,1),(2,1),(3,1),(2,2),(4,1),(1,2)\big) \in \mathbb{A}([4] \times [2])$ we have $\bar{\varepsilon} = \big((1,1),(2,1),(2,2),(1,2)\big) \in \mathbb{A}([2])$. Using these notations we want to formulate a corollary to Proposition 5.2.4.

5.2.6 Corollary (to Proposition 5.2.4). Let $m \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N} \setminus [k-1]$ and assume $\varepsilon = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in (E^{(m)}_{n,k} \cap \mathbb{A}([k] \times [m]))$, where the set $E^{(m)}_{n,k}$ has been defined in equation (5.2.12). Then, we have

$$\alpha^{(\varepsilon)}_{\max} = \alpha^{(\tilde{\varepsilon})}_{\max} \cdot \alpha^{(\bar{\varepsilon})}_{\max}. \tag{5.2.19}$$

Proof: We just refer to the proof of Corollary 5.1.8 which displays the same formal steps of calculation, where we just need to apply the appropriate assertions for the case $m \geq 1$. □

5.2.7 Definition (Induced m-colored partition). Let $m \in \mathbb{N}$. We set

$\forall\, k \in \mathbb{N},\ \forall\, n \in \mathbb{N} \setminus [k-1],\ \forall\, \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$:

$$T_{\sigma,k} := \{\, (\gamma_i, \sigma_i)_{i\in[n]} \in ([k] \times [m])^{\times n} \mid |\mathsf{set}(\gamma_i)_{i\in[n]}| = k \,\} \tag{5.2.20}$$

and

$$T^{(m)} := \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus[k-1]} \bigcup_{\sigma\in[m]^{\times n}} T_{\sigma,k}. \tag{5.2.21}$$

Now, we want that an n-tuple $\varepsilon \in T_{\sigma,k}$ induces a partition $\pi \in \mathsf{Part}_{\sigma,k}$ with k blocks. Let $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus [k-1]$ and $\varepsilon \in T_{\sigma,k}$. We set

$$\forall\, i \in [k]\colon \beta_{\varepsilon,i} := \{\, (\ell, \sigma_\ell) \in \mathbb{N} \times \mathbb{N} \mid \mathsf{type}(\varepsilon_\ell) = i \,\}. \tag{5.2.22}$$

And by this we set

$$\mathsf{indPart}(\varepsilon) := \bigcup_{i\in[k]} \{\, \mathsf{block}(\beta_{\varepsilon,i}) \,\} \in \mathsf{Part}_{\mathsf{col}(\varepsilon),k}\,. \tag{5.2.23}$$

Thus, we define

$$\mathsf{indPart}\colon \begin{cases} T^{(m)} \longrightarrow \bigcup_{\varepsilon\in T^{(m)}} \mathsf{Part}_{\mathsf{col}(\varepsilon)} \\ \varepsilon \longmapsto \mathsf{indPart}(\varepsilon). \end{cases} \tag{5.2.24}$$

5.2.8 Remark. The above defined map is not injective but surjective. Moreover, for each $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus [k-1]$ and $\sigma \in [m]^{\times n}$ we have

$$\mathbb{A}([k] \times [m]) \cap T_{\sigma,k} = \mathsf{indPart}^{-1}\big(\mathrm{red}(\mathsf{Part}_{\sigma,k})\big). \tag{5.2.25}$$

Or in other words, we can say that $\mathsf{indPart}(\varepsilon)$ is reduced as an (m-colored) partition (Definition 4.2.28 **(a)**) if and only if ε is reduced as a tuple (Convention 2.5.8).

5.2.9 Lemma. Let $\odot$ be a symmetric u.a.u.-product in the category $\mathsf{AlgP_m}$, which has the right-ordered monomials property. Let $\varepsilon = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in T_{\sigma,k}$ for some $k \in \mathbb{N}$, $n \in \mathbb{N}\setminus[k-1]$, $\sigma \in [m]^{\times n}$. Define the set

$$\mathcal{N}_\varepsilon := \{\, \varepsilon' = (\varepsilon'_{i,1}, \varepsilon_{i,2})_{i\in[n]} \in \mathbb{N}\times\mathbb{N} \mid \exists f \in S_k \colon \forall i \in [n] \colon \varepsilon'_{i,1} = f(\varepsilon_{i,1}) \,\}. \tag{5.2.26}$$

If we set $\pi := \mathsf{indPart}(\varepsilon) \in \mathsf{Part}_{\mathsf{col}(\varepsilon),k}$, then we have

$$\mathcal{N}_\varepsilon = (\mathsf{indPart})^{-1}(\pi) \tag{5.2.27}$$

$$\forall\, \varepsilon' \in (\mathsf{indPart})^{-1}(\pi) \colon \alpha_{\max}^{(\mathrm{red}\,\varepsilon')} = \alpha_{\max}^{(\mathrm{red}\,\varepsilon)}. \tag{5.2.28}$$

Proof: The proof of Lemma 5.1.12 can be formally carried over to the case $m \geq 1$. We only have to apply the regarding lemmas for the case $m \geq 1$ (for instance we need equation (5.2.7)). □

Similar to Convention 5.1.13 we now want to use a diagrammatic approach to better handle highest coefficients.

5.2.10 Convention. Let $m \in \mathbb{N}$. We set

$$\forall\, \varepsilon \in T^{(m)},\ \forall \pi \in \mathsf{Part}_\varepsilon \colon \alpha_\pi := \alpha_{\max}^{\mathrm{red}(\varepsilon')} \text{ for some } \varepsilon' \in (\mathsf{indPart})^{-1}(\pi). \tag{5.2.29}$$

This convention is well-defined since $\mathsf{indPart}\colon T^{(m)} \longrightarrow \mathsf{Part}$ is surjective and because of equation (5.2.28). Due to Remark 5.2.8, this convention implies $\forall\, \varepsilon \in T^{(m)},\ \forall \pi \in \mathsf{Part}_{\mathsf{col}(\varepsilon)} \colon \alpha_\pi = \alpha_{\mathrm{red}\,\pi}$.

We want to apply this convention to the statement of Corollary 5.2.6 as we did in Corollary 5.1.14 for the case $m = 1$.

5.2.11 Corollary (to Proposition 5.2.4). Let $m \in \mathbb{N}$, $k \in \mathbb{N}\setminus\{1\}$, $n \in \mathbb{N}\setminus[k-1]$ and $\sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$. For all $\varepsilon = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in \cap T_{\sigma,k}$ with $\pi := \mathsf{indPart}(\varepsilon) \in \mathsf{Part}_{\sigma,k}^{(\ell+1)\vee(\ell+2)}$ holds

$$\alpha_\pi = \alpha_{(\mathsf{pr}_1 \circ \mathsf{UMem}_{\sigma,k}^{\ell+1})(\pi)} \cdot \alpha_{(\mathsf{pr}_2 \circ \mathsf{UMem}_{\mathsf{pr}_2(\varepsilon),k}^{\ell+1})(\pi)}. \tag{5.2.30}$$

In other words, for all $(\tilde{\varepsilon}, \bar{\varepsilon}) \in (T_{\sigma,k-1} \times T_{\cdot,2})$ such that $(\tilde{\pi}, \bar{\pi}) := (\mathsf{indPart}(\tilde{\varepsilon}), \mathsf{indPart}(\bar{\varepsilon})) \in \mathrm{sub}\big(\mathsf{Part}_{n,k-1}^{(\ell+1)\wedge(\ell+2)}\big)$ holds

$$\alpha_{\tilde{\pi}} \cdot \alpha_{\bar{\pi}} = \alpha_{\mathsf{split}_{\sigma,k-1}^{\ell+1}(\tilde{\pi},\bar{\pi})}. \tag{5.2.31}$$

PROOF: The proof is done analogously to the proof of Corollary 5.1.14. □

5.2.12 Lemma. Let $m \in \mathbb{N}$ and let $\mathcal{P}$ be an m-colored universal class of partitions. Let us denote the reversed refinement order for partitions by $\leq$. Then,

$\forall n \in \mathbb{N},\ \forall k \in [n],\ \forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in T_{\sigma,k},\ \forall (\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[m]}, \varphi_i)_{i\in[k]} \in \big(\mathrm{Obj}(\mathsf{AlgP_m})\big)^{\times k},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathcal{A}^{(\sigma)}_{\mathsf{type}(\varepsilon_i)}$:

$$(\varphi_1 \odot_{\mathcal{P}} \cdots \odot_{\mathcal{P}} \varphi_k)(a_1 \cdot \dots \cdot a_n) = \begin{cases} \displaystyle\sum_{\pi \leq \mathsf{indPart}(\varepsilon)} \prod_{b\in\pi} \Big(\sum_{i=1}^k (\log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i))\big(\overrightarrow{\bigotimes_{i\in\mathsf{set}(\mathsf{type}(b))}} a_i\big)\Big) & \text{for } \mathsf{indPart}(\varepsilon) \in \mathcal{P} \\ \displaystyle\sum_{j=k+1}^n \sum_{\substack{\pi\in\mathcal{P}_{\sigma,j}:\\ \forall b\in\pi,\\ \exists b'\in\mathsf{indPart}(\varepsilon):\\ \mathsf{set}(\mathsf{type}(b))\subseteq\mathsf{set}(\mathsf{type}(b'))}} \prod_{b\in\sigma} \Big(\sum_{i=1}^k (\log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i))\big(\overrightarrow{\bigotimes_{i\in\mathsf{set}(\mathsf{type}(b))}} a_i\big)\Big) & \text{for } \mathsf{indPart}(\varepsilon) \notin \mathcal{P}. \end{cases} \tag{5.2.32}$$

PROOF: As in the proof for the case $m = 1$, i. e., the proof of Lemma 5.1.20, the main observation is the following. Given the assumptions from the above assertion let b denote a block of a partition $\pi \in \mathsf{Part}_{\sigma,k}$ with the property that there exist $i, j \in [k]$ with $i \neq j$ such that $i, j \in \mathsf{set}(\mathsf{type}(b))$ and $a_i \in \mathcal{A}^{(\sigma_i)}_{\mathsf{type}\ \varepsilon_i} \neq \mathcal{A}^{(\sigma_j)}_{\mathsf{type}\ \varepsilon_j} \ni a_j$. For such a block b we have

$$\Big(\sum_{i=1}^k \log_{\mathcal{P}}(\varphi_i \circ \mathrm{j}_i)\Big)\big(\overrightarrow{\bigotimes_{i\in\mathsf{set}(\mathsf{type}(b))}} a_i\big) = 0.$$

This follows from Lemma 3.4.23 and the definition of the map j_i in equation (3.4.64). The remaining steps of the proof are formally the same as in the proof of Lemma 5.1.20 and we therefore omit the details. □

5.2.13 Lemma. Let $m \in \mathbb{N}$ and let $\mathcal{P}$ be an m-colored universal class of partitions. Consider $\odot_{\mathcal{P}}$ and let us denote its associated operation $\boxdot$ defined in equation (5.1.5) by $\boxdot_{\mathcal{P}}$. Then,

$\forall n \in \mathbb{N},\ \forall k \in [n],\ \forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in T_{\sigma,k},\ \forall (\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[m]}, \varphi_i)_{i\in[k]} \in \big(\mathrm{Obj}(\mathsf{AlgP_m})\big)^{\times k},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^n \mathcal{A}^{(\sigma))}_{\mathsf{type}(\varepsilon_i)}$:

$$(\varphi_1 \boxdot_{\mathcal{P}} \cdots \boxdot_{\mathcal{P}} \varphi_k)(a_1 \cdot \dots \cdot a_n) = \begin{cases} \displaystyle\prod_{b\in\mathsf{indPart}(\varepsilon)} \Big(\sum_{i=1}^k (\varphi_i \circ \mathrm{j}_i)(a_b)\Big) & \text{for } \mathsf{indPart}(\varepsilon) \in \mathcal{P} \\ 0 & \text{for } \mathsf{indPart}(\varepsilon) \notin \mathcal{P}. \end{cases} \tag{5.2.33}$$

PROOF: There is formally no difference to the proof of the single-colored version of this assertion, i. e., the proof of Lemma 5.1.21. □

5.2.14 Lemma. Let $\mathcal{P}$ be an m-colored universal class of partitions, where $m \in \mathbb{N}$. According to Theorem 3.4.32 **(e)** $\odot_{\mathcal{P}}$ satisfies the right-ordered monomials property. Therefore we may speak of highest-coefficients for $\odot_{\mathcal{P}}$. Then, all the nonzero highest coefficients w.r.t $\odot_{\mathcal{P}}$ are equal to 1 and the induced universal class of partitions w.r.t $\odot_{\mathcal{P}}$ is again $\mathcal{P}$. In other words, if we denote the highest coefficients belonging to $\odot_{\mathcal{P}}$ by α_π then

$$\forall \pi \in \bigcup_{\varepsilon \in T^{(m)}} \mathsf{Part}_{\mathsf{col}(\varepsilon)} \colon \alpha_\pi \neq 0 \implies \alpha_\pi = 1, \tag{5.2.34a}$$

$$\mathfrak{P}_{\odot_{\mathcal{P}}} = \mathcal{P} \tag{5.2.34b}$$

Proof: The proof is similar to case $m = 1$, i. e., to the proof of Lemma 5.1.23. □

At this point we are not able to extend the assertion of Proposition 5.1.16 to all symmetric u.a.u.-products with the right-ordered monomials property in the category $\mathsf{AlgP_m}$. There we have shown that nonzero highest coefficients can be expressed as a product of highest coefficients which belong to two-block partitions. Although partition induced universal products $\odot_{\mathcal{P}}$ have this property by design, as we will see later, we need a stronger version of the right-ordered monomials property for general universal products which might not be partition induced. We will see later, that we are able to extend the assertion of Proposition 5.1.16 if we demand for the product $\odot$ being positive. We will see that a positive and symmetric u.a.u.-product allows us to apply the "change color"-axiom (Definition 3.4.9 **(f)**) to nonzero highest coefficients. If we think of the extension of Proposition 5.1.16 to the m-colored case as a reformulation of Proposition 4.2.11 on the level of highest coefficients, then we may convince ourselves that the "change color"-axiom is truly needed. We need the following two lemmas when we want to prove that an m-faced positive and symmetric u.a.u.-product induces an m-colored universal class of partitions.

5.2.15 Lemma. Let $k \in \mathbb{N}$ and $\big(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i\big) \in \mathsf{Obj}(\mathsf{AlgP_m})^{\times k}$. We assume that on each algebra $\mathscr{A}_i$ an involution $*_i \colon \mathscr{A}_i \longrightarrow \mathscr{A}_i$ is defined, which turns $\mathscr{A}_i$ into an $*$-algebra, where each subalgebra $\mathscr{A}_i^{(j)}$ is a $*_i$-subalgebra and each linear functional φ_i is strongly positive. Assume $\forall i \in [k],\ \forall r, s \in [m],\ r \neq s$ that there exists an algebra isomorphism $\mathsf{iso}_i^{(r),(s)} \colon \mathscr{A}_i^{(r)} \longrightarrow \mathscr{A}_i^{(s)}$ such that

$\forall i \in [k],\ \forall r, s \in [m],\ r \neq s$:

$$*_i \circ \mathsf{iso}_i^{(r),(s)} = \mathsf{iso}_i^{(r),(s)} \circ *_i\restriction_{\mathscr{A}_i^{(r)}}, \tag{5.2.35}$$

$$\varphi_i \circ \iota_{\mathscr{A}_i^{(s)}} \circ \mathsf{iso}_i^{(r),(s)} = \varphi_i \circ \iota_{\mathscr{A}_i^{(r)}} \tag{5.2.36}$$

$$\varphi_i \circ \mu_{\mathscr{A}_i} \circ \big((\iota_{\mathscr{A}_i^{(s)}} \circ \mathsf{iso}_i^{(r),(s)}) \otimes \iota_{\mathscr{A}_i^{(r)}}\big) = \varphi_i \circ \mu_{\mathscr{A}_i} \circ \big(\iota_{\mathscr{A}_i^{(r)}} \otimes (\iota_{\mathscr{A}_i^{(s)}} \circ \mathsf{iso}_i^{(r),(s)})\big), \tag{5.2.37}$$

where $\iota_{\mathscr{A}_i^{(r)}} \colon \mathscr{A}_i^{(r)} \hookrightarrow \bigsqcup_{j=1}^m \mathscr{A}_i^{(j)} \cong \mathscr{A}_i$ is the canonical insertion algebra homomorphism. Let $\odot$ be a positive u.a.u.-product and abbreviate $\varphi_1 \odot \cdots \odot \varphi_k$ by $\bigodot_{i=1}^k \varphi_i$.
For all $r, s \in [m]$ with $r \neq s$,$\varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k])$, $\forall i \in [n]$: $a_i \in \mathscr{A}_{\varepsilon_i}$ such that $a_1 \in \mathscr{A}_{\varepsilon_1}^{(r)}$ holds

$$\left(\bigodot_{i=1}^k \varphi_i\right)(a_1 \cdot a_2 \cdot \cdots \cdot a_n) = \left(\bigodot_{i=1}^k \varphi_i\right)\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1) \cdot a_2 \cdot \cdots \cdot a_n\big). \tag{5.2.38}$$

Proof: Since $\odot$ is positive, we deduce that $\bigodot_{i=1}^{k}\varphi_i$ is a strongly positive linear functional on $(\bigsqcup_{i=1}^{k}\mathscr{A}_i,*)$, where the involution $*$ is similarly defined as in Convention 2.1.14. By Remark 1.1.28 we know that $\bigodot_{i=1}^{k}\varphi_i$ is positive. Now, we can calculate

$$\begin{aligned}
&\left|\left(\bigodot_{i=1}^{k}\varphi_i\right)\Big(\underbrace{\big(a_1-\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)}_{=:\alpha^*\in\mathscr{A}_{\varepsilon_1}}\cdot\underbrace{a_2\cdot\dots\cdot a_n}_{=:\beta}\Big)\right|^2=\left|\left(\bigodot_{i=1}^{k}\varphi_i\right)(\alpha^*\cdot\beta)\right|^2\\
&\le\left(\bigodot_{i=1}^{k}\varphi_i\right)(\alpha^*\cdot\alpha)\underbrace{\left(\bigodot_{i=1}^{k}\varphi_i\right)(\beta^*\cdot\beta)}_{=:c\in\mathbb{C}}\quad[\![\text{ eq. (1.1.29) }]\!]\\
&=c\cdot\varphi_{\varepsilon_1}(\alpha^*\cdot\alpha)\quad[\![\text{ unitality of }\odot\ ]\!]\\
&=c\cdot\varphi_{\varepsilon_1}\Big(\big(a_1-\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)\big(a_1-\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^*\Big)\quad[\![\text{ def. of }\alpha\ ]\!]\\
&=c\cdot\varphi_{\varepsilon_1}\Big(a_1\cdot a_1^*-\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\cdot a_1^*+a_1\cdot\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^*-\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\cdot\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^*\Big)\\
&=c\cdot\varphi_{\varepsilon_1}(a_1\cdot a_1^{*_{\varepsilon_1}})-c\cdot\varphi_{\varepsilon_1}\Big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\cdot\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^{*_{\varepsilon_1}}\Big)\\
&\quad-c\cdot\varphi_{\varepsilon_1}\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\cdot a_1^*\big)+c\cdot\varphi_{\varepsilon_1}\big(a_1\cdot\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^*\big)\quad[\![\ \varphi_{\varepsilon_1}\text{ is linear, def. of }*\ ]\!]\\
&=c\cdot\varphi_{\varepsilon_1}(a_1\cdot a_1^{*_{\varepsilon_1}})-c\cdot\varphi_{\varepsilon_1}\Big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}\big(a_1\cdot a_1^{*_{\varepsilon_1}}\big)\Big)\\
&\quad-c\cdot\varphi_{\varepsilon_1}\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\cdot a_1^*\big)+c\cdot\varphi_{\varepsilon_1}\big(a_1\cdot\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^*\big)\\
&\qquad[\![\text{ eq. (5.2.35), }\mathsf{iso}_{\varepsilon_1}^{(r),(s)}\text{ is homomorphism }]\!]\\
&=0-c\cdot\varphi_{\varepsilon_1}\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\cdot a_1^*\big)+c\cdot\varphi_{\varepsilon_1}\big(a_1\cdot\big(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1)\big)^*\big)\\
&\qquad[\![\ (\mathscr{A}_{\varepsilon_1}^{(r)})^{*_{\varepsilon_1}}\subseteq\mathscr{A}_{\varepsilon_1}^{(r)},\text{ eq. (5.2.36) }]\!]\\
&=c\cdot\Bigg(\Big(\varphi_{\varepsilon_1}\circ\mu_{\mathscr{A}_{\varepsilon_1}}\circ\big((\iota_{\mathscr{A}_{\varepsilon_1}^{(s)}}\circ\mathsf{iso}_{\varepsilon_1}^{(r),(s)})\otimes\iota_{\mathscr{A}_{\varepsilon_1}^{(r)}}\big)\Big)\\
&\qquad-\Big(\varphi_{\varepsilon_1}\circ\mu_{\mathscr{A}_{\varepsilon_1}}\circ\big(\iota_{\mathscr{A}_{\varepsilon_1}^{(r)}}\otimes(\iota_{\mathscr{A}_{\varepsilon_1}^{(s)}}\circ\mathsf{iso}_{\varepsilon_1}^{(r),(s)})\big)\Big)\Bigg)(a_1\otimes a_1^{*_{\varepsilon_1}})\\
&\qquad[\![\text{ eq. (5.2.35), }\mathsf{iso}_{\varepsilon_1}^{(r),(s)}\text{ is homomorphism, }(\mathscr{A}_{\varepsilon_1}^{(j)})^{*_{\varepsilon_1}}\subseteq\mathscr{A}_{\varepsilon_1}^{(j)}\ ]\!]\\
&=0\quad[\![\text{ eq. (5.2.37) }]\!].
\end{aligned}$$

The above calculation implies that

$$\left(\bigodot_{i=1}^{k}\varphi_i\right)\big((a_1-\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1))\cdot a_2\cdot\dots\cdot a_n\big)=0$$

$$\implies \left(\bigodot_{i=1}^{k} \varphi_i\right)\left(a_1 \cdot a_2 \cdot \dots \cdot a_n\right) = \left(\bigodot_{i=1}^{k} \varphi_i\right)\left(\mathsf{iso}_{\varepsilon_1}^{(r),(s)}(a_1) \cdot a_2 \cdot \dots \cdot a_n\right)$$

$$\left[\!\left[\left(\bigodot_{i=1}^{k} \varphi_i\right) \text{ is linear} \right]\!\right].$$

This shows that equation (5.2.38) holds. ◻

5.2.16 Lemma. Let $k \in \mathbb{N}$ and $\left(\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[m]}, \varphi_i\right) \in \mathsf{Obj}(\mathsf{AlgP_m})^{\times k}$. We assume that on each algebra $\mathcal{A}_i$ an involution $*_i \colon \mathcal{A}_i \longrightarrow \mathcal{A}_i$ is defined which turns $\mathcal{A}_i$ into an $*$-algebra and each subalgebra $\mathcal{A}_i^{(j)}$ is a $*_i$-subalgebra. Then, there exists an k-tuple $\left(\tilde{\mathcal{A}}_i, (\tilde{\mathcal{A}}_i^{(j)})_{j\in[m]}, \tilde{\varphi}_i\right)_{i\in[k]} \in \mathsf{Obj}(\mathsf{AlgP_m})^{\times k}$ such that

(a) there exist injective homomorphisms

$$\forall\, i \in [k]\colon\ \mathsf{inc}_i \in \mathsf{Morph}_{\mathsf{Alg_m}}\Big(\big(\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[m]}\big), \big(\tilde{\mathcal{A}}_i, (\tilde{\mathcal{A}}_i^{(j)})_{j\in[m]}\big)\Big) \tag{5.2.39}$$

with the property

$$\forall\, i \in [k]\colon \tilde{\varphi}_i \circ \mathsf{inc}_i = \varphi_i, \tag{5.2.40}$$

(b) $\forall\, i \in [k],\ \forall\, \varphi,\ \varphi' \in \mathrm{Lin}(\mathcal{A}_i, \mathbb{C}),\ \forall\, c \in \mathbb{R}$:

$$\widetilde{\varphi + c\varphi'} = \tilde{\varphi} + c\widetilde{\varphi'}, \tag{5.2.41}$$

(c) $\forall\, i \in [k]$ there exist involutions $\tilde{*}_i \colon \tilde{\mathcal{A}}_i \longrightarrow \tilde{\mathcal{A}}_i$, which turn each $\tilde{\mathcal{A}}_i$ into a $*$-algebra such that $(\tilde{\mathcal{A}}_i^{(j)})^{\tilde{*}_i} \subseteq \tilde{\mathcal{A}}_i^{(j)}$,

(d) $\forall\, i \in [k],\ \forall\, r,\ s \in [m],\ r \neq s$ there exist algebra isomorphisms $\mathsf{iso}_i^{(r),(s)} \colon \tilde{\mathcal{A}}_i^{(r)} \longrightarrow \tilde{\mathcal{A}}_i^{(s)}$ with the properties

$$\tilde{*}_i \circ \mathsf{iso}_i^{(r),(s)} = \mathsf{iso}_i^{(r),(s)} \circ \tilde{*}_i, \tag{5.2.42}$$

$$\tilde{\varphi}_i \circ \iota_{\tilde{\mathcal{A}}_i^{(s)}, \tilde{\mathcal{A}}_i} \circ \mathsf{iso}_i^{(r),(s)} = \tilde{\varphi}_i \circ \iota_{\tilde{\mathcal{A}}_i^{(r)}, \tilde{\mathcal{A}}_i}, \tag{5.2.43}$$

$$\tilde{\varphi}_i \circ \mu_{\tilde{\mathcal{A}}_i} \circ \big((\iota_{\tilde{\mathcal{A}}_i^{(s)}, \tilde{\mathcal{A}}_i} \circ \mathsf{iso}_i^{(r),(s)}) \otimes \iota_{\tilde{\mathcal{A}}_i^{(r)}, \tilde{\mathcal{A}}_i}\big) = \tilde{\varphi}_i \circ \mu_{\tilde{\mathcal{A}}_i} \circ \big(\iota_{\tilde{\mathcal{A}}_i^{(r)}, \tilde{\mathcal{A}}_i} \otimes (\iota_{\tilde{\mathcal{A}}_i^{(s)}, \tilde{\mathcal{A}}_i} \circ \mathsf{iso}_i^{(r),(s)})\big), \tag{5.2.44}$$

$$\tilde{\varphi}_i \circ \mu_{\tilde{\mathcal{A}}_i} \circ \Big(\big(\iota_{\tilde{\mathcal{A}}_i^{(s)}, \tilde{\mathcal{A}}_i} \circ \mathsf{iso}_i^{(r),(s)}\big) \otimes \mathsf{id}_{\tilde{\mathcal{A}}_i}\Big) = \tilde{\varphi}_i \circ \mu_{\tilde{\mathcal{A}}_i} \circ \big(\iota_{\tilde{\mathcal{A}}_i^{(r)}, \tilde{\mathcal{A}}_i} \otimes \mathsf{id}_{\tilde{\mathcal{A}}_i}\big), \tag{5.2.45}$$

where $\iota_{\tilde{\mathcal{A}}_i^{(r)}, \tilde{\mathcal{A}}_i} \colon \tilde{\mathcal{A}}_i^{(r)} \hookrightarrow \bigsqcup_{j=1}^{m} \tilde{\mathcal{A}}_i^{(j)} \cong \tilde{\mathcal{A}}_i$ denotes the canonical homomorphic insertion maps.

(e) If $\varphi_i \colon \mathcal{A}_i \longrightarrow \mathbb{C}$ is assumed to be strongly positive, then $\tilde{\varphi}_i \colon \tilde{\mathcal{A}}_i \longrightarrow \mathbb{C}$ is also strongly positive.

Proof: We define

$$\forall\, i \in [k],\ \forall\, j \in [m]\colon \quad \tilde{\mathcal{A}}_i^{(j)} := \bigsqcup_{\ell=1}^{m} \mathcal{A}_i^{(\ell)},$$

$$\forall i \in [k]: \qquad \tilde{\mathscr{A}}_i := \bigsqcup_{j=1}^{m} \tilde{\mathscr{A}}_i^{(j)}. \tag{I}$$

Hence, each algebra $\tilde{\mathscr{A}}_i^{(j)}$ is just a copy of $\mathscr{A}_i$ and each copy is labeled by the index j. Thus, the faces of $\tilde{\mathscr{A}}_i$ are isomorphic to each other. Furthermore we define algebra homomorphisms by

$$\forall i \in [k], \forall j \in [m]: \nu_i^{(j)}: \begin{cases} \tilde{\mathscr{A}}_i^{(j)} \longrightarrow \mathscr{A}_i \\ a \longmapsto a. \end{cases} \tag{II}$$

By this, we define linear functionals

$$\forall i \in [k]: \tilde{\varphi}_i: \begin{cases} \tilde{\mathscr{A}}_i \longrightarrow \mathbb{C} \\ a \longmapsto \left(\varphi_i \circ \left(\bigsqcup_{j=1}^{m} \nu_i^{(j)}\right)\right)(a). \end{cases} \tag{III}$$

We obtain $\left(\tilde{\mathscr{A}}_i, (\tilde{\mathscr{A}}_i^{(j)})_{j\in[m]}, \tilde{\varphi}_i\right) \in \mathsf{Obj}(\mathsf{AlgP_m})^{\times k}$.

Ad **(a)**: We define

$$\forall i \in [k], \forall j \in [m]: \mathsf{inc}_i^{(j)}: \begin{cases} \mathscr{A}_i^{(j)} \longrightarrow \tilde{\mathscr{A}}_i^{(j)} \\ a \longmapsto (\iota_{\mathscr{A}_i^{(j)}, \mathscr{A}_i})(a), \end{cases} \tag{IV}$$

where $\iota_{\mathscr{A}_i^{(j)}, \mathscr{A}_i}: \mathscr{A}_i^{(j)} \hookrightarrow \tilde{\mathscr{A}}_i^{(j)}$ denotes the canonical homomorphic insertion map for each $i \in [k]$, $j \in [m]$. By this, we define

$$\forall i \in [k]: \mathsf{inc}_i: \begin{cases} \mathscr{A}_i \cong \bigsqcup_{j=1}^{m} \mathscr{A}_i^{(j)} \longrightarrow \bigsqcup_{j=1}^{m} \tilde{\mathscr{A}}_i^{(j)} \\ a \longmapsto \left(\coprod_{j=1}^{m} \mathsf{inc}_i^{(j)}\right)(a). \end{cases} \tag{V}$$

We claim that the map inc_i is an injective algebra homomorphism with the property

$$\forall i \in [k]: \tilde{\varphi}_i \circ \mathsf{inc}_i = \varphi_i.$$

Since the maps $\mathsf{inc}_i^{(j)}$ from equation (IV) are injective, we obtain that the map from equation (I) is also injective. Furthermore, since each map inc_i is an algebra homomorphism, the map from equation (I) is an algebra homomorphism, too. We have

$$\forall i \in [k]: \mathsf{inc}_i \in \mathsf{Morph}_{\mathsf{Alg_m}}\left((\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}), (\tilde{\mathscr{A}}_i, (\tilde{\mathscr{A}}_i^{(j)})_{j\in[m]})\right),$$

because $\forall i \in [k]: \mathsf{inc}_i(\mathscr{A}_i^{(j)}) \subseteq \tilde{\mathscr{A}}_i^{(j)}$ which is shown by the universal mapping property of $\sqcup$ in the category $\mathsf{Alg_m}$. Now, we observe that

$$\tilde{\varphi}_i \circ \mathsf{inc}_i$$

$$= \left(\varphi_i \circ \left(\bigsqcup_{\ell=1}^{m} \nu_i^{(\ell)}\right)\right) \circ \left(\coprod_{j=1}^{m} \mathsf{inc}_i^{(j)}\right) \quad [\![\text{ def. of } \tilde{\varphi}_i \text{ in eq. (III), def. of } \mathsf{inc}_i \text{ in eq. (V) }]\!]$$

$$= \varphi_i \circ \bigsqcup_{j=1}^{m} \iota_{\mathscr{A}_i^{(j)},\mathscr{A}_i} \qquad [\![\text{ universal property of } \sqcup, \text{ def. of } \nu_i^{(\ell)}, \mathsf{inc}_i^{(j)}]\!]$$

$$= \varphi_i \circ \mathsf{id}_{\mathscr{A}_i} \qquad [\![\mathscr{A}_i \cong \textstyle\bigsqcup_{i=1}^{m} \mathscr{A}_i^{(j)}]\!],$$

This finishes the proof for the statements of **(a)**.
AD **(b)**: This statement follows immediately from the definition of $\tilde{\varphi}_i$ in equation (III).
AD **(c)**: We define involutions by

$$\forall\, i \in [k],\, \forall\, j \in [m] \colon \tilde{*}_i^{(j)} \colon \begin{cases} \tilde{\mathscr{A}}_i^{(j)} \longrightarrow \tilde{\mathscr{A}}_i^{(j)} \\ a \longmapsto a^{*_i}. \end{cases}$$

Once again the maps $\tilde{*}_i^j$ are just copies of the maps $*_i$ acting on the regarding copy of $\mathscr{A}_i$ labeled by j. By this, we define for each $n \in \mathbb{N}$, $(\varepsilon_i)_{i\in[n]} \in \mathbb{A}([k])$ and $(a_i)_{i\in[n]} \in \prod_{i=\varepsilon_1}^{\varepsilon_n} \mathscr{A}_i$ an involution on generators

$$\forall\, i \in [k] \colon \tilde{*}_i \colon \begin{cases} \tilde{\mathscr{A}}_i \longrightarrow \tilde{\mathscr{A}}_i \\ \tilde{\iota}_{\varepsilon_1}(a_1) \cdot \dots \cdot \tilde{\iota}_{\varepsilon_n}(a_n) \longmapsto \tilde{\iota}_{\varepsilon_n}\Big(\big(\tilde{*}_i^{(\varepsilon_n)}\big)(a_n)\Big) \cdot \dots \cdot \tilde{\iota}_{\varepsilon_1}\Big(\big(\tilde{*}_i^{(\varepsilon_1)}\big)(a_1)\Big). \end{cases} \tag{VI}$$

Herein $\tilde{\iota}_i \colon \tilde{\mathscr{A}}_i^{(j)} \hookrightarrow \tilde{\mathscr{A}}_i$ denotes the canonical homomorphic insertion map for each $i \in [k]$. We can see by the above definitions made that the stated properties of **(c)** are fulfilled.
AD **(d)**: We claim that if we set

$$\forall\, i \in [k],\, \forall\, r, s \in [m],\, r \neq s \colon \mathsf{iso}_i^{(r),(s)} \colon \begin{cases} \tilde{\mathscr{A}}_i^{(r)} \longrightarrow \tilde{\mathscr{A}}_i^{(s)} \\ a \longmapsto a, \end{cases}$$

then the stated equations are fulfilled. Equation (5.2.42) is obviously satisfied since $\mathsf{iso}_i^{(r),(s)}$ just acts like the identity map but embeds elements differently. For the proof of equation (5.2.43) we let $a \in \tilde{\mathscr{A}}_i^{(r)}$ and calculate

$$\big(\tilde{\varphi}_i \circ \iota_{\tilde{\mathscr{A}}_i^{(s)},\tilde{\mathscr{A}}_i} \circ \mathsf{iso}_i^{(r),(s)}\big)(a)$$

$$= \big(\tilde{\varphi}_i \circ \iota_{\tilde{\mathscr{A}}_i^{(s)},\tilde{\mathscr{A}}_i}\big)(\underbrace{a}_{\in \tilde{\mathscr{A}}_i^{(s)}}) \qquad [\![\text{ def. of } \mathsf{iso}_i^{(r),(s)}]\!]$$

$$= \left(\varphi_i \circ \left(\bigsqcup_{j=1}^{m} \nu_i^{(j)}\right) \circ \iota_{\tilde{\mathscr{A}}_i^{(s)},\tilde{\mathscr{A}}_i}\right)(a) \qquad [\![\text{ def. of } \tilde{\varphi}_i]\!]$$

$$= (\varphi_i \circ \nu_i^{(s)})(a) \qquad [\![\text{ universal mapping property of } \sqcup]\!]$$

$$= (\varphi_i)(a) \qquad [\![\nu_i^{(s)} \text{ acts as identity map}]\!]$$

$$= (\varphi_i \circ \nu_i^{(r)})(\underbrace{a}_{\in \tilde{\mathscr{A}}_i^{(r)}}) \qquad [\![\nu_i^{(r)} \text{ acts as identity map}]\!]$$

$$= \left(\varphi_i \circ \left(\bigsqcup_{j=1}^{m} \nu_i^{(j)}\right) \circ \iota_{\tilde{\mathscr{A}}_i^{(s)},\tilde{\mathscr{A}}_i}\right)(a) \quad [\![\text{ universal mapping property of } \sqcup]\!]$$

$$= \left(\tilde{\varphi}_i \circ \iota_{\tilde{\mathscr{A}}_i^{(r)},\tilde{\mathscr{A}}_i}\right)(a) \qquad [\![\text{ def. of } \tilde{\varphi}_i]\!].$$

A similar calculation holds for equation (5.2.44) and (5.2.45), where it is important to use $\tilde{\varphi}_i = \varphi_i \circ \left(\bigsqcup_{j\in[m]} \nu_i^{(j)}\right)$. This concludes the proof of **(d)**.

Ad **(e)**: For the implication to hold, we need to show that $\forall\, i \in [k]$ that $\tilde{\varphi}_i \colon \tilde{\mathscr{A}}_i \longrightarrow \mathbb{C}$ is strongly positive with respect to the involution $\tilde{*}_i$ defined in equation (VI) on the algebra $\tilde{\mathscr{A}}_i$ defined in equation (I). By Definition 1.1.27 this is equivalent to

$$\forall\, i \in [k]\ \forall\, a \in (\tilde{\mathscr{A}}_i)^{\mathbb{1}} : (\tilde{\varphi}_i)^{\mathbb{1}}\left(a^{(\tilde{*}_i)^{\mathbb{1}}} \cdot a\right) \geq 0, \tag{VII}$$

where $(\tilde{*}_i)^{\mathbb{1}} : (\tilde{\mathscr{A}}_i)^{\mathbb{1}} \longrightarrow (\tilde{\mathscr{A}}_i)^{\mathbb{1}}$ is the canonical unital extension of $\tilde{*}_i \colon \tilde{\mathscr{A}}_i \longrightarrow \tilde{\mathscr{A}}_i$. We fix an element $i \in [k]$. Let us denote the canonical unital extension of $\nu_i^{(j)}$, defined in equation (II), by

$$(\nu_i^j)^{\mathbb{1}} : (\tilde{\mathscr{A}}_i^{(j)})^{\mathbb{1}} \longrightarrow (\mathscr{A}_i)^{\mathbb{1}}.$$

From the isomorphism stated in equation (2.1.9) we obtain

$$(\tilde{\varphi}_i)^{\mathbb{1}} = (\varphi_i)^{\mathbb{1}} \circ \bigsqcup_{j=1}^{m}{}_{\mathbb{1}} (\nu_i^{(j)})^{\mathbb{1}} \tag{VIII}$$

up to isomorphism holds. By definition of the involved involutions, we can easily see that for each $j \in [m]$ the map $\nu_i^{(j)}$ is a homomorphism of algebras with involution. We have

$$(\tilde{\mathscr{A}}_i)^{\mathbb{1}} \cong \bigsqcup_{j=1}^{m}{}_{\mathbb{1}} (\tilde{\mathscr{A}}_i^{(j)})^{\mathbb{1}}.$$

from Lemma 2.1.5. In other words, we know that $(\tilde{\mathscr{A}}_i)^{\mathbb{1}}$ is generated by $\bigcup_{i=1}^{m} (\tilde{\mathscr{A}}_i^{(j)})^{\mathbb{1}}$. So, any arbitrary element $a \in (\tilde{\mathscr{A}}_i)^{\mathbb{1}}$ can be generated by elements from $\bigcup_{i=1}^{m} (\tilde{\mathscr{A}}_i^{(j)})^{\mathbb{1}}$. If we apply equation (VIII) to an element $a^{(\tilde{*}_i)^{\mathbb{1}}} a$, use the universal mapping property of the free product of unital algebras and that each map $\nu_i^{(j)}$ is a homomorphism of algebras with involution, then the claim of equation (VII) follows since φ_i has been assumed to be strongly positive, in other words $(\varphi_i)^{\mathbb{1}}$ is positive. □

Let $m \in \mathbb{N}$. Like we did in the single-colored case, we now use all the nonzero highest coefficients for assigning to each m-faced universal product $\odot$ with the right-ordered monomials property a set of m-colored partitions, i. e., we set

$$\mathfrak{P}_{\odot} := \left\{ \pi \in \bigcup_{\varepsilon \in T^{(m)}} \mathsf{Part}_{\mathsf{col}(\varepsilon)} \;\middle|\; \alpha_\pi \neq 0 \right\}. \tag{5.2.46}$$

5.2.17 Theorem. Let $\odot$ be a partition induced universal product or a positive and symmetric u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. We set

$\forall n \in \mathbb{N} \setminus \{1\}$, $\forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$, $\forall \ell \in \{0, \ldots, n-2\} \subseteq \mathbb{N}$:

$$(\mathfrak{P}_\odot)_\sigma^{(\ell+1)\wedge(\ell+2)} := \mathfrak{P}_\odot \cap \mathsf{Part}_{\sigma,k}^{(\ell+1)\wedge(\ell+2)}, \tag{5.2.47}$$

$$\forall n \in \mathbb{N},\ \forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n} : (\mathfrak{P}_\odot)_\sigma := \mathfrak{P}_\odot \cap \mathsf{Part}_\sigma\,. \tag{5.2.48}$$

Then,

(a) $\forall n \in \mathbb{N}$, $\forall \sigma = (\sigma_i)_{i\in[n]} \in \mathbb{A}(\{1\} \times [m])$:

$$\alpha_{\mathsf{indPart}(\sigma)} = 1, \tag{5.2.49}$$

(b) $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$, $\forall \ell \in \{0, \ldots, n-2\} \subseteq \mathbb{N}$, $\forall \pi \in (\mathfrak{P}_\odot)_\sigma^{(\ell+1)\wedge(\ell+2)}$:

$$\sigma_{\ell+1} = \sigma_{\ell+2} \implies \alpha_\pi = \alpha_{\mathsf{delete}_{\sigma,\ell+2}(\pi)}, \tag{5.2.50}$$

(c) $\forall n \in \mathbb{N}$, $\forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$, $\forall \ell \in \{0, \ldots, n-1\} \subseteq \mathbb{N}$, $\forall \pi \in (\mathfrak{P}_\odot)_\sigma$:

$$\alpha_\pi = \alpha_{\mathsf{double}_{\sigma,\ell+1}(\pi)}, \tag{5.2.51}$$

(d) $\forall n \in \mathbb{N}$, $\forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$, $\forall \pi \in (\mathfrak{P}_\odot)_\sigma$:

$$(\alpha_\pi)^* = \alpha_{\mathsf{mirror}(\pi)}, \tag{5.2.52}$$

(e) $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n}$, $\forall \delta, \delta' \in [m]$, $\forall \pi \in (\mathfrak{P}_\odot)_\sigma$:

$$\alpha_\pi = \alpha_{\mathsf{cCol}_{\sigma,(\delta,\delta')}(\pi)}. \tag{5.2.53}$$

Moreover, the induced set of partitions $\mathfrak{P}_\odot$ is an m-colored universal class of partitions.

PROOF: Let us first assume that $\odot$ is a partition induced universal product. Equation (5.2.34) states that $\mathfrak{P}_\odot = \mathcal{P}$ and that all nonzero highest coefficients belonging to $\odot$ can only have the value 1. Equations (5.2.49)–(5.2.52) are then only restatements of the axioms which a universal class of partitions needs to satisfy. But $\odot$ is assumed to be a partition induced universal product and therefore there exists a universal class of partitions $\mathcal{P}$ and the claim follows.

For the rest of the proof we assume that $\odot$ is an m-faced positive and symmetric u.a.u.-product.

AD **(a)**: Assume that $n \in \mathbb{N}$ and let $((\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[m]}, \varphi_i))_{i\in\{1,2\}} \in (\mathsf{Obj}(\mathsf{AlgP_m}))^{\times 2}$. Then, we have homomorphic insertion maps $\mathcal{A}_i^{(j)} \xhookrightarrow{\iota_i^{(j)}} \mathcal{A}_i \xhookrightarrow{\iota_i} \mathcal{A}_1 \sqcup \mathcal{A}_2$. Let $\sigma = (\sigma_i)_{i\in[n]} \in \mathbb{A}(\{1\} \times [m])$. Then, we can calculate for $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathcal{A}_1^{(\mathsf{col}(\sigma_i))}$

$$\begin{aligned}
&(\varphi_1 \odot \varphi_2)\big((\iota_1 \circ \iota_1^{(\sigma_{1,2})})(a_1) \cdot \cdots \cdot (\iota_1 \circ \iota_1^{(\sigma_{n,2})})(a_n)\big) \\
&\quad = (\varphi_1 \odot \varphi_2)\Big(\iota_1\big(\iota_1^{(\sigma_{1,2})}(a_1) \cdot \cdots \cdot \iota_1^{(\sigma_{n,2})}(a_n)\big)\Big) \quad [\![\ \iota_1 \text{ is homomorphism of algebras}\ ]\!] \\
&\quad = \underbrace{1}_{=\alpha_{\max}^{(\varepsilon)}\ [\![\text{ coefficient lemma }]\!]} \cdot \varphi_1\big(\iota_{\sigma_{1,2},1}(a_1) \cdot \cdots \cdot \iota_{\sigma_{n,2},1}(a_n)\big) \quad [\![\ \odot \text{ is unital}\ ]\!]
\end{aligned}$$

If we choose $(a_i)_{i\in[n]}$ such that the right-hand side of the above calculation is $\neq 0$, then we have shown that $\alpha_{\max}^{(\varepsilon)} = \alpha_{\mathsf{indPart}(\varepsilon)} = 1$.

AD **(b)**: To show that equation (5.2.50) holds is similar to the proof of equation (5.1.38) and therefore is omitted.

AD **(c)**: The proof of equation (5.2.51) is a bit different than in the case $m = 1$, i. e., for equation (5.1.39). Let us fix $\varepsilon = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in ([k] \times [m])^{\times n} \cap T^{(m)}$ such that $\mathsf{indPart}(\varepsilon) \in \mathfrak{P}_{\odot}$, i.e. $\alpha^{(\varepsilon)}_{\max} \neq 0$. Furthermore, let us assume that $\varepsilon = \operatorname{red}\varepsilon$. If the tuple ε is not reduced, then due to Lemma 5.2.2 the proof in this case is similar to the following one. According to Lemma 5.2.3 **(b)** we can find m-faced algebras and linear functionals $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$ and $(a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}^{(\varepsilon_{i,2})}_{\varepsilon_{i,1}}$ such that

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \alpha^{(\varepsilon)}_{\max} \underbrace{\prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)}_{=1} \neq 0. \tag{I}$$

Let us denote the canonical homomorphic embedding by $\iota_i^{(j)} : \mathscr{A}_i^{(j)} \hookrightarrow (\mathscr{A}_i^{(j)})^{\mathbb{1}}$ and the unit of $(\mathscr{A}_i^{(j)})^{\mathbb{1}}$ by $\mathbb{1}_i^{(j)}$. For each $i \in [k]$ we set

$$\tilde{\varphi}_i := \varphi_i^{\mathbb{1}} \circ \mathsf{can}_i \circ \mathsf{pr}_i, \tag{II}$$

wherein we denote natural homomorphic projection maps (differently defined to equation (4.1.1))

$$\mathsf{pr}_i : \coprod_{j=1}^{m} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \longrightarrow \textstyle\coprod_{j=1}^{m} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \Big/ \langle j \in [m-1] : \mathbb{1}_i^{j} - \mathbb{1}_i^{j+1} \rangle,$$

and canonical isomorphisms (Remark 2.1.4 **(a)**, Lemma 2.1.5)

$$\mathsf{can}_i : \textstyle\coprod_{j=1}^{m} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \Big/ \langle j \in [m-1] : \mathbb{1}_i^{j} - \mathbb{1}_i^{j+1} \rangle \longrightarrow \displaystyle\coprod_{j=1}^{m}{}_{\mathbb{1}} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \longrightarrow (\mathscr{A}_i)^{\mathbb{1}}.$$

If we set

$$\forall\, i \in [m] : \tilde{\mathscr{A}}_i := \coprod_{j=1}^{m} (\mathscr{A}_i^{(j)})^{\mathbb{1}},$$

then we can see that

$$\forall\, i \in [k] : \Big(\tilde{\mathscr{A}}_i, \big((\mathscr{A}_i^{(j)})^{\mathbb{1}}\big)_{j\in[m]}, \tilde{\varphi}_i\Big) \in \mathsf{AlgP_m}\,.$$

Using the universality of $\odot$ (Remark 2.1.10) we obtain

$$\begin{aligned}(\tilde{\varphi}_1 \odot \cdots \odot \tilde{\varphi}_k) \circ \coprod_{i=1}^{k} \coprod_{j=1}^{m} \iota_i^{(j)} &= \Big(\big(\tilde{\varphi}_1 \circ \coprod_{j=1}^{m} \iota_1^{(j)}\big) \odot \cdots \odot \big(\tilde{\varphi}_k \circ \coprod_{j=1}^{m} \iota_k^{(j)}\big)\Big) \\ &= \varphi_1 \odot \cdots \odot \varphi_k \quad [\![\text{ Lem. } 2.1.5]\!].\end{aligned} \tag{III}$$

Now, we can calculate

$$\begin{aligned}&\alpha^{(\varepsilon)}_{\max} \prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \\ &\quad = \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) \quad [\![\text{ eq. (I) }]\!]\end{aligned}$$

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ eq. (5.1.4) }]\!]$$

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big(\big(t_1\tilde{\varphi}_1 \odot \cdots \odot t_k\tilde{\varphi}_k\big)\big(\iota^{(\varepsilon_{1,2})}_{\varepsilon_{1,1}}(a_1) \cdot \cdots \cdot \iota^{(\varepsilon_{n,2})}_{\varepsilon_{n,1}}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ eq. (III) }]\!]$$

$$= \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big(\big(t_1\tilde{\varphi}_1 \odot \cdots \odot t_k\tilde{\varphi}_k\big)$$

$$\big(\iota^{(\varepsilon_{1,2})}_{\varepsilon_{1,1}}(a_1) \cdot \cdots \cdot \iota^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}}(a_{\ell+1}) \cdot \iota^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}}(\mathbb{1}^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}}) \cdot \cdots \cdot \iota^{(\varepsilon_{n,2})}_{\varepsilon_{n,1}}(a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

$[\![(\mathscr{A}^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}})^{\mathbb{1}}$ is unital algebra $]\!]$

$$= \big(\tilde{\varphi}_1 \boxdot \cdots \boxdot \tilde{\varphi}_k\big)\big(\iota^{(\varepsilon_{1,2})}_{\varepsilon_{1,1}}(a_1) \cdot \cdots \cdot \iota^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}}(a_{\ell+1}) \cdot \iota^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}}(\mathbb{1}^{(\varepsilon_{\ell+1,2})}_{\varepsilon_{\ell+1,1}}) \cdot \cdots \cdot \iota^{(\varepsilon_{n,2})}_{\varepsilon_{n,1}}(a_n)\big)$$

$$= \alpha^{(\mathrm{red}\,\tilde{\varepsilon})}_{\max} \prod_{i=1}^{k} \tilde{\varphi}_i\Big(j\big(\iota^{(\varepsilon_{1,2})}_{\varepsilon_{1,1}}(a_1), \ldots, \iota^{(\varepsilon_{n,2})}_{\varepsilon_{n,1}}(a_n)\big)(\mathbb{1}_i)\Big)$$

$$\left[\!\!\left[\begin{array}{l}\tilde{\varepsilon} := \big((\varepsilon_{1,1}, \varepsilon_{1,2}), \ldots, (\varepsilon_{\ell+1,1}, \varepsilon_{\ell+1,2}), (\varepsilon_{\ell+1,1}, \varepsilon_{\ell+1,2}), \ldots, (\varepsilon_{n,1}, \varepsilon_{n,2})\big) \\ \in ([k] \times [m])^{\times(n+1)} \cap T^{(m)}, \text{ Lem. 5.2.3 } \textbf{(a)}\end{array}\right]\!\!\right]$$

$$= \alpha^{(\mathrm{red}\,\tilde{\varepsilon})}_{\max} \prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \quad [\![\text{ eq. (II), } (\varphi_i)^{\mathbb{1}} \text{ is unital extension }]\!].$$

From the above calculation we obtain $\alpha_{\mathsf{indPart}(\tilde{\varepsilon})} = \alpha_\pi$ and $\mathsf{double}_{\mathsf{col}(\varepsilon),\ell+1}(\pi) = \mathsf{indPart}(\tilde{\varepsilon})$.

AD **(d)**: Assume $n \in \mathbb{N}$ and $k \in [n]$. Now, let us fix $\varepsilon = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in ([k]\times[m])^{\times n} \cap T^{(m)}$ such that $\mathsf{indPart}(\varepsilon) \in \mathfrak{P}_\odot$, i. e., $\alpha^{(\varepsilon)}_{\max} \neq 0$. Without loss of generality we can assume that ε is reduced. If the tuple ε is not reduced, then the proof in this case is similar to the following one due to Lemma 5.2.2. According to Lemma 5.2.3 **(b)** we can find m-faced algebras and linear functionals $(\mathscr{A}_i, (\mathscr{A}^{(j)}_i)_{j\in[m]}, \varphi_i)_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times k}$ and $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}^{(\varepsilon_{i,2})}_{\varepsilon_{i,1}}$ such that

$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \alpha^{(\varepsilon)}_{\max} \underbrace{\prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)}_{=1} \neq 0. \tag{IV}$$

By Lemma 5.2.3 **(b)**, we can also assume that each algebra $\mathscr{A}_i$ is a $*_i$-algebra such that $j \in [m]$ $\mathscr{A}^{(j)}_i$ is a $*_i$-subalgebra. Furthermore, we can assume that the linear functionals $(\varphi_i)_{i\in[k]} \in \prod_{i\in[k]} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ are strongly positive and not equal to the zero map. Now, we can calculate

$$\Big(\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n)\Big)^*$$

$$= \left(\frac{\partial^n}{\partial t_1 \dots \partial t_n}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n)\Big)\Big|_{t_1=\ldots=t_k=0}\right)^* \quad [\![\text{ def. of } T_{k\boxdot} \text{ in eq. (5.1.4) }]\!]$$

$$= \left(\frac{\partial^n}{\partial t_1 \dots \partial t_n}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)^{\mathbb{1}}\underbrace{(a_1 \cdot \cdots \cdot a_n)}_{\in (\bigsqcup_{i\in[k]} \mathscr{A}_i)^{\mathbb{1}}}\Big)\Big|_{t_1=\ldots=t_k=0}\right)^*$$

⟦ unique unital extension ⟧

$$= \frac{\partial^n}{\partial t_1 \ldots \partial t_n}\Big(\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)^{\mathbb{1}}\Big)^*(a_1 \cdot \cdots \cdot a_n)\Big)\Big|_{t_1=\ldots=t_k=0}$$

⟦ derivative and complex conjugation are linear ⟧

$$= \frac{\partial^n}{\partial t_1 \ldots \partial t_n}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)^{\mathbb{1}}\big((a_1 \cdot \cdots \cdot a_n)^*\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

⟦ positivity of $\odot$ implies by Def. 2.1.15 that $\bigodot_{i\in[k]} \varphi_i$ is strongly positive on $\bigsqcup_{i\in[k]} \mathscr{A}_i$, $\big(\bigodot_{i\in[k]} \varphi_i\big)^{\mathbb{1}}$ is hermitian by Lem. 1.1.26 ⟧

$$= \frac{\partial^n}{\partial t_1 \ldots \partial t_n}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)\big(\underbrace{(a_1 \cdot \cdots \cdot a_n)^*}_{\in \bigsqcup_{i\in[k]} \mathscr{A}_i}\big)\Big)\Big|_{t_1=\ldots=t_k=0}$$

⟦ unique unital extension ⟧

$$= \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)\big(\underbrace{(a_n)^{*_{\varepsilon_n}}}_{\in \mathscr{A}^{(\mathsf{col}(\varepsilon_n))}_{\mathsf{type}(\varepsilon_n)}} \cdot \cdots \cdot \underbrace{(a_1)^{*_{\varepsilon_1}}}_{\in \mathscr{A}^{(\mathsf{col}(\varepsilon_1))}_{\mathsf{type}(\varepsilon_1)}}\big) \quad \text{⟦ eq. (I), } \mathscr{A}^{(j)}_i \text{ is } *_i\text{-subalgebra ⟧}$$

$$= \alpha^{(\varepsilon')}_{\max} \prod_{i=1}^{k} \varphi_i\big(j(a_n^*, \ldots, a_1^*)(\mathbb{1}_i)\big) \quad \text{⟦ } \varepsilon' = (\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_1)\text{, Lem. 5.2.3 } \textbf{(a)} \text{ ⟧}$$

$$= \alpha^{(\varepsilon')}_{\max} \prod_{i=1}^{k} \Big(\varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)\Big)^*$$

⟦ each φ_i is strongly positive on $\mathscr{A}_i$, also hermitian by Remark 1.1.28 ⟧

$$= \alpha^{(\varepsilon')}_{\max} \Big(\prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)\Big)^*$$

If we take the complex conjugate of equation (IV) and compare this with the above calculation, then we obtain $(\alpha^{(\varepsilon)}_{\max})^* = (\alpha_{\mathsf{indPart}(\varepsilon)})^* = \alpha^{(\varepsilon')}_{\max}$ and $\mathsf{mirror}(\mathsf{indPart}(\varepsilon)) = \varepsilon'$.
Ad **(e)**: We claim that it suffices to show that
$\forall n \in \mathbb{N} \setminus \{1\},\ \forall \sigma = (\sigma_i)_{i\in[n]} \in [m]^{\times n},\ \forall \delta \in [m],\ \forall \pi \in (\mathfrak{P}_\odot)_\sigma$:

$$\alpha_\pi = \alpha_{\mathsf{cCol}_{\sigma,(\delta,\sigma_n)}(\pi)}, \tag{V}$$

i. e., we can change the color of the first leg in the partition π. If we want to change the color of the last leg in the partition π, then we can apply mirror to π, i. e., $(\alpha_\pi)^* = \alpha_{\mathsf{mirror}(\pi)}$, change the color of the first leg of $\mathsf{mirror}(\pi)$, i. e., use equation (V) and apply the map mirror once more. Now let us show equation (V).

Assume $n \in \mathbb{N}\setminus\{1\}$ and $k \in [n]$. Now, let us fix $\varepsilon = ((\varepsilon_{i,1}, \varepsilon_{i,2}))_{i\in[n]} \in ([k]\times[m])^{\times n} \cap T^{(m)}$ such that $\mathsf{indPart}(\varepsilon) \in \mathfrak{P}_\odot$, i. e., $\alpha^{(\varepsilon)}_{\max} \neq 0$. Without loss of generality we can assume that ε is reduced. If the tuple ε is not reduced, then the proof in this case is similar to the following one due to Lemma 5.2.2. As discussed in the proof of **(d)** we can find $(\mathscr{A}_i, (\mathscr{A}^{(j)}_i)_{j\in[m]})_{i\in[k]} \in \big(\mathsf{Obj}(\mathsf{Alg_m})\big)^{\times k}$, where each algebra $\mathscr{A}_i$ is a $*_i$-algebra and $\mathscr{A}_i^{(j)}$ is a $*_i$-subalgebra for all $j \in [m]$. Furthermore we can find linear functionals $(\varphi_i)_{i\in[k]} \in \prod_{i\in[k]} \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ which are strongly positive and not equal to the zero map. Then, we can find $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$ such that

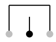
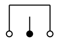
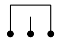
$$\big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) = \alpha_{\max}^{(\varepsilon)} \underbrace{\prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big)}_{=1} \neq 0. \tag{VI}$$

From Lemma 5.2.16 we obtain the existence of an k-tuple $\big(\tilde{\mathscr{A}}_i, (\tilde{\mathscr{A}}_i^{(j)})_{j\in[m]}, \tilde{\varphi}_i\big) \in \mathsf{Obj}(\mathsf{AlgP_m})^{\times k}$, which satisfies the prerequisites of Lemma 5.2.15. Therefore, we can calculate

$$\begin{aligned}
&\alpha_{\max}^{(\varepsilon)} \prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \\
&= \big(\varphi_1 \boxdot \cdots \boxdot \varphi_k\big)(a_1 \cdot \cdots \cdot a_n) \quad [\![\text{ eq. (VI) }]\!] \\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\big)(a_1 \cdot \cdots \cdot a_n)\Big)\Big|_{t_1=\ldots=t_k=0} \quad [\![\text{ eq. (5.1.4) }]\!] \\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\tilde{\varphi}_1) \odot \cdots \odot (t_k\tilde{\varphi}_k)\big)\big(\coprod_{i=1}^{k} \mathsf{inc}_i(a_1 \cdot \cdots \cdot a_n)\big)\Big)\Big|_{t_1=\ldots=t_k=0} \\
&\quad [\![\text{ Lem. 5.2.16 } \textbf{(a)} \text{ \& } \textbf{(b)}, \mathsf{inc}_i \in \mathsf{Morph}_{\mathsf{Alg_m}}(\mathscr{A}_i, \tilde{\mathscr{A}}_i) \text{ universality of } \odot]\!] \\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big(\big((t_1\tilde{\varphi}_1) \odot \cdots \odot (t_k\tilde{\varphi}_k)\big)\big(\underbrace{\mathsf{inc}_{\mathsf{type}(\varepsilon_1)}(a_1)}_{=:\tilde{a}_1 \in \tilde{\mathscr{A}}_{\mathsf{type}(\varepsilon_1)}^{(\mathsf{col}(\varepsilon_1))}} \cdot \cdots \cdot \underbrace{\mathsf{inc}_{\mathsf{type}(\varepsilon_n)}(a_n)}_{=:\tilde{a}_n \in \tilde{\mathscr{A}}_{\mathsf{type}(\varepsilon_n)}^{(\mathsf{col}(\varepsilon_n))}}\big)\Big)\Big|_{t_1=\ldots=t_k=0} \\
&\quad [\![\; \mathsf{inc}_i \text{ is homomorphism of algebras }]\!] \\
&= \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Bigg(\big((t_1\tilde{\varphi}_1) \odot \cdots \odot (t_k\tilde{\varphi}_k)\big)\Big(\big(\mathsf{iso}_{\mathsf{type}(\varepsilon_1)}^{(\mathsf{col}(\varepsilon_1)),(\delta)}(\tilde{a}_1) \cdot \tilde{a}_2 \cdot \cdots \cdot \tilde{a}_n\big)\Big)\Bigg)\Bigg|_{t_1=\ldots=t_k=0} \\
&\quad [\![\text{ eq. (5.2.38) may be applied to } \textstyle\bigodot_{i\in[k]} t_i\tilde{\varphi}_i \text{ by Lem. 5.2.16 }]\!] \\
&= \big(\tilde{\varphi}_1 \boxdot \cdots \boxdot \tilde{\varphi}_k\big)\Big(\big(\mathsf{iso}_{\mathsf{type}(\varepsilon_1)}^{(\mathsf{col}(\varepsilon_1)),(\delta)}(\tilde{a}_1) \cdot \tilde{a}_2 \cdot \cdots \cdot \tilde{a}_n\big)\Big) \quad [\![\text{ eq. (5.1.4) }]\!] \\
&= \alpha_{\max}^{(\mathrm{red}\,\tilde{\varepsilon})} \prod_{i=1}^{k} \tilde{\varphi}_i\Big(j\big(\mathsf{iso}_{\mathsf{type}(\varepsilon_1)}^{(\mathsf{col}(\varepsilon_1)),(\delta)}(\tilde{a}_1), \ldots, \tilde{a}_n\big)(\mathbb{1}_i)\Big) \\
&\quad [\![\; \tilde{\varepsilon} = \big((\mathsf{type}(\varepsilon_1), \delta), \varepsilon_2, \ldots, \varepsilon_n\big) \in ([k] \times [m])^{\times n} \cap T^{(m)}, \text{ Lem. 5.2.3 } \textbf{(a)} \;]\!] \\
&= \alpha_{\max}^{(\mathrm{red}\,\tilde{\varepsilon})} \prod_{i=1}^{k} \varphi_i\big(j(a_1, \ldots, a_n)(\mathbb{1}_i)\big) \quad [\![\text{ def. of } \tilde{a}_i, \text{ eq. (5.2.45), eq. (5.2.40) }]\!]
\end{aligned}$$

From the above calculation we obtain $\alpha_\pi = \alpha_{\max}^{(\varepsilon)} = \alpha_{\max}^{(\mathrm{red}\,\tilde{\varepsilon})}$ and $\mathsf{cCol}_{\mathsf{type}(\varepsilon),(\delta,\mathsf{col}(\varepsilon_n))}(\mathsf{indPart}(\varepsilon)) = \mathsf{indPart}(\tilde{\varepsilon})$.

The statement that $\mathfrak{P}_\odot$ respects the axioms of an m-colored universal class of partitions, i. e., the defining properties of Definition 3.4.9 are satisfied now follows from the equations (5.2.49) – (5.2.52) and Corollary 5.2.11. □

5.2.18 Remark. The class of bi-interval partitions defined by Gu and Skoufranis in [GS19, Def. 3.5] is not a two-colored universal class of partitions. The partition  is only a bi-interval partition for certain color combinations of the first and last leg. For instance, the partition  can be considered a bi-interval partition, while the partition  fails to be bi-interval. The corresponding bi-boolean product is a symmetric u.a.u.-product which is not positive as discussed in [GHS20]. This implies one cannot drop the assumption of positivity in the multi-faced setting and is in contrast to the single-faced case (Theorem 5.1.15), where we did not demand positivity of the universal product. Furthermore, we can see that if one wants to classify all the symmetric two-faced u.a.u.-products, the two-colored universal classes of partitions are not enough.

Now, we are ready to prove the version of Proposition 5.1.16 for the m-colored case.

5.2.19 Proposition. Let $\odot$ be a partition induced universal product or a positive and symmetric u.a.u.-product in the category $\mathsf{AlgP_m}$ for some $m \in \mathbb{N}$. Let $k \in \mathbb{N} \setminus \{1\}$, $n \in \mathbb{N} \setminus [k-1]$. If $\pi \in (\mathfrak{P}_\odot)_\varepsilon$ with $|\pi| = k$ for some $\varepsilon \in [m]^{\times n}$ and $\alpha_\pi \neq 0$, then there exists a $(k-1)$-tuple of two-block partitions $(\delta_i)_{i\in[k-1]} \in ((\mathfrak{P}_\odot)_{\cdot,2})^{\times(k-1)}$ such that

$$\alpha_\pi = \prod_{i=1}^{k-1} \alpha_{\delta_i}. \tag{5.2.54}$$

Proof: We prove this assertion by induction over $k \in \mathbb{N} \setminus \{1\}$. For the induction base $k = 2$, the statement follows, if we set $\delta_1 = \pi$ and by using $\pi \in \mathfrak{P}_\odot \implies \alpha_\pi \neq 0$.

For the induction step $k \longrightarrow k+1$, we assume that the assertion holds for $k \in \mathbb{N} \setminus \{1\}$ with $\pi \in (\mathfrak{P}_\odot)_{\varepsilon,k}$ for some $\varepsilon \in [m]^{\times n}$ with $n \in \mathbb{N} \setminus [k-1]$. We calculate

$$\begin{aligned}
\alpha_\pi &= \alpha_{\tilde{\pi}} \cdot \alpha_{\bar{\pi}} && [\![\, \mathfrak{P}_\odot \text{ is u.c.p. by Prop. 5.2.17, Lem. 4.2.3, eq. (5.2.30)} \,]\!] \\
&= \alpha_{\tilde{\pi}} \cdot \alpha_{\delta_k} && [\![\, \delta_k := \bar{\pi} \in (\mathfrak{P}_\odot)_{\cdot,2} \text{ since } \pi \in \mathfrak{P}_\odot \implies \alpha_\pi \neq 0 \,]\!] \\
&= \Big(\prod_{i=1}^{k-1} \alpha_{\delta_i}\Big) \cdot \alpha_{\delta_k} && [\![\, \text{induction hypothesis applied to } \tilde{\pi} \in \mathfrak{P}_\odot,\ |\pi| = k \,]\!].
\end{aligned}$$

The existence of the tuple $(\delta_i)_{i\in[k]}$ of two-block partitions in $\mathfrak{P}_\odot$ proves the statement of equation (5.1.40). □

5.2.20 Theorem. Let $m \in \mathbb{N}$. Assume that $\odot, \tilde{\odot}$ are partition induced universal products or positive and symmetric u.a.u.-products in $\mathsf{AlgP_m}$. If

$$\forall \pi \in \mathrm{red}(\mathsf{Part}_{\cdot,2})\colon\ \alpha_\pi = \tilde{\alpha}_\pi, \tag{5.2.55}$$

then $\odot = \tilde{\odot}$ as bifunctors.

Proof: The proof is formally the same as for Theorem 5.1.17 in the single-faced case. The starting point is Theorem 2.5.13 which basically tells that any u.a.u.-product $\odot$ with the right-ordered monomials property is uniquely determined by its highest coefficients. Thanks to Theorem 5.2.17 **(a)** and **(c)** we have $\alpha_\pi = 1$ for any 1-block partition $\pi \in \mathfrak{P}_\odot$. For any partition

$\pi \in \mathfrak{P}_{\odot}$ which has more than one block, the assertion follows from Proposition 5.2.19. □

The following statement is well-known.

5.2.21 Lemma. Let $\mathscr{A}$ be a $*$-algebra and $\varphi\colon \mathscr{A} \longrightarrow \mathbb{C}$ a positive linear functional. If we set $\forall n \in \mathbb{N}, \forall i \in [n], \forall x_i \in \mathscr{A}$:

$$A := (A_{ij})_{i,j\in[n]} := \big(\varphi(x_i^* x_j)\big)_{i,j\in[n]} \in M_n(\mathbb{C}), \tag{5.2.56}$$

then the complex matrix A is positive semidefinite.

Proof: Fix some $n \in \mathbb{N}$, an n-tuple $(x_i)_{i\in[n]} \in \mathscr{A}^{\times n}$ and $\mathbf{c} := (c_i)_{i\in[n]} \in \mathbb{C}^n$. Set $y := \sum_{i=1}^n c_i x_i$. Then, we have

$$0 \le \varphi(y^* y) \iff 0 \le \sum_{i=1}^n \sum_{j=1}^n \bar{c}_i \varphi(x_i^* x_j) c_j \iff 0 \le \langle \mathbf{c}^*, A\mathbf{c}\rangle.$$

Since $\mathbf{c} \in \mathbb{C}^{\times n}$ has been arbitrarily chosen, we have $\forall \mathbf{c} \in \mathbb{C}^{\times n}\colon \langle \mathbf{c}^*, A\mathbf{c}\rangle \ge 0$. The complex matrix A is hermitian because of equation (1.1.28). This shows that the hermitian matrix A is positive semidefinite. □

5.2.22 Proposition. Let $\odot$ be a positive and symmetric u.a.u.-product in the category $\mathsf{AlgP_m}$ for $m \in \mathbb{N}$. Then,

$$\forall (\sigma_1, \sigma_2) \in ([m])^{\times 2}\colon \alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))} = 1. \tag{5.2.57}$$

Proof: We shall apply Lemma 5.2.21. Denote by $\mathbb{C}\langle X\rangle$ the free associative algebra over the set X. For each $j \in [m]$ set $X^{(j)} := \{x_j, x_j^*\}$ with indeterminates x_j and x_j^* and $Y^{(j)} := \{y_j, y_j^*\}$ with indeterminates y_j and y_j^*. Then we set

$$\forall j \in [m]\colon \quad \begin{aligned} \mathscr{A}_1^{(j)} &= \mathbb{C}\langle X^{(j)}\rangle \\ \mathscr{A}_2^{(j)} &= \mathbb{C}\langle Y^{(j)}\rangle. \end{aligned} \tag{I}$$

The algebras $\mathscr{A}_i^{(j)}$ become $*$-algebras in the canonical way. Set $\mathscr{A}_i = \bigsqcup_{j\in[m]} \mathscr{A}_i^{(j)}$ for $i \in [2]$. Let $\varphi_i\colon \mathscr{A}_i \longrightarrow \mathbb{C}$ be a $*$-algebra homomorphisms such that $\varphi_1(x_j) = 1 = \varphi_1(x_j^*)$ for $j \in [m]$ and $\varphi_2(y_j) = 1 = \varphi_2(y_j^*)$ for $j \in [m]$. Then, $\varphi_1 \odot \varphi_2\colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathbb{C}$ is well defined and $\mathscr{A}_1 \sqcup \mathscr{A}_2 \cong \mathbb{C}\langle x_j, x_j^*, y_j, y_j^* \mid j \in [m]\rangle$. Now, let $(\sigma_1, \sigma_2) \in ([2])^{\times 2}$. Then, from the universal coefficient theorem follows

$$\begin{aligned} (\varphi_1 \odot \varphi_2)(x_{\sigma_1} y_{\sigma_2}) &= \alpha_{\mathsf{max}}^{((1,\sigma_1),(2,\sigma_2))} \varphi_1(x_{\sigma_1})\varphi_2(y_{\sigma_2}), \\ (\varphi_1 \odot \varphi_2)(y_{\sigma_2} x_{\sigma_1}) &= \alpha_{\mathsf{max}}^{((2,\sigma_2),(1,\sigma_1))} \varphi_1(x_{\sigma_1})\varphi_2(y_{\sigma_2}). \end{aligned}$$

Since φ_i is a homomorphism of algebras, it follows that φ_i is strongly positive. Moreover, we have assumed that $\odot$ is positive and therefore we have

$$\forall a, b, c \in \mathbb{C}\colon (\varphi_1 \odot \varphi_2)^{\mathbb{1}}\Big(\big(a\mathbb{1} + bx_{\sigma_1} + cy_{\sigma_2}\big)^* \underbrace{\big(a\mathbb{1} + bx_{\sigma_1} + cy_{\sigma_2}\big)}_{\in(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{\mathbb{1}}}\Big) \ge 0.$$

This is equivalent to $\forall a, b, c \in \mathbb{C}$:

$$0 \le (\varphi_1 \odot \varphi_2)^{\mathbb{1}}\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}^* \begin{pmatrix} \mathbb{1} & x_{\sigma_1} & y_{\sigma_2} \\ x_{\sigma_1}^* & x_{\sigma_1}^* x_{\sigma_1} & x_{\sigma_1}^* y_{\sigma_2} \\ y_{\sigma_2}^* & y_{\sigma_2}^* x_{\sigma_1} & y_{\sigma_2}^* y_{\sigma_2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)$$

$$= \begin{pmatrix} a \\ b \\ c \end{pmatrix}^* \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))} \\ 1 & \left(\alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))}\right)^* & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad [\![\text{ eq. (5.2.52) }]\!].$$

By Lemma 5.2.21 the above matrix is positive semidefinite which implies that its determinant is greater or equal than zero and therefore for the characteristic polynomial holds

$$\alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))} + \left(\alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))}\right)^* - \alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))}\left(\alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))}\right)^* - 1 \ge 0. \quad \text{(II)}$$

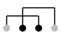
In the proof of [Voß13, Satz 1.7.7] it is discussed that this complex polynomial has its global maximum in the point $(1, 0)$ and there takes the value 0. Thus, the only solution of equation (II) is $\alpha_{\mathsf{indPart}((1,\sigma_1),(2,\sigma_2))} = 1$ which we had to show. □

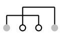
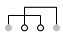
From now on we will only consider two-faced algebras, i. e., we set $m = 2$. By our classification result, we are be able to determine possible values for the highest coefficients which belong to two-block partitions.

5.2.23 Lemma. Let $\odot$ be a positive and symmetric u.a.u.-product in the category AlgP_2. Then,

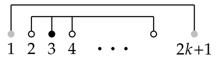
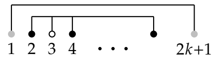
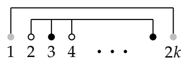
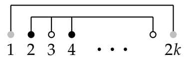
(a) α  $\neq 0 \implies \alpha$  $= 1$,

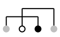
(b) α  $\neq 0 \implies \alpha$  $= 1$,

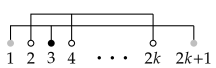
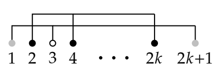
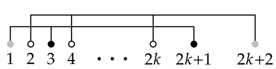
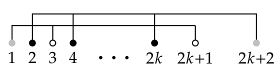
(c) α  $\neq 0 \implies \alpha$  $= 1$,

(d) α  $\neq 0 \implies \alpha$  $= 1$,

(e) $q := \alpha$  $\neq 0 \implies \forall k \in \mathbb{N} \setminus \{1\}$:

$\alpha_\pi =$
- $|q|^{2(k-1)}$ for $\pi = \mathrm{red}(\pi) \in \mathsf{Part}_{\{\circ,\bullet\}}$ $\wedge$ ($\pi =$  $\vee\ \pi =$ )
- $q|q|^{2(k-2)}$ for $\pi = \mathrm{red}(\pi) =$  $\in \mathsf{Part}_{\{\circ,\bullet\}}$
- $q^*|q|^{2(k-2)}$ for $\pi = \mathrm{red}(\pi) =$  $\in \mathsf{Part}_{\{\circ,\bullet\}}$, (5.2.58)

(f) $q := \alpha$  $\neq 0 \implies \forall k \in \mathbb{N}$:

$\alpha_\pi =$
- $|q|^{2(k-1)}$ for $k \neq 1, \pi = \mathrm{red}(\pi) \in \mathsf{Part}_{\{\circ,\bullet\}}$ $\wedge$ ($\pi =$  $\vee\ \pi =$ )
- $q|q|^{2(k-1)}$ for $\pi = \mathrm{red}(\pi) =$  $\in \mathsf{Part}_{\{\circ,\bullet\}}$
- $q^*|q|^{2(k-1)}$ for $\pi =$  $\in \mathsf{Part}_{\{\circ,\bullet\}}$. (5.2.59)

(g) $\alpha_{\sqcap} \neq 0 \wedge \alpha_{\sqcap} \neq 0 \implies \alpha_{\sqcap} = \alpha_{\sqcap} \in \mathbb{C} \setminus \{0\}$.

Proof: Throughout the proof we use that $\mathfrak{P}_{\odot}$ is a two-colored universal class of partitions according to Theorem 5.2.17

Ad **(a)**: We will follow the proof of Lemma 4.2.33 **(a)**

$$\left(\alpha_{\sqcap}\right)^2 = \alpha_{\sqcap}\alpha_{\sqcap} \quad [\![\text{ eq. (5.2.51) }]\!]$$
$$= \alpha_{\sqcap} \quad [\![\text{ eq. (5.2.31) }]\!]$$
$$= \alpha_{\sqcap}\alpha_{\downarrow\downarrow} \quad [\![\text{ eq. (5.2.30) }]\!]$$
$$= \alpha_{\sqcap}\alpha_{\downarrow\downarrow} \quad [\![\text{ eq. (5.2.50) }]\!]$$

The assertion now follows from the fact that $\alpha_{\downarrow\downarrow} = \alpha_{\downarrow\downarrow} = 1$ which has been shown in Proposition 5.2.22.

Ad **(b)**: The proof is similar to the proof of **(a)**.

Ad **(c)**: Like in the proof of Proposition 5.1.18 we obtain the following equation

$$\left(\alpha_{\sqcap}\right)^2 = \alpha_{\sqcap}\alpha_{\sqcap}.$$

We just need to attach black labels to the legs of the partitions and the same steps of reasoning hold. The assertion now follows from **(a)**.

Ad **(d)**: The proof is similar to the proof of **(c)**.

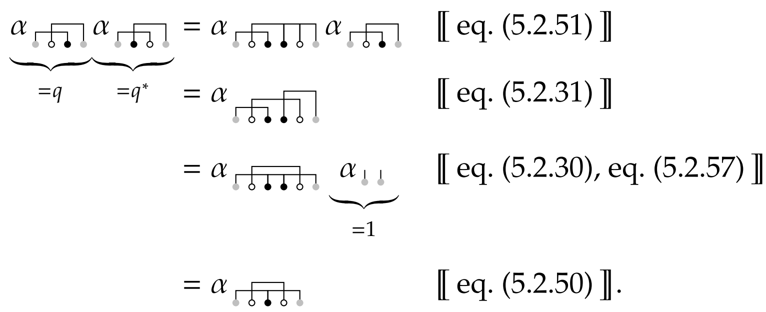
Ad **(e)**: We can prove the equations by induction over $k \in \mathbb{N} \setminus \{1\}$. For the induction base $k = 1$ we need to show the three cases from equation (2.1.19). The cases that the last leg is at position 4 are clear, because $\alpha_{\sqcap} = (\alpha_{\sqcap})^*$. For the other case we calculate

$$|q|^2 = \underbrace{\alpha_{\sqcap}}_{=q^*}\underbrace{\alpha_{\sqcap}}_{=q} = \alpha_{\sqcap}\alpha_{\sqcap} \quad [\![\text{ eq. (5.2.51) }]\!]$$
$$= \alpha_{\sqcap} \quad [\![\text{ eq. (5.2.31) }]\!]$$
$$= \alpha_{\sqcap}\underbrace{\alpha_{\sqcap}}_{=1} \quad [\![\text{ eq. (5.2.30), eq. (5.2.50), eq. (5.2.57) }]\!]$$
$$= \alpha_{\sqcap}$$

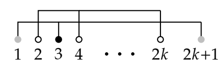
(compare this to the proof of Lemma 4.2.36 **(a)**). For the induction step $k \to k+1$, we will only show one of the three cases, since they all share similar steps of reasoning. So let us assume $\pi = \sqcap$ (legs $1\ 2\ 3\ 4\ \cdots\ 2k$). The proofs for the other case are all similar to the following calculation:

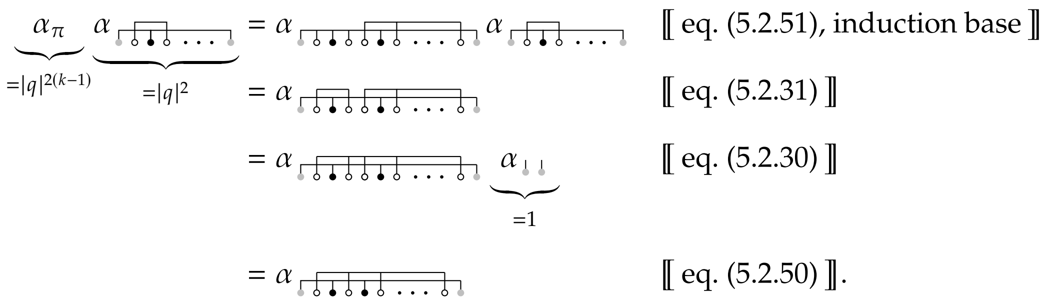
$$\underbrace{\alpha_{\pi}}_{=q|q|^{2(k-2)}}\underbrace{\alpha_{\sqcap}}_{=|q|^2} = \alpha_{\sqcap}\alpha_{\sqcap} \quad [\![\text{ induction base, eq. (5.2.51) }]\!]$$
$$= \alpha_{\sqcap} \quad [\![\text{ eq. (5.2.31) }]\!]$$
$$= \alpha_{\sqcap}\underbrace{\alpha_{\downarrow\downarrow}}_{=1} \quad [\![\text{ eq. (5.2.30), eq. (5.2.57) }]\!]$$
$$= \alpha_{\sqcap} \quad [\![\text{ eq. (5.2.50) }]\!].$$

Ad **(f)**: We can prove the equations by induction over $k \in \mathbb{N}$. The induction base $k = 1$ for the two lower cases in equations (5.2.59) is clear and for the induction base $k = 2$ for the first case in equation (5.2.59) we can calculate

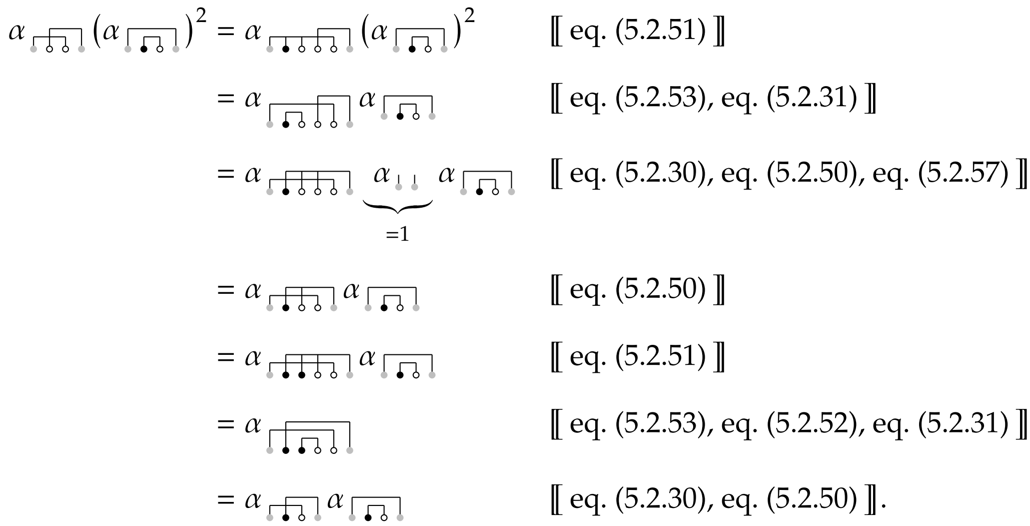


For the induction step $k \to k+1$ we calculate for $\pi :=$

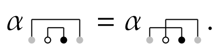




The other cases use similar arguments for the induction step and we therefore omit them.
Ad **(g)**: From Lemma 4.2.43 **(a)** we can conclude that $\{\ ,\ \} \subseteq \mathsf{Gen}(\{\ ,\ \})$. Therefore, we may conclude that $\alpha \neq 0$. If we follow the calculation of its proof, then we obtain the following calculation for the regarding highest coefficients

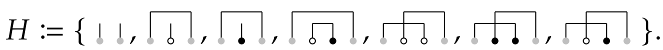


In **(d)** we have shown that $\alpha \neq 0 \implies \alpha = 1$ and therefore the above calculation and equation (5.2.52) imply



□

In preparation for the next lemma we define



(5.2.60)

5.2.24 Lemma. Let $\odot$ be a partition induced universal product or a positive and symmetric u.a.u.-product in the category AlgP_2. Let

$$G = \{\gamma_1, \gamma_n\} \subseteq H \tag{5.2.61}$$

for $n \in \{1,2\}$ be the (according to Theorem 5.2.17) associated subset of generators of $\mathfrak{P}_\odot$, i. e., $\mathfrak{P}_\odot = \mathsf{Gen}(G)$. If $q_i := \alpha_{\gamma_i} \in \mathbb{C} \setminus \{0\}$ for all $i \in [n]$, then

$$\forall \pi \in \mathsf{Gen}(G),\ \exists \text{ polynomial } p \in \mathbb{C}_0[x_1, x_1^*, x_n, x_n^*]\colon \alpha_\pi = \mathsf{eval}_{q_1,q_n}(p), \tag{5.2.62}$$

wherein $\mathbb{C}[x_1, x_1^*, x_n, x_n^*]$ denotes the set of polynomials in commuting indeterminates $\{x_1, x_1^*, x_n, x_n^*\}$ and $\mathsf{eval}_{q_1,q_n}\colon \mathbb{C}[x_1, x_1^*, x_n, x_n^*] \longrightarrow \mathbb{C}$ is the canonical evaluation homomorphism. Furthermore, the set of coefficients $\{q_1, q_n\}$ uniquely determines the product $\odot$.

PROOF: It is clear that the assertion holds for partition induced universal products $\odot_\mathcal{P}$ where $\mathcal{P}$ is a two-colored universal class of partitions. Because, in Lemma 5.2.14 we have seen that these products have the property that their nonzero highest coefficients are 1. The diagram of Figure 4.1 shows us which set of generators G we need to choose for $\mathfrak{P}_{\odot_\mathcal{P}} = \mathcal{P}$.

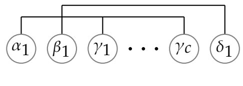
Now, assume that $\odot$ is a positive and symmetric two-faced u.a.u.-product. Thanks to Theorem 5.2.17, we know that $\mathfrak{P}_\odot$ is a two-colored universal class of partitions. Because of our classification result of two-colored universal classes of partitions in Theorem 4.2.44 $\odot_\mathcal{P}$ must match one of the universal class displayed in the diagram of Figure 4.1. But it is not possible that $\odot_\mathcal{P} = 1\mathsf{B}_{\{\circ,\bullet\}}$. Because of Proposition 5.2.22, we have α  $= 1$. Now, the procedure is the following: assume $\mathfrak{P}_\odot$ is any instance of a two-colored universal class of partitions from the diagram of Figure 4.1 with its set of generators G displayed there. By Proposition 5.2.19 it suffices to show

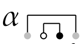
$$\forall \pi \in \mathsf{red}\big(\mathsf{Gen}(G)_{\cdot,2}\big),\ \exists \text{ polynomial } p \in \mathbb{C}[q_1, q_1^*, q_n, q_n^*]\colon \alpha_\pi = p(q_1, q_1^*, q_n, q_n^*).$$

Thus, it suffices to show equation (5.2.62) for reduced two-block partitions of $\mathsf{Gen}(G)$. Throughout the proofs of Lemma 4.2.32 to Lemma 4.2.43 we have determined all the reduced two-block partitions for the several cases of $\mathfrak{P}_\odot$. Lemma 5.2.23 then shows us that all these reduced two-block partitions are indeed polynomials depending on q_1, q_1^*, q_n, q_n^*. By this and from Theorem 5.2.20 then follows that the set of coefficients $\{q_1, q_n\}$ uniquely determines the product $\odot$.

But the case $\mathfrak{P}_\odot = \mathsf{Part}_{\{\circ,\bullet\}}$ is not covered by Lemma 5.2.23. So, let us assume $\mathfrak{P}_\odot = \mathsf{Part}_{\{\circ,\bullet\}}$. We recall that in the proof of Lemma 4.2.43 **(c)** we have introduced a so-called "minimal two-block crossing partition" $\mathsf{m2bC}(\alpha, \beta, \gamma, \delta)$. The first thing to notice is: if

$\forall c \in \mathbb{N},\ \forall \alpha_1,\ \beta_1,\ \gamma := (\gamma_i)_{i\in[c]},\ \delta_1 \in \mathbb{A}(\{\circ,\bullet\})$:

$\pi := \mathsf{m2bC}(\alpha_1, \beta_1, \gamma, \delta_1) =$  ,

then α_π is a polynomial in the indeterminates q and q^*, where $q = \alpha$  $= \alpha$  (look at Lemma 5.2.23 **(g)**). There are some cases we have to consider but we do not further elaborate on this. The necessary steps should be clear from the proof of Lemma 4.2.43 **(c)**. In particular, we can see from there that a generic reduced two-block partition with arbitrary many crossings is generated by $\mathsf{m2bC}(\alpha, \beta, \gamma, \delta)$. This shows that also those partitions are a polynomial depending on q and q^*. □

In preparation for the next theorem we denote by U the set as the union of the set of all partition induced universal products in the category AlgP_2 and the set of all positive and symmetric u.a.u.-products in the category AlgP_2. We denote the set of all two-colored universal classes of partitions by P. We set

$$f\colon \begin{cases} U \longrightarrow P, \\ \odot \longmapsto \mathfrak{P}_\odot \end{cases} \tag{5.2.63}$$

and

$$\tilde{U} = f^{-1}(\mathsf{NC}_{\{\circ,\bullet\}}) \cup f^{-1}(\mathsf{biNC}) \cup f^{-1}(\mathsf{Part}_{\{\circ,\bullet\}}) \tag{5.2.64}$$

($\tilde{U}$ is the union of the preimage of the union $\mathsf{NC}_{\{\circ,\bullet\}} \cup \mathsf{biNC} \cup \mathsf{Part}_{\{\circ,\bullet\}}$ under f).

5.2.25 Theorem. Let $\odot$ be a partition induced universal product or a positive and symmetric u.a.u.-product in the category AlgP_2. Then, the map $f\colon U \longrightarrow P$ is surjective and

$$\tilde{f} := f\restriction_{U\setminus\tilde{U}}\colon U \setminus \tilde{U} \longrightarrow P \tag{5.2.65}$$

is injective. Furthermore,

$$\forall \odot \in U \setminus \tilde{U},\ \forall \pi \in \mathsf{Part}_{\{\circ,\bullet\}}\colon (\alpha_\pi \neq 0 \implies \alpha_\pi = 1). \tag{5.2.66}$$

Proof: First, we show surjectivity of the map $f\colon U \longrightarrow P$. We need to show that a universal product $\odot \in U$ with the property that $\mathfrak{P}_\odot = \mathcal{P}$ for any $\mathcal{P} \in P$ exists. We claim that $\odot_\mathcal{P}$ does the job. Clearly, we have $\odot_\mathcal{P} \in U$. From Lemma 5.2.14 we obtain $f(\odot_\mathcal{P}) = \mathcal{P}$ for any $\mathcal{P} \in P$.

Next, we show that $\tilde{f} := f\restriction_{U\setminus\tilde{U}}$ is injective. By the statement that $\mathfrak{P}_\odot$ is a two-colored universal class of partitions we have $\operatorname{im}(\tilde{f}) \subseteq P \setminus \{\mathsf{NC}_{\{\circ,\bullet\}}, \mathsf{biNC}, \mathsf{Part}_{\{\circ,\bullet\}}\}$. Let $\odot \in U \setminus \tilde{U}$, then from our classification result of P from Theorem 4.2.44 and the diagram of Figure 4.1 we can see that the two-colored universal class of partitions $\tilde{f}(\odot)$ is generated by a certain subset of $\Gamma := \{\gamma_1, \gamma_n\} \subseteq \{$$\}$ for $n \in [2]$. From Lemma 5.2.24, Proposition 5.2.22 and Lemma 5.2.23 **(a)** – **(d)** we can see

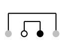
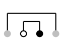
$$\forall \pi \in \mathsf{Gen}(\Gamma)\colon \alpha_\pi = 1. \tag{I}$$

Now, we can calculate

$$\begin{aligned}
&\tilde{f}(\odot) = \tilde{f}(\tilde{\odot}) \\
&\quad\iff f(\odot) = f(\tilde{\odot}) \quad [\![\text{ def. of } \tilde{f}\ ]\!] \\
&\quad\iff \mathfrak{P}_\odot = \mathfrak{P}_{\tilde{\odot}} \quad [\![\text{ def. of } f\ ]\!] \\
&\quad\iff \{\pi \in \mathsf{Part}_{\{\circ,\bullet\}} \mid \alpha_\pi \neq 0\} = \{\pi \in \mathsf{Part}_{\{\circ,\bullet\}} \mid \tilde{\alpha}_\pi \neq 0\} \quad [\![\text{ def. of } \mathfrak{P}_\odot \text{ in eq. (5.2.46) }]\!] \\
&\quad\iff \{\pi \in\mid \alpha_\pi \neq 0\} = \mathsf{Gen}(\Gamma) = \{\pi \in \mathsf{Part}_{\{\circ,\bullet\}} \mid \tilde{\alpha}_\pi \neq 0\} \\
&\quad\implies \forall \pi \in \mathsf{Part}_{\{\circ,\bullet\}}\colon \alpha_\pi = \tilde{\alpha}_\pi \quad [\![\text{ eq. (I) }]\!] \\
&\quad\iff \odot = \tilde{\odot} \quad [\![\text{ Thm. 2.5.13 }]\!].
\end{aligned}$$

□

5.2.26 Proposition. Let $\odot$ be a partition induced universal product or a positive and symmetric u.a.u.-product.

(a) If α $\neq 0$, then $0 < |\alpha$$| \leq 1$.

(b) If $\alpha_{\{\{1,3\},\{2,4\}\}} \neq 0$, then $0 < |\alpha_{\{\{1,3\},\{2,4\}\}}| \leq 1$.

Proof: Ad **(a)**: If $\odot$ is a partition induced universal product the assertion follows from Lemma 5.2.14. Now assume that $\odot$ is a positive and symmetric u.a.u.-product. Denote by $\mathbb{C}_0[X]$ the commutative, non-unital polynomial algebra over the set X. Assume that $X = \{x\}$ for some indeterminate x. Assume $\mathscr{A}^\circ$, $\mathscr{A}^\bullet$, $\mathscr{B}^\circ$, $\mathscr{B}^\bullet$ are algebras such that

$$\mathscr{A}^\circ \cong \mathscr{A}^\bullet \cong \mathscr{B}^\circ \cong \mathscr{A}^\bullet \cong \mathbb{C}_0[x].$$

Set $\mathscr{A} := \mathscr{A}^\circ \sqcup \mathscr{A}^\bullet$ and $\mathscr{B} = \mathscr{B}^\circ \sqcup \mathscr{B}^\bullet$. Then, $\mathscr{A}$ and $\mathscr{B}$ are two-faced algebras, i. e., $\mathscr{A}, \mathscr{B} \in \mathsf{Alg}_2$. Define $\varphi_1 \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$ and $\varphi_2 \in \mathrm{Lin}(\mathscr{B}, \mathbb{C})$ by

$$\varphi_i(x^n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{I}$$

We define an involution on $\mathbb{C}_0[x]$ by $x^* = x$. Then it is clear that φ_i is strongly positive for each $i \in [2]$. Since we have assumed that $\odot$ is positive $\varphi_1 \odot \varphi_2 \colon \mathscr{A} \sqcup \mathscr{B} \longrightarrow \mathbb{C}$ is strongly positive. Let $a_\circ \in \mathscr{A}^\circ$, $a_\bullet \in \mathscr{A}_\bullet$, $b_\circ \in \mathscr{B}_\circ$, $b_\bullet \in \mathscr{B}^\bullet$, then we set $y_1 := b_\bullet a_\circ \in \mathscr{A} \sqcup \mathscr{B}$ and $y_2 := b_\circ a_\bullet \in \mathscr{A} \sqcup \mathscr{B}$. We obtain that the matrix

$$\begin{pmatrix} (\varphi_1 \odot \varphi_2)(a_\circ b_\bullet b_\bullet a_\circ) & (\varphi_1 \odot \varphi_2)(a_\circ b_\bullet b_\circ a_\bullet) \\ (\varphi_1 \odot \varphi_2)(a_\bullet b_\circ b_\bullet a_\circ) & (\varphi_1 \odot \varphi_2)(a_\bullet b_\circ b_\circ a_\bullet) \end{pmatrix}$$

must be positive-semidefinite, where we have applied Lemma 5.2.21 to (y_1, y_2) instead of (x_1, x_2). If we use the universal coefficient theorem, equation (I), equation (5.2.52) and Lemma 5.2.23 **(a)**–**(b)** we have that the matrix

$$\begin{pmatrix} 1 & (\alpha_{\{\{1,4\},\{2,3\}\}})^* \\ \alpha_{\{\{1,4\},\{2,3\}\}} & 1 \end{pmatrix}$$

is positive-semidefinite. For a positive-semidefinite matrix its determinant needs to be non-negative and the assertion follows from this implication.
Ad **(b)**: If we take the same construction as in **(a)**, but apply Lemma 5.2.21 with $y_1 := a_\bullet b_\circ \in \mathscr{A} \sqcup \mathscr{B}$ and $y_2 := b_\circ a_\circ \in \mathscr{A} \sqcup \mathscr{B}$, then we have that the matrix

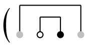
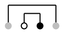
$$\begin{pmatrix} (\varphi_1 \odot \varphi_2)(b_\circ a_\bullet a_\bullet b_\circ) & (\varphi_1 \odot \varphi_2)(b_\circ a_\bullet b_\circ a_\circ) \\ (\varphi_1 \odot \varphi_2)(a_\circ b_\circ a_\bullet b_\circ) & (\varphi_1 \odot \varphi_2)(a_\circ b_\circ b_\circ a_\circ) \end{pmatrix}$$

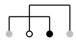
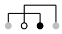
must be positive-semidefinite. By the same reasoning as in **(a)** we obtain that the matrix

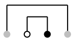
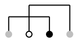
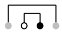
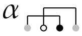
$$\begin{pmatrix} 1 & (\alpha_{\{\{1,3\},\{2,4\}\}})^* \\ \alpha_{\{\{1,3\},\{2,4\}\}} & 1 \end{pmatrix}$$

must be positive-semidefinite. For a positive-semidefinite matrix its determinant needs to be non-negative and the assertion follows from this implication. □

5.2.27 Remark. By Theorem 4.2.44 and the diagram of Figure 4.1 P has been classified in terms of sets of certain generators, i. e., $\forall \mathcal{P} \in P$, $\exists G \subseteq \mathsf{Part}_{\{\circ,\bullet\}} \colon \mathcal{P} = \mathsf{Gen}(G)$ (possible cases for G displayed in the diagram of Figure 4.1). In other words, by Lemma 5.2.24 for any $\mathcal{P} \in U$ there exists a G from the diagram of Figure 4.1 such that $\mathfrak{P}_\odot = \mathsf{Gen}(G)$ and $\mathcal{P}$ is uniquely determined by the set of coefficients $\{\, \alpha_\pi \in \mathbb{C} \mid \pi \in \mathsf{Gen}(G) \,\}$. Then,

$$\forall \odot \in f^{-1}(\mathsf{NC}_{\{\circ,\bullet\}}) = f^{-1}\Big(\mathsf{Gen}\big(\ $$  $$\ \big)\Big) \colon \alpha_{\ }$$  $$\in \{\, q \in \mathbb{C} \mid 0 < |q| \le 1 \,\}, \quad (5.2.67)$$

$$\forall \odot \in f^{-1}(\mathsf{biNC}) = f^{-1}\Big(\mathsf{Gen}\big(\ $$  $$\ \big)\Big) \colon \alpha_{\ }$$  $$\in \{\, q \in \mathbb{C} \mid 0 < |q| \le 1 \,\}, \quad (5.2.68)$$

$$\forall \odot \in f^{-1}(\mathsf{Part}_{\{\circ,\bullet\}}) = f^{-1}\Big(\mathsf{Gen}\big(\{\ $$  $$,\ $$  $$\ \}\big)\Big) \colon \alpha_{\ }$$  $$= \alpha_{\ }$$  $$\quad (5.2.69)$$

$$\in \{\, q \in \mathbb{C} \mid 0 < |q| \le 1 \,\}.$$

This is just a restatement of Lemma 5.2.23 **(e)** – **(g)** and Proposition 5.2.26. Thanks to Lemma 5.2.24, we have

$$\forall \odot \in \tilde{U},\ \forall \pi \in \mathsf{Part}_{\{\circ,\bullet\}} \colon (\alpha_\pi \neq 0 \implies \alpha_\pi \in \{\, q \in \mathbb{C} \mid 0 < |q| \le 1 \,\}), \quad (5.2.70)$$

where the set $\tilde{U}$ is defined in equation (5.2.64).

5.2.28 Remark (A list of all "candidates" of positive and symmetric two-faced u.a.u.-products). The above assertions and the classification of two-colored universal classes of partitions leads to the existence of a well-defined mapping

$$\left\{ \begin{array}{l} \text{positive and symmetric two-faced u.a.u.-product} \\ \text{with all nonzero highest coefficients } (\alpha_\pi)_{\pi \in \mathsf{Part}_{\{\circ,\bullet\}}} \in (\mathbb{C} \setminus \{0\})^{\mathsf{Part}_{\{\circ,\bullet\}}} \end{array} \right\} \ni \odot$$

$$\longmapsto \mathfrak{P}_\odot \in \{\text{two-colored universal class of partitions}\} \quad (5.2.71)$$

and therefore to the diagram of Figure 5.1, which necessarily contains all possible types of a positive and symmetric two-faced u.a.u.-products, characterized in terms of its associated highest coefficients. This is justified by Theorem 2.5.13 and Theorem 5.2.17 tells us that the highest coefficients behave as two-colored universal class of partitions. Then, we may use the classification result of Theorem 4.2.44 (or for a more compact view use the diagram of Figure 4.1) to decide how the (nonzero) highest coefficients can be grouped.

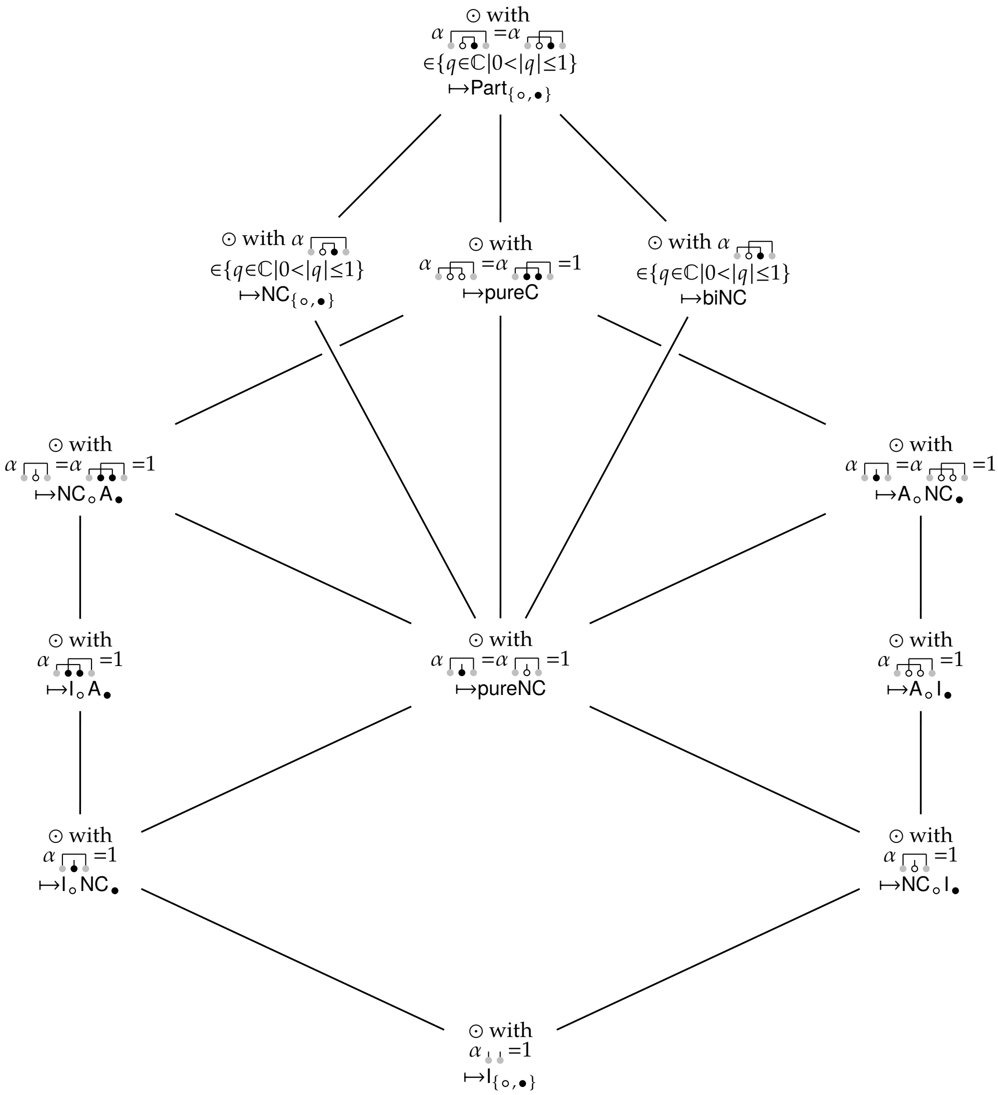


Figure 5.1: A possible way to look at the preimage of the mapping of equation (5.2.70) with necessary nonzero values for the highest coefficients of a symmetric two-faced u.a.u.-product such that it is positive.

One might ask how the general prescription of a positive and symmetric two-faced u.a.u.-product in terms of its nonzero highest coefficients looks like (compare this with the statement of Theorem 2.5.13). In particular, how are these "deformation parameters" from Remark 5.2.27 and Proposition 5.2.26 incorporated, which could lead to the definition of a positive and symmetric two-faced u.a.u.-product? A possible strategy to answer this question is the following. We start with equation (2.5.29) and borrow all the notation and prerequisites introduced in Lemma 2.5.12. Let $\psi \in \mathrm{Lin}\big(\bigsqcup_{i=1}^{m} \mathrm{T}(V_i), \mathbb{C}\big)$. We want to find an expression for $\exp\psi$ in the case $\odot$ is a two-faced positive and symmetric u.a.u.-product. Start from equation (2.5.29) with $\delta = (\delta_i)_{i\in[n]} \in [2]^{\times n}$. Then, we calculate

$$(\exp_{\odot}\psi)\big(\mathsf{can}(v_{i_1}^{(\delta_1)} \otimes \cdots \otimes v_{i_n}^{(\delta_n)})\big)$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k!} \sum_{\substack{\varepsilon=(\varepsilon_i)_{i\in[n]} \\ \in[k]^{\times n}, \\ |\mathrm{set}\,\varepsilon|=k}} \left(\alpha_{\max}^{(\mathrm{red}(((\varepsilon_i,\delta_i))_{i\in[n]}))} \prod_{i=1}^{k} \psi\big(\mathsf{can}(\tilde{v}_{b_i})\big) \right) \right)$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k!} \sum_{\substack{(\varepsilon_i,\delta_i)_{i\in[n]} \\ \in\mathrm{red}(T_{\delta,k})}} \left(\alpha_{\max}^{((\varepsilon_i,\delta_i)_{i\in[n]})} \prod_{i=1}^{k} \psi\big(\mathsf{can}(\tilde{v}_{b_i})\big) \right) \right)$$

⟦ def. of $T_{\delta,k}$ in eq. (5.2.20), eq. (2.5.9) ⟧

$$= \sum_{k=1}^{n} \left(\frac{1}{k!} \sum_{\substack{(\varepsilon_i,\delta_i)_{i\in[n]} \\ \in\mathsf{indPart}^{-1}(\mathrm{red}(\mathsf{Part}_{\delta,k}))}} \left(\alpha_{\max}^{((\varepsilon_i,\delta_i)_{i\in[n]})} \prod_{i=1}^{k} \psi\big(\mathsf{can}(\tilde{v}_{b_i})\big) \right) \right)$$

⟦ def. of the map indPart in eq. (5.2.24), $\mathsf{Part}_{\delta,k}$ defined in Def. 3.4.1 **(d)**
definition of reduced partition in Def. 4.2.28 **(a)** ⟧

$$= \sum_{k=1}^{n} \sum_{\substack{\pi \\ \in\mathrm{red}(\mathsf{Part}_{\delta,k})}} \left(\alpha_\pi \prod_{b\in\pi} \psi\big(\mathsf{can}(\tilde{v}_b)\big) \right) \quad [\![\text{ Conv. 5.2.10, Conv. 3.4.14 }]\!]$$

$$= \sum_{k=1}^{n} \sum_{\substack{\pi \\ \in\mathrm{red}((\mathfrak{P}_\odot)_{\delta,k})}} \left(\alpha_\pi \prod_{b\in\pi} \psi\big(\mathsf{can}(\tilde{v}_b)\big) \right) \quad [\![\ (\mathfrak{P}_\odot)_{\delta,k} \text{ defined in eq. (5.2.46) }]\!] \tag{5.2.72}$$

We can compare the above result to the expression of equation (2.5.29). The difference is that in equation (5.2.72) the sum runs only over all nonzero coefficients and all summands where the highest coefficient is zero are neglected. Thus, we may say that equation (5.2.72) is a more refined version of equation (2.5.29). We claim that a similar calculation holds for the logarithm $\log_\odot \psi$. By the above expression for the exponential on the dual of $\bigsqcup_{i=1}^{k} \mathrm{T}(V_i)$ we obtain for $m = 2$

$\forall n \in \mathbb{N},\ \forall \varepsilon = (\varepsilon_i)_{i\in[n]} \in \mathbb{A}([2]\times[m]),\ \forall (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[m]}, \varphi_i)_{i\in[2]} \in \big(\mathsf{Obj}(\mathsf{AlgP_m})\big)^{\times 2},\ \forall (a_i)_{i\in[n]} \in \prod_{i=1}^{n} \mathscr{A}_{\mathsf{type}(\varepsilon_i)}^{(\mathsf{col}(\varepsilon_i))}$:

$$(\varphi_1 \odot \varphi_2)(\underbrace{a_1 \cdot \dots \cdot a_n}_{\in \bigsqcup_{j=1}^{m} \bigsqcup_{i=1}^{k} \mathscr{A}_i^{(j)}})$$

$$= \left(\left(\exp_\odot\Big(\mathsf{BCH}_{\boxplus}\big(\log_\odot \tilde{\varphi}_1, \log_\odot \tilde{\varphi}_2\big) \Big) \right) \circ \mathsf{inc}_{\mathscr{A}_1\sqcup\mathscr{A}_2, \bigsqcup_{j=1}^{m} \mathrm{T}(V_j)} \right) (a_1 \cdot \dots \cdot a_n)$$

⟦ moment-cummulant formula for $\odot$ in eq. (2.4.21), notations from Prop. 2.4.12 ⟧

$$= \left(\Big(\exp_\odot\big(\log_\odot \tilde{\varphi}_1 + \log_\odot \tilde{\varphi}_2\big) \Big) \circ \mathsf{inc}_{\mathscr{A}_1\sqcup\mathscr{A}_2, \bigsqcup_{j=1}^{m} \mathrm{T}(V_j)} \right) (a_1 \cdot \dots \cdot a_n)$$

⟦ $\odot$ is symmetric, Lem. 2.4.7 ⟧

$$= \sum_{k=1}^{n} \sum_{\substack{\pi \\ \in \mathrm{red}((\mathfrak{P}_\odot)_{\delta,k})}} \Big(\alpha_\pi \prod_{b\in\pi} \big(\log_\odot \tilde{\varphi}_1 + \log_\odot \tilde{\varphi}_2\big)\big(\mathsf{can}(\tilde{a}_b)\big)\Big) \quad [\![\text{ eq. (5.2.72) }]\!]$$

$$= \sum_{k=1}^{n} \sum_{\substack{\pi \le \mathsf{indPart}(\varepsilon) \\ \in \mathfrak{P}_\odot}} \Big(\alpha_\pi \prod_{b\in\pi} \big(\log_\odot \tilde{\varphi}_1 + \log_\odot \tilde{\varphi}_2\big)\big(\mathsf{can}(\tilde{a}_b)\big)\Big)$$

$$+ \sum_{k=1}^{n} \sum_{\substack{\pi \not\le \mathsf{indPart}(\varepsilon) \\ \in \mathfrak{P}_\odot}} \Big(\alpha_\pi \prod_{b\in\pi} \big(\log_\odot \tilde{\varphi}_1 + \log_\odot \tilde{\varphi}_2\big)\big(\mathsf{can}(\tilde{a}_b)\big)\Big) \quad [\![\text{ Lem. 4.2.48 }]\!]$$

$$= \sum_{k=1}^{n} \sum_{\substack{\pi \le \mathsf{indPart}(\varepsilon) \\ \in \mathfrak{P}_\odot}} \Big(\alpha_\pi \prod_{b\in\pi} \big(\log_\odot \tilde{\varphi}_1 + \log_\odot \tilde{\varphi}_2\big)\big(\mathsf{can}(a_b)\big)\Big) + 0 \tag{5.2.73}$$

[[def. of $\tilde{\varphi}_i$ in eq. (2.4.21), def. of $\tilde{a}_b$ in Lem. 2.5.12 **(a)** (v replaced by a)]]

To determine the occurring highest coefficients in the above formula for the computation of $(\varphi_1 \odot \varphi_2)(a_1 \cdot \dots \cdot a_n)$ we can use Lemma 5.2.24 and Lemma 5.2.23. It remains the open question which values of the "deformation parameters" lead to a positive and symmetric u.a.u.-products? Furthermore, we need to answer the question which partition induced universal products are actually positive? We give partial answers to these kind of questions in the next chapter.
We want to make a conjecture, where the proof of this conjecture remains an open task. We want to present this conjecture at this point, since it might provide a different perspective on the diagram of Figure 5.1. In fact, equation (5.2.72) and (5.2.73) might give rise to a definition of a symmetric u.a.u.-product only in terms of partitions. So far, our partition induced universal products have been designed in that way that all their corresponding nonzero highest coefficients are one. In Theorem 3.4.32 we have shown that such universal products satisfy the properties of a symmetric u.a.u.-product. Now, is it possible to generalize the notion of a partition induced universal product for a given m-colored universal class of partitions, where the nonzero highest coefficients are not necessarily one? We think that this is possible, following a similar strategy of Section 3.4, while in particular using equation (5.2.72) and (5.2.73) as a leitmotiv. We conjecture that the conditions of Corollary 5.2.11 and Theorem 5.2.17 **(a)**– **(c)** for the highest coefficients of the generalized partition induced universal product are sufficient to obtain a symmetric m-faced u.a.u.-product. In this sense by the diagram of Figure 5.1, we could say that any positive and symmetric two-faced u.a.u.-products is necessarily a generalized partition induced universal product. Here we use Theorem 2.5.13, since such generalized partition induced universal products would again have the right-ordered monomials property (like in Theorem 3.4.32 **(e)**). To determine all positive and symmetric two-faced u.a.u.-products is now equivalent to show that the corresponding (generalized) partition induced universal is positive or not.

Chapter 6

Further investigations on positivity

In Remark 5.2.28 we have found a mapping which helps us to find a list of all "candidates" of positive and symmetric two-faced u.a.u.-products. This list is actually the preimage of the mapping defined in equation (5.2.63) for elements of the classification from Theorem 4.2.44. At this point we do not know, if some of these preimages are empty if we intersect with the set of all positive and symmetric two-faced u.a.u.-products. This addresses two questions:

- Which two-colored universal classes of partitions $\mathcal{P}$ are allowed such that the partition induced universal product $\odot_{\mathcal{P}}$ is a positive and symmetric two-faced u.a.u.-product?
- With respect to Proposition 5.2.26 what choices for the values for the highest coefficients $\alpha_{\sqcap\sqcap} \in \mathbb{C}$ and $\alpha_{\sqcap\!\sqcap\!\sqcap} \in \mathbb{C}$ of a positive and symmetric u.a.u.-product are allowed such that $\mathfrak{P}_{\odot} \in \{\mathsf{NC}_{\{\circ,\bullet\}}, \mathsf{biNC}, \mathsf{Part}_{\{\circ,\bullet\}}\}$ and equation (5.2.73) is the prescription for a positive and symmetric two-faced u.a.u.-product?

In this chapter we aim to show that some preimages of the mapping of equation (5.2.63) are actually not empty when we intersect with the set of all positive and symmetric two-faced u.a.u.-products. For this, we need to introduce some terminology what we do in Section 6.1. In Sections 6.2, 6.3 we prove positivity for the "boolean-tensor" product $\odot_{\mathsf{I}_\circ\mathsf{A}_\bullet}$. In Sections 6.4, 6.5 among other things we try to find some necessary conditions on the deformation parameters for the deformed bifree and free case to yield positive products.

6.1 General facts about representations of algebras

6.1.1 Convention. Let V be a vector space and assume there exist two linear subspaces $V_1 = \mathbb{C}\Omega$ for $\Omega \in V$ and $V_2 \subseteq V$ such that $V = V_1 \oplus V_2$. Then, we set

$$\mathrm{P}_{\mathbb{C}\Omega} \colon \begin{cases} V = \mathbb{C}\Omega \oplus V_2 \longrightarrow \mathbb{C}\Omega, \\ v = \lambda\Omega + v_2 \longmapsto \lambda\Omega \end{cases} \tag{6.1.1}$$

and we mean the projection onto $\mathbb{C}\Omega$ with respect to the decomposition $V = \mathbb{C}\Omega \oplus V_2$. Furthermore, we set

$$\mathsf{coord}_{\Omega} \colon \begin{cases} V = V_1 \oplus V_2 \longrightarrow \mathbb{C}, \\ v = \lambda\Omega + v_2 \longmapsto \lambda \end{cases} \tag{6.1.2}$$

and we mean the coordinate along the vector Ω in V with respect to the decomposition $V = \mathbb{C}\Omega \oplus V_2$.

6.1.2 Definition (Representation of an algebra, algebraically cyclic representation [Wil08, Def. 3.1.3]). Let $\mathscr{A}$ be an algebra and V be a vector space. A representation π of $\mathscr{A}$ on V is a homomorphism of algebras $\pi\colon \mathscr{A} \longrightarrow \mathrm{Lin}(V)$. Here, $\mathrm{Lin}(V)$ is an algebra in the canonical way. We say that a representation $\pi\colon \mathscr{A} \longrightarrow \mathrm{Lin}(V)$ is *algebraically cyclic* or just *cyclic* if and only if there exists a vector $\Omega \in V$ such that

$$V = \{\, \pi(a)\Omega \mid a \in \mathscr{A} \,\} + \mathbb{C}\Omega. \tag{6.1.3}$$

In this case Ω is called *algebraically cyclic vector* or just *cyclic vector* for the representation π.

6.1.3 Definition (Pointed representation [Ger17], vacuum cyclic representation). Let $\mathscr{A}$ be an algebra. A *pointed representation* for the algebra $\mathscr{A}$ is an ordered triple $(\pi, \Omega, \hat{V})$, consisting of a vector space $\hat{V}$, a representation $\pi\colon \mathscr{A} \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V})$ and the vector $\Omega := (1, 0) \in \mathbb{C} \oplus \hat{V}$. A *vacuum cyclic representation* is a pointed representation $(\pi, \Omega, \hat{V})$, such that Ω is an algebraically cyclic vector for the representation π. The vector Ω is also called the *vacuum vector*.

6.1.4 Convention. For any pointed representation $(\pi, \Omega, \hat{V})$ for an algebra $\mathscr{A}$ we define the *vacuum state* w.r.t. Ω by

$$\mathsf{vac}_\Omega(\pi)\colon \begin{cases} \mathscr{A} \longrightarrow \mathbb{C}, \\ a \longmapsto \mathsf{coord}_\Omega\big(\pi(a)\Omega\big). \end{cases} \tag{6.1.4}$$

It can be easily seen that $\mathsf{vac}_\Omega(\pi) \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$.

6.1.5 Definition (Invariant subspace for a representation [Wil08, Def. 3.1.6]). Let $\mathscr{A}$ be an algebra and $\pi\colon \mathscr{A} \longrightarrow \mathrm{Lin}(V)$ be a representation. We say that a vector subspace $W \subseteq V$ is *invariant* for π if and only if

$$\pi(\mathscr{A})W := \{\, \pi(a)w \mid a \in \mathscr{A},\ w \in W \,\} \subseteq W. \tag{6.1.5}$$

6.1.6 Definition (Restricted representation [Wil08, Def. 3.1.8]). Let $\mathscr{A}$ be an algebra and $\pi\colon \mathscr{A} \longrightarrow \mathrm{Lin}(V)$ be a representation. Let $\mathscr{B} \subseteq \mathscr{A}$ be a subalgebra of $\mathscr{A}$. Then, the *restricted representation* $\pi\!\upharpoonright_{\mathscr{B}}$ is defined as the map

$$\pi\!\upharpoonright_{\mathscr{B}}\colon \begin{cases} \mathscr{B} \longrightarrow \mathrm{Lin}(V), \\ b \longmapsto \pi(b). \end{cases} \tag{6.1.6}$$

Let $W \subseteq V$ be a π-invariant subspace, then for each element $a \in \mathscr{A}$ the element $\pi(a)$ is a linear map from W to W. Thus, we can define the *restricted representation* $\pi\!\upharpoonright_W$ as the map

$$\pi\!\upharpoonright_W\colon \begin{cases} \mathscr{A} \longrightarrow \mathrm{Lin}(W), \\ a \longmapsto \pi(a)\!\upharpoonright_W. \end{cases} \tag{6.1.7}$$

6.1.7 Remark. One can show that $\pi\!\upharpoonright_{\mathscr{B}}\colon \mathscr{B} \longrightarrow \mathrm{Lin}(V)$ and $\pi\!\upharpoonright_W\colon \mathscr{A} \longrightarrow \mathrm{Lin}(W)$ from the above definition are representations.

6.1.8 Proposition (Unital GNS-construction [Wil08, Satz 3.2.1]). Let $\mathscr{A}$ be a unital algebra with unit $\mathbb{1}_{\mathscr{A}} \in \mathscr{A}$ and $\varphi \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$ be a linear functional such that $\varphi(\mathbb{1}_{\mathscr{A}}) = 1$.

(a) The set

$$N_\varphi := \{\, a \in \mathscr{A} \mid \forall a' \in \mathscr{A}\colon \varphi(a'a) = 0 \,\} \tag{6.1.8}$$

is a left ideal in $\mathscr{A}$. It holds

$$N_\varphi \subseteq \ker \varphi. \tag{6.1.9}$$

Denote by $\eta := \eta_\varphi \colon \mathscr{A} \longrightarrow \mathscr{A}/N_\varphi$ the canonical quotient map.

(b) The following decomposition of vector spaces is satisfied

$$\eta(\mathscr{A}) = \mathbb{C}\eta(\mathbb{1}_\mathscr{A}) \oplus \eta(\ker \varphi). \tag{6.1.10}$$

(c) The map

$$\rho_\varphi \colon \begin{cases} \mathscr{A} \longrightarrow \operatorname{Lin}\big(\eta(\mathscr{A})\big), \\ a \longmapsto \rho_\varphi(a) \colon \begin{cases} \eta(\mathscr{A}) \longrightarrow \eta(\mathscr{A}), \\ \eta(a') \longmapsto \eta(aa') \end{cases} \end{cases} \tag{6.1.11}$$

is well-defined and $\big(\rho_\varphi, \eta(\mathbb{1}_\mathscr{A}), \eta(\ker \varphi)\big)$ is a vacuum cyclic representation for $\mathscr{A}$. In particular, ρ_φ is a homomorphism of unital algebras.

(d) The linear functional φ can be realized as a vacuum state with respect to the vacuum vector $\Omega := \eta(\mathbb{1}_\mathscr{A})$, i. e.,

$$\varphi = \mathsf{vac}_\Omega(\rho_\varphi). \tag{6.1.12}$$

(e) Let $K \subseteq \eta(\ker \varphi)$ be a linear subspace such that $\forall\, a \in \mathscr{A} \colon \rho_\varphi(a)(K) \subset K$, then this implies $K = \{0\}$. Or in other words, there does not exist a non-trivial ρ_φ-invariant subspace of $\eta(\ker \varphi)$.

Proof: We will mostly present only the ideas for the proof. Many parts of the proofs are similar to the standard textbook case for the GNS-construction, where the algebra $\mathscr{A}$ carries an involutive structure and the linear functional is assumed to be positive.

Ad **(a)**: It is a standard task to show that the set $N_\varphi \subseteq \ker \varphi$ is a vector subspace and that it has a left ideal property.

Ad **(b)**: We notice that for any $a \in \mathscr{A}$ holds

$$a = \underbrace{\varphi(a)\mathbb{1}_\mathscr{A}}_{\in \mathbb{C}\mathbb{1}_\mathscr{A}} + \underbrace{\big(a - \varphi(a)\mathbb{1}_\mathscr{A}\big)}_{\in \ker \varphi}.$$

Since the map η is linear we have $\eta(a) = \varphi(a)\eta(\mathbb{1}_\mathscr{A}) + \eta(a - \varphi(a)\mathbb{1}_\mathscr{A})$ for any $a \in \mathscr{A}$ and therefore

$$\eta(\mathscr{A}) = \mathbb{C}\eta(\mathbb{1}_\mathscr{A}) + \eta(\ker \varphi).$$

It remains to show that this decomposition is direct, i. e., $\mathbb{C}\eta(\mathbb{1}_\mathscr{A}) \cap \eta(\ker \varphi) = \{0\}$. If we assume that $n \in \mathbb{C}\eta(\mathbb{1}_\mathscr{A}) \cap \eta(\ker \varphi)$, then there exist $\lambda \in \mathbb{C}$ and $a \in \ker \varphi$, such that $\mathbb{C}\eta(\mathbb{1}_\mathscr{A}) \ni \lambda\eta(\mathbb{1}_\mathscr{A}) = n = \eta(a) \in \eta(\ker \varphi)$. Since $\lambda\mathbb{1}_\mathscr{A} - a \in \ker \eta$, we have $\lambda\mathbb{1}_\mathscr{A} - a \in N_\varphi$. Because $a \in \ker \varphi$ and $N_\varphi \subseteq \ker \varphi$, we obtain

$$\lambda = \lambda\varphi(\mathbb{1}_\mathscr{A}) - \underbrace{\varphi(a)}_{=0} = \varphi(\underbrace{\lambda\mathbb{1}_\mathscr{A} - a}_{\in N_\varphi}) = 0.$$

Ad **(c)**: To show that ρ_φ is well-defined actually means that the map $\rho_\varphi(a)$ is well-defined for any $a \in \mathscr{A}$. This holds because by the left ideal property of N_φ one can show that for c_1, $c_2 \in \mathscr{A}$ with $\eta(c_1) = \eta(c_2)$ yields $\eta(ac_1) = \eta(ac_2)$. One then easily shows that ρ_φ is a linear map and that $\rho_\varphi(\mathbb{1}_\mathscr{A})$ is the identity map and therefore ρ_φ is a representation of $\mathscr{A}$ on the vector space $\eta(\mathscr{A})$. It remains to show that $(\eta(\ker\varphi), \rho_\varphi, \eta(\mathbb{1}_{\mathscr{A}})$ is a vacuum cyclic representation for $\mathscr{A}$. Because of **(b)** the triple $(\eta(\ker\varphi), \rho_\varphi, \eta(\mathbb{1}_{\mathscr{A}}))$ is a pointed representation for the unital algebra $\mathscr{A}$. It remains to show that $\Omega := \eta(\mathbb{1}_{\mathscr{A}})$ is an algebraically cyclic vector for π. Thus, we need to show

$$\eta(\mathscr{A}) = \{ \rho_\varphi \eta(\mathbb{1}_{\mathscr{A}}) \mid a \in \mathscr{A} \} + \mathbb{C}\Omega.$$

This equation holds because for any $a \in \mathscr{A}$ we have $\eta(a) = \eta(a\mathbb{1}_{\mathscr{A}}) = \rho_\varphi \eta(\mathbb{1}_{\mathscr{A}})$.
Ad **(d)**: Since we have shown that $\big(\rho_\varphi, \eta(\mathbb{1}_{\mathscr{A}}), \eta(\ker\varphi)\big)$ is a vacuum cyclic representation, it makes sense to speak about the coordinate $\mathsf{coord}_{\eta(\mathbb{1}_{\mathscr{A}})}(\,\cdot\,) \in \mathbb{C}$ with respect to the direct sum decomposition of $\eta(\mathscr{A})$. For any $a \in \mathscr{A}$ we can calculate

$$\begin{aligned}\mathsf{coord}_\Omega\big(\rho_\varphi(a)\Omega\big) &= \mathsf{coord}_{\eta(\mathbb{1}_{\mathscr{A}})}\big(\eta(a)\big) \quad [\![\, \Omega = \eta(\mathbb{1}_{\mathscr{A}}) \text{ \& def. of } \rho_\varphi \text{ in eq. (6.1.11)}]\!]\\ &= \mathsf{coord}_{\eta(\mathbb{1}_{\mathscr{A}})}\Big(\eta\big(\underbrace{\varphi(a)\mathbb{1}_{\mathscr{A}}}_{\in\mathbb{C}\eta(\mathbb{1}_{\mathscr{A}})} + \underbrace{(a-\varphi(a)\mathbb{1}_{\mathscr{A}})}_{\in\eta(\ker\varphi)}\big)\Big)\\ &= \varphi(a) \quad [\![\text{ eq. (6.1.2) }]\!].\end{aligned}$$

Ad **(e)**: Let K be an ρ_φ-invariant subspace of $\eta(\ker\varphi)$. For any element in $k \in K$, there needs to exist an element $a_0 \in \ker\varphi$ such that $k = \eta(a_0)$. Since K is ρ_φ-invariant we have $\forall\, a \in \mathscr{A} : \rho_\varphi(a)\eta(a_0) = \eta(aa_0) \in K \subseteq \eta(\ker\varphi)$. Therefore, we have by equation (6.1.12)

$$\forall\, a \in \mathscr{A} : 0 = \varphi(aa_0) = \mathsf{coord}_{\eta(\mathbb{1}_{\mathscr{A}})}\big(\eta(aa_0)\big).$$

This shows that $a_0 \in N_\varphi$ which implies $\eta(a_0) = 0$. □

6.1.9 Lemma ([Wil08, Lem. 3.3.1]). Let $\mathscr{A}$ be an algebra and $a' \in \mathscr{A}$.

Tfae: **(a)** $a' \in N_{\varphi^{\mathbb{1}}}$,

(b) $\varphi^{\mathbb{1}}(a') = 0$ and $\forall\, a \in \mathscr{A} : \varphi(aa') = 0$.

Proof: We omit the proof since this is done by straightforward calculation. □

We also have a version of the statement of Proposition 6.1.8 for algebras which are not necessarily unital. The above lemma then helps us to prove that there does not exist a non-trivial ρ_φ-invariant subspace.

6.1.10 Lemma (General GNS-construction [Wil08, Satz 3.3.2]). Let $\mathscr{A}$ be an algebra and $\varphi \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$. If we set $\rho_\varphi := \rho_{\varphi^{\mathbb{1}}}\restriction_{\mathscr{A}}$, i. e., ρ_φ is the restricted representation of $\rho_{\varphi^{\mathbb{1}}}$ to $\mathscr{A} \subseteq \mathscr{A}^{\mathbb{1}}$, then the triple

$$\big(\rho_\varphi, \eta(\mathbb{1}_{\mathscr{A}^{\mathbb{1}}}), \eta(\ker\varphi^{\mathbb{1}})\big) \tag{6.1.13}$$

is a vacuum cyclic representation for $\mathscr{A}$, the subspace $\eta(\ker\varphi^{\mathbb{1}})$ has no non-trivial ρ_φ-invariant subspace and

$$\varphi = \mathsf{vac}_\Omega(\rho_\varphi). \tag{6.1.14}$$

Proof: We use the fact that $\mathscr{A}^{\mathbb{1}} = \mathbb{C} \oplus \mathscr{A}$. We will only discuss that there does not exist a non-trivial subspace. We can similarly to the proof of Proposition 6.1.8 **(e)** show that if $\eta(a_0') \in K$ for $K \subsetneq \eta(\ker\varphi^{\mathbb{1}})$ being an arbitrary ρ_φ-invariant subspace, this implies $\varphi^{\mathbb{1}}(a_0') = 0$. Then, by ρ_φ-invariance of K we have

$$\forall\, a \in \mathscr{A} : \rho_\varphi(a)\eta(a_0') = \eta(aa_0') \in K \subseteq \eta(\ker\varphi^{\mathbb{1}})$$

and this by equation (6.1.14) implies $\varphi^{\mathbb{1}}(aa_0') = 0$. Now, we can use Lemma 6.1.9 and it follows that $a_0' \in N_{\varphi^{\mathbb{1}}}$ and therefore $\eta(a_0') = 0$. □

6.1.11 Definition (GNS-triple [Wil08, Def. 3.4.1]). Let $\mathscr{A}$ be an algebra. A *GNS-triple* for the algebra $\mathscr{A}$ is an ordered triple $(\pi, \Omega, \hat{V})$, consisting of a representation $\pi\colon \mathscr{A} \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V})$, the vector $\Omega := (1, 0) \in \mathbb{C} \oplus \hat{V}$ and a vector space $\hat{V}$ such that

(a) $(\pi, \Omega, \hat{V})$ is a vacuum cyclic representation for $\mathscr{A}$,

(b) there does not exist a non-trivial π-invariant subspace of $\hat{V}$, i. e.,

$$\forall \text{ vector subspaces } K \subseteq \hat{V}\colon \pi(\mathscr{A})K \subseteq K \implies K = \{0\}. \tag{6.1.15}$$

6.1.12 Remark. It is obvious that the vacuum cyclic representations of Proposition 6.1.8 and Lemma 6.1.9 are GNS-triples.

6.1.13 Convention. Due to Lemma 6.1.10 for any algebra $\mathscr{A}$ and $\varphi \in \mathrm{Lin}(\mathscr{A}, \mathbb{C})$ we obtain an associated representation ρ_φ, a vacuum vector $\eta(\mathbb{1}_{\mathscr{A}^{\mathbb{1}}})$ and a subspace $\eta(\ker \varphi^{\mathbb{1}})$. We set

$$\mathsf{GNS}(\varphi) := \rho_\varphi, \tag{6.1.16}$$

$$\Omega_\varphi := \eta(\mathbb{1}_{\mathscr{A}^{\mathbb{1}}}), \tag{6.1.17}$$

$$\hat{V}_\varphi := \eta(\ker \varphi^{\mathbb{1}}). \tag{6.1.18}$$

We introduce the notion of "equivalent GNS-triples" ([Wil08, Def. 3.4.5]) resp. "homomorphic vacuum cyclic representations" ([Wil08, Def. 3.5.2]). In fact, we do not need these definitions for the following sections. We only need them in addition with Lemma 6.1.15 for some heuristics in order to argue, why a certain definition might not be the best and better seek for another one. The reader can skip Definition 6.1.14 and Lemma 6.1.15 and can come back to them when we discuss a possible definition of a universal product induced by a representation for the so-called "boolean-tensor product" in Section 6.3.

6.1.14 Definition (Homomorphic vacuum cyclic representations [Wil08, Def. 3.5.2], equivalent GNS-triples [Wil08, Def. 3.4.5]).

(a) Let $\mathscr{A}$ be an algebra. Let $(\pi, \Omega, \hat{V})$ and $(\sigma, \Theta, \hat{W})$ be two vacuum cyclic representations for the algebra $\mathscr{A}$. We say that π *is homomorphic to* σ if and only if a linear and surjective map $\Phi\colon \mathbb{C}\Omega \oplus \hat{V} \longrightarrow \mathbb{C}\Theta \oplus \hat{W}$ exists such that

$$\Phi\Omega = \Theta, \tag{6.1.19a}$$

$$\Phi(\hat{V}) = \hat{W}, \tag{6.1.19b}$$

$$\forall a \in \mathscr{A} : \Phi \circ \pi(a) = \sigma(a) \circ \Phi, \tag{6.1.19c}$$

We say that π and σ are *homomorphic* vacuum cyclic representations if and only if π is homomorphic to σ or σ is homomorphic to π.

(b) We say that two GNS-triples $(\pi, \Omega, \hat{V})$ and $(\sigma, \Theta, \hat{W})$ for the algebra $\mathscr{A}$ are *equivalent* if and only if they are homomorphic as vacuum cyclic representations and the map Φ from **(a)** is additionally injective.

The name "equivalent" is chosen by intention because it can be shown that this leads to an equivalence relation for GNS-triples.

6.1.15 Lemma ([Wil08, Satz 3.4.9]). Let $\mathscr{A}$ be an algebra and $(\pi, \Omega, \hat{V})$ a GNS-triple for the $\mathscr{A}$ in the sense of Definition 6.1.11. If we set

$$\varphi := \mathsf{vac}_{\Omega}(\pi), \tag{6.1.20}$$

then the GNS-triple $(\pi, \Omega, \hat{V})$ is equivalent to the GNS-triple $\big(\mathsf{GNS}(\varphi), \Omega_\varphi, \hat{V}_\varphi\big)$, which is obtained from Lemma 6.1.10 in the general GNS-construction. In other words, there exists a linear and bijective map $\Phi\colon V_\varphi := \Omega_\varphi \oplus \hat{V}_\varphi \longrightarrow \mathbb{C}\Omega \oplus \hat{V}$ with the properties

- **(a)** $\Phi \circ \mathrm{P}_{\mathbb{C}\Omega_\varphi} = \mathrm{P}_{\mathbb{C}\Omega} \circ \Phi$
- **(b)** $\Phi(\hat{V}_\varphi) = \hat{V}$
- **(c)** $\forall a \in \mathscr{A} : \Phi \circ \mathsf{GNS}(\varphi)(a) = \pi(a) \circ \Phi$

Proof: We will only outline the main steps of the proof and refer for more details to the proof of [Wil08, Satz 3.4.9]. We set

$$\Phi_0 : \begin{cases} \mathsf{GNS}(\varphi)(\mathscr{A})\Omega_\varphi \longrightarrow \pi(\mathscr{A})\Omega \\ \mathsf{GNS}(\varphi)(a)\Omega_\varphi \longmapsto \pi(a)\Omega. \end{cases} \tag{I}$$

We omit the proof for linearity of Φ_0 which can be directly done. Next, we claim that Φ_0 is well-defined linear mapping. Therefor, we consider the following calculation for any $a \in \mathscr{A}$ such that

$$
\begin{aligned}
&\mathsf{GNS}(\varphi)(a)\Omega_\varphi = 0 \\
&\iff \mathsf{GNS}(\varphi)(a)\eta(\mathbb{1}_{\mathscr{A}^{\mathbb{1}}}) = 0 \quad [\![\text{ Conv. 6.1.13 }]\!] \\
&\iff \eta(a) \quad [\![\text{ def. of } \mathsf{GNS}(\varphi) = \rho_\varphi = \rho_{\varphi^{\mathbb{1}}}\restriction_{\mathscr{A}} \text{ in eq. (6.1.11) }]\!] \\
&\iff a \in N_{\varphi^{\mathbb{1}}} \quad [\![\eta\colon \mathscr{A}^{\mathbb{1}} \longrightarrow \mathscr{A}^{\mathbb{1}}/N_{\varphi^{\mathbb{1}}}]\!] \\
&\iff \forall a' \in \mathscr{A}^{\mathbb{1}} : \varphi^{\mathbb{1}}(a'a) = 0 \quad [\![N_\varphi \text{ defined in eq. (6.1.8) }]\!] \\
&\iff \varphi(a) = 0 \wedge \big(\forall a' \in \mathscr{A} : \varphi(a'a) = 0\big) \quad [\![\text{ Lem. 6.1.9 }]\!] \\
&\iff \mathsf{coord}_\Omega\, \pi(a)\Omega = 0 \wedge \big(\forall a' \in \mathscr{A} : \mathsf{coord}_\Omega\, \pi(a'a)\Omega = 0\big) \\
&\qquad [\![\text{ def. of } \varphi \text{ in eq. (6.1.20) }]\!] \\
&\iff \pi(a)\Omega \in \hat{V} \wedge \big(\forall a' \in \mathscr{A} : \pi(a'a)\Omega \in \hat{V}\big) \quad [\![\operatorname{im} \pi(a) \subseteq \mathbb{C}\Omega \oplus \hat{V}]\!] \\
&\iff \mathbb{C}\big(\pi(a)\Omega\big) \subseteq \hat{V} \text{ is } \pi\text{-invariant subspace} \\
&\iff \pi(a)\Omega = 0 \quad [\![\text{ eq. (6.1.15) }]\!].
\end{aligned}
$$

Now, we can see that the prescription of equation (I) satisfies $\Phi_0(0) = 0$, because by the above we have shown that $\mathsf{GNS}(\varphi)(a)\Omega_\varphi = 0 \implies \pi(a)\Omega = 0$. It is easily seen, that Φ_0 is now a well-defined linear map. Because, whenever $\mathsf{GNS}(\varphi)(a_1)\Omega_\varphi = \mathsf{GNS}(\varphi)(a_2)\Omega_\varphi$, then $\Phi_0(\mathsf{GNS}(\varphi)(a_1 - a_2)\Omega_\varphi) = \Phi(0) = 0$ and therefore $\Phi_0(\mathsf{GNS}(\varphi)(a_1)\Omega_\varphi) = \Phi_0(\mathsf{GNS}(\varphi)(a_2)\Omega_\varphi)$.

Injectivity for Φ_0 for this linear map means to show that $\ker\Phi_0 = \{0\}$. This is indeed true because from the above we can see

$$\pi(a)\Omega = 0 \implies \mathsf{GNS}(\varphi)(a)\Omega_\varphi = 0. \tag{II}$$

The map Φ_0 is surjective because Ω is algebraically cyclic for π, and thus we have $\mathbb{C}\Omega \oplus \hat{V} = \mathbb{C}\Omega + \pi(\mathscr{A})\Omega$. Hence, we have shown that Φ_0 is a bijective linear map.

At this point it is an open problem how to treat the vectors Ω_φ and Ω. For instance, it can be the case that $\Omega_\varphi \in \eta(\mathscr{A})$ and $\Omega \notin \pi(\mathscr{A})\Omega$. But the following assertion helps us to prevent such a dilemma, because it holds that

$$\forall a \in \mathscr{A} : \Big(\eta(a)\Omega_\varphi = \Omega_\varphi \iff \pi(a)\Omega = \Omega\Big). \tag{III}$$

For the $\Leftarrow$-direction of the proof the first to show is that $\pi(a)\Omega = \Omega$ implies $\mathsf{coord}_{\Omega_\varphi}(\eta(a)-\Omega_\varphi) = 0$. This can be shown using the properties of the GNS-construction from Lemma 6.1.10 and Proposition 6.1.8. Now, we can conclude $\eta(a)-\eta(\mathbb{1}_{\mathscr{A}^\mathbb{1}}) \in \ker(\varphi^\mathbb{1})$, since $V_\varphi = \mathbb{C}\Omega_\varphi \oplus \hat{V}_\varphi$ and $\hat{V}_\varphi = \eta(\ker\varphi^\mathbb{1})$. Using that Φ_0 has an inverse we can obtain that $\forall a' \in \mathscr{A}: \mathsf{GNS}(\varphi)(a')(\eta(a)-\eta(\mathbb{1}_{\mathscr{A}^\mathbb{1}})) = 0$. This in turn implies that $\mathbb{C}(\eta(a)-\eta(\mathbb{1}_{\mathscr{A}^\mathbb{1}}))$ is a $\mathsf{GNS}(\varphi)$-invariant subspace of $\ker(\varphi^\mathbb{1})$ which means that $\eta(a) - \eta(\mathbb{1}_{\mathscr{A}^\mathbb{1}}) = \{0\}$ and therefore $\eta(a) = \eta(\mathbb{1}_{\mathscr{A}^\mathbb{1}}) \equiv \Omega_\varphi$. The other direction of the proof of equation (III) uses a similar scheme.

Equation (II) now enables us to distinguish between two cases.

- $\Omega_\varphi \in \eta(\mathscr{A})$ and $\Omega \in \pi(\mathscr{A})\Omega$,
- $\Omega_\varphi \notin \eta(\mathscr{A})$ and $\Omega \notin \pi(\mathscr{A})\Omega$.

In the first case we set $\Phi := \Phi_0 : V_\varphi \longrightarrow \mathbb{C}\Omega + \pi(\mathscr{A})\Omega$ and in the second case we use the linear extension of the map Φ_0 to define a bijective linear map $\Phi := \Phi_0 : V_\varphi \longrightarrow \mathbb{C}\Omega + \pi(\mathscr{A})\Omega = \mathbb{C}\Omega \oplus \hat{V}$ by $\Phi\eta(a) := \Phi_0\eta(a)$ for all $a \in \mathscr{A}$ and $\Phi\Omega_\varphi := \Omega$. This shows **(a)**.

Ad **(b)**: We need to show that

$$\forall \lambda \in \mathbb{C}, \forall a \in \mathscr{A} : \Big(\lambda\Omega_\varphi + \eta(a) \in \eta(\ker\varphi^\mathbb{1}) \iff \lambda\Omega + \pi(a)\Omega \in \hat{V}\Big).$$

We skip the calculation. The above equation implies $\Phi(\hat{V}_\varphi) = \hat{V}$.

Ad **(c)**: We calculate for any $a \in \mathscr{A}$, $a_0' = \lambda\mathbb{1}_{\mathscr{A}^\mathbb{1}} + a_0 \in \mathscr{A}^\mathbb{1}$

$$\begin{aligned}\Phi\big(\mathsf{GNS}(\varphi)(a)\eta(a_0')\big) &= \Phi\big(\eta(\lambda a) + \eta(aa_0)\big) = \pi(\lambda a)\Omega + \pi(aa_0)\Omega \quad [\![\, \textbf{(a), (b)} \,]\!] \\ &= \pi(a)\big(\lambda\Omega + \pi(a_0)\Omega\big) = \big(\pi(a)\circ\Phi\big)\big(\lambda\Omega_\varphi + \mathsf{GNS}(\varphi)(a_0)\Omega_\varphi\big) \\ &= \big(\pi(a)\circ\Phi\big)\big(\eta(a_0')\big).\end{aligned}$$

□

6.1.16 Lemma. Let $(\pi, \Omega, \hat{V})$ be a pointed representation for an algebra $\mathscr{A}$. Assume $W \subseteq \mathbb{C}\Omega\oplus\hat{V}$ is a π-invariant subspace and $\Omega \in W$, then $(\pi\restriction_W, \Omega, \hat{V}\cap W)$ is a pointed representation for $\mathscr{A}$.

Proof: The restricted representation $\pi\restriction_W : \mathscr{A} \longrightarrow \operatorname{Lin}(W)$ is a representation for the algebra $\mathscr{A}$. It remains to show $W = \mathbb{C}\Omega\oplus(\hat{V}\cap W)$. First we show $W = \mathbb{C}\Omega+(\hat{V}\cap W)$ as sets. Let $w \in W$, then since $w \in W \subseteq V = \mathbb{C}\Omega \oplus \hat{V}$ there exists unique $\lambda \in \mathbb{C}$ and $\hat{v} \in \hat{V}$ such that $w = \lambda\Omega + \hat{v}$. Because $w - \lambda\Omega = \hat{v}$, $\Omega \in W$ and W is vector subspace, we obtain $\hat{v} \in W$. Thus, we have shown that $W \subseteq \mathbb{C}\Omega + (\hat{V}\cap W)$. The other direction $\mathbb{C}\Omega + (\hat{V}\cap W) \subseteq W$ is clear. Moreover, by associativity of intersection we have

$$\mathbb{C}\Omega \cap (\hat{V}\cap W) = (\mathbb{C}\Omega\cap\hat{V})\cap W = \{0\}\cap W = \{0\}.$$

Thus, we have shown $W = \mathbb{C}\Omega \oplus (\hat{V} \cap W)$. □

6.1.17 Lemma ([Wil08, Lem. 3.5.5]). Let $(\pi, \Omega, \hat{V})$ be a vacuum cyclic representation for an algebra $\mathscr{A}$ and assume we are given a homomorphism of algebras $j\colon \mathscr{B} \longrightarrow \mathscr{A}$. If we set $\hat{K} := \hat{V} \cap (\pi \circ j)(\mathscr{B})\Omega$. Then, we have

$$\mathbb{C}\Omega + (\pi \circ j)(\mathscr{B})\Omega = \mathbb{C}\Omega \oplus \hat{K}. \tag{6.1.21}$$

Moreover, $((\pi \circ j)\restriction_{\mathbb{C}\Omega+(\pi\circ j)(\mathscr{B})\Omega}, \Omega, \hat{K})$ is a vacuum cyclic representation for $\mathscr{B}$.

Proof: The triple $((\pi \circ j)\restriction_{\mathbb{C}\Omega+(\pi\circ j)(\mathscr{B})\Omega}, \Omega, \hat{K})$ is a pointed representation for the algebra $\mathscr{B}$ by Lemma 6.1.16. By definition of the subspace $\hat{K}$ this representation is algebraically cyclic for the vacuum vector $\Omega \in \mathbb{C} \oplus \hat{V}$. □

6.1.18 Convention. Let $(\pi, \Omega, \hat{V})$ be a vacuum cyclic representation for an algebra $\mathscr{A}$ and $j\colon \mathscr{B} \longrightarrow \mathscr{A}$ a homomorphism of algebras. Then, by the assertion of Lemma 6.1.17 we denote by $\mathsf{prep}(\pi \circ j)$ a representation defined by

$$\mathsf{prep}(\pi \circ j) := (\pi \circ j)\restriction_{\mathbb{C}\Omega+(\pi\circ j)(\mathscr{B})\Omega}. \tag{6.1.22}$$

We can think of $\mathsf{prep}(\pi \circ j)$ as something like an induced pointed representation due to Lemma 6.1.17 and that is why we abbreviate it by prep.

6.1.19 Remark (GNS-construction in the positive case).

(a) If we are in the setting of a unital $*$-algebra $\mathscr{A}$ and $\varphi \in \operatorname{Lin}(\mathscr{A}, \mathbb{C})$, where $*$ is an involution on $\mathscr{A}$ and $\varphi \in \operatorname{Lin}(\mathscr{A}, \mathbb{C})$ is positive, then the GNS-construction ([Pal01, Thm. 9.4.7]) gives us the structure of a pre-Hilbert space. The main reason for this richer structure lies in Lemma 1.1.26 and therefore by equation (1.1.29)

$$\big(\forall a' \in \mathscr{A}\colon \varphi(a'a) = 0\big) \iff \varphi(a^*a) = 0. \tag{6.1.23}$$

For positive functionals on the algebra $\mathscr{A}$ we obtain for the set N_φ, defined in equation (6.1.8), another equivalent characterization, namely

$$N_\varphi = \{\, a \in \mathscr{A} \mid \varphi(a^*a) = 0 = 0 \,\}. \tag{6.1.24}$$

(b) We say that an ordered triple $(\pi, \Omega, \hat{V})$ is a *GNS-triple* for a $*$-algebra $\mathscr{A}$ if and only if $V = \mathbb{C}\Omega \oplus \hat{V}$ is a pre-Hilbert space, i. e., a vector space with complex inner product $\langle\cdot\,,\cdot\rangle\colon V\times V \longrightarrow V$ with linearity in the second argument, $\Omega := (1, 0) \in \mathbb{C}\oplus\hat{V}$, $(\mathbb{C}\Omega)^\perp = \hat{V}$ and $\pi\colon \mathscr{A} \longrightarrow \operatorname{Adj}(V)$ is a $*$-homomorphism. Here, we set

$$\operatorname{Adj}(V) := \{A \in \operatorname{Lin}(V) \mid \exists C \in \operatorname{Lin}(V),\ \forall v, w \in V\colon \langle v, Aw\rangle = \langle Cv, w\rangle\,\} \tag{6.1.25}$$

as the subalgebra of adjointable operators in $\operatorname{Lin}(V)$. The statement of [Pal01, Thm. 9.4.7] shows us that for any $*$-algebra $\mathscr{A}$ and $\varphi \in \operatorname{Lin}(\mathscr{A}, \mathbb{C})$, where $*$ is an involution on $\mathscr{A}$ and $\varphi \in \operatorname{Lin}(\mathscr{A}, \mathbb{C})$ is strongly positive, a GNS-triple $(\pi, \Omega, \hat{V})$ for the $*$-algebra $\mathscr{A}$ exists such that

$$\forall a \in \mathscr{A}\colon \varphi(a) = \mathsf{vac}_\Omega\big(\pi(a)\big). \tag{6.1.26}$$

(c) We claim that any GNS-triple $(\pi, \Omega, \hat{V})$ for a $*$-algebra $\mathscr{A}$ is also a GNS-triple for $\mathscr{A}$ in the sense of Definition 6.1.11. In other words, we claim that the condition of Definition 6.1.11 **(b)** is automatically satisfied. To show this, we assume that $(\pi, \Omega, \hat{V})$ is a GNS-triple for a $*$-algebra $\mathscr{A}$ with $\langle\Omega, \Omega\rangle = 1$. Without loss of generality we can assume that $\mathscr{A}$ is unital, otherwise we canonically extend the $*$-representation to $\mathscr{A}^{\mathbb{1}}$, which again defines a GNS-triple but now for $\mathscr{A}^{\mathbb{1}}$. It can be shown that both GNS-triples are homomorphic (Definition 6.1.14). So, let us assume $\mathscr{A}$ is unital. Assume that $0 \neq v \in \hat{V}$ and since Ω is algebraically cyclic, there exists an element $a \in \mathscr{A}$ such that $v = \pi(a)\Omega$. But the following calculation shows that if we choose $a' = a^*$ then $\mathrm{P}_{\mathbb{C}\Omega}(\pi(a')v) \neq 0$, because

$$\mathsf{coord}_\Omega\big(\pi(a^*)\pi(a)\Omega\big) = \langle\Omega, \pi(a^*)\pi(a)\Omega\rangle = \langle\pi(a)\Omega, \pi(a)\Omega\rangle = \langle v, v\rangle \neq 0. \tag{6.1.27}$$

Since $0 \neq v \in \hat{V}$ was chosen arbitrarily we have shown that the subspace $\hat{V}$ has no non-trivial π-invariant subspaces.

(d) The orthogonal projection $\mathrm{P}_{\mathbb{C}\Omega}$ for the complete subspace $\mathbb{C}\Omega$ is self-adjoint [Hei11, 2. Aufgabe zu 3.6]. Furthermore, for any element $x \in \mathbb{C}\Omega \oplus \hat{V}$ the Fourier-coefficients determine $\mathrm{P}_{\mathbb{C}\Omega}(x)$, i. e.,

$$\mathrm{P}_{\mathbb{C}\Omega}(x) = \langle\Omega, x\rangle\Omega. \tag{6.1.28}$$

6.2 Boolean-tensor product of representations

The following definition resembles [Ger17, Def. 1].

6.2.1 Definition (Boolean-tensor product of representations). For $i \in [2]$, let $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2)$ and let $(\pi_i, \Omega_i, \hat{V}_i)$ be pointed representations for the algebra $\mathscr{A}_i$. Set

$$\forall\, i \in [2]\colon V_i := \mathbb{C}\Omega_i \oplus \hat{V}_i. \tag{6.2.1}$$

Then, we define

$$\gamma_{\pi_1}^{(1)}\colon \begin{cases} \mathscr{A}_1^{(1)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \pi_1(a) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}, \end{cases} \tag{6.2.2}$$

$$\gamma_{\pi_2}^{(1)}\colon \begin{cases} \mathscr{A}_2^{(1)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \mathrm{P}_{\mathbb{C}\Omega_1} \otimes \pi_2(a), \end{cases} \tag{6.2.3}$$

$$\gamma_{\pi_1}^{(2)}\colon \begin{cases} \mathscr{A}_1^{(2)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \pi_1(a) \otimes \mathsf{id}_{V_2}, \end{cases} \tag{6.2.4}$$

$$\gamma_{\pi_2}^{(2)}\colon \begin{cases} \mathscr{A}_2^{(2)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \mathsf{id}_{V_1} \otimes \pi_2(a). \end{cases} \tag{6.2.5}$$

The maps $\gamma_{\pi_i}^{(j)}$ are morphisms of algebras. Thus, for any $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2)$ for $i \in [2]$

we may define

$$\pi_1 \bowtie \pi_2 \colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \cong (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(1)} \sqcup (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(2)} \longrightarrow \operatorname{Lin}(V_1 \otimes V_2), \\ a \longmapsto \big(\underbrace{(\pi_1 \bowtie \pi_2)\restriction_{(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(1)}}}_{=:\gamma^{(1)}_{\pi_1} \sqcup \gamma^{(1)}_{\pi_2}} \sqcup \underbrace{(\pi_1 \bowtie \pi_2)\restriction_{(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(2)}}}_{=:\gamma^{(2)}_{\pi_1} \sqcup \gamma^{(2)}_{\pi_2}} \big)(a). \end{cases} \tag{6.2.6}$$

6.2.2 Lemma.

(a) $\forall i \in [2],\ \forall (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2),\ \forall$ pointed rep. $(\pi_i, \Omega_i, \hat{V}_i)$ for $\mathscr{A}_i$:

$$(\pi_1 \bowtie \pi_2, \Omega_1 \otimes \Omega_2, \widehat{V_1 \otimes V_2}) \tag{6.2.7}$$

is a pointed representation for the algebra $\mathscr{A}_1 \sqcup \mathscr{A}_2$ with

$$\pi_1 \bowtie \pi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2$$

$$\longrightarrow \operatorname{Lin}\Big(\overbrace{\mathbb{C}(\Omega_1 \otimes \Omega_2) \oplus \underbrace{\big((\mathbb{C}\Omega_1 \otimes \hat{V}_2) \oplus (\hat{V}_2 \otimes \mathbb{C}\Omega_2) \oplus (\hat{V}_1 \otimes \hat{V}_2)\big)}_{=:\widehat{V_1 \otimes V_2}}}^{=V_1 \otimes V_2} \Big), \tag{6.2.8}$$

(b) $\forall i \in [2],\ \forall$ involutive algebras $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2)$,
$\forall$ pointed $*$-rep. $(\pi_i, \Omega_i, \hat{V}_i)$ for $\mathscr{A}_i$:

$$(\pi_1 \bowtie \pi_2, \Omega_1 \otimes \Omega_2, \widehat{V_1 \otimes V_2}) \tag{6.2.9}$$

is a pointed $*$-representation for the $*$-algebra $\mathscr{A}_1 \sqcup \mathscr{A}_2$ and in particular the map $\pi_1 \bowtie \pi_2$ is a homomorphism of $*$-algebras.

(c) $\forall i \in [2],\ \forall (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2),\ \forall$ pointed rep. $(\pi_i, \Omega_i, \hat{V}_i)$ for $\mathscr{A}_i$, $\forall a \in \mathscr{A}_i$:

$$(\pi_1 \bowtie \pi_2)\big(\iota_i(a)\big) \circ \mathsf{inc}_{V_i, V_1 \otimes V_2} = \mathsf{inc}_{V_i, V_1 \otimes V_2} \circ \pi_i(a), \tag{6.2.10}$$

where $\forall i \in [2]$: $\iota_i \colon \mathscr{A}_i \hookrightarrow \mathscr{A}_1 \sqcup \mathscr{A}_2$ is the canonical insertion homomorphism.

(d) $\forall i \in [2],\ \forall (\mathscr{B}_i, (\mathscr{B}^{(j)})_{j\in[2]}), (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2)$,
$\forall j_i \in \mathsf{Morph}_{\mathsf{Alg}_2}\Big((\mathscr{B}_i, (\mathscr{B}^{(j)})_{j\in[2]}), (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]})\Big)$, $\forall$ pointed rep. $(\pi_i, \Omega_i, \hat{V}_i)$ for $\mathscr{A}_i$:

$$\Big(\mathsf{prep}\big(\pi_1 \circ j_1\big) \bowtie \mathsf{prep}\big(\pi_2 \circ j_2\big)\Big)\restriction_{\big((\pi_1 \bowtie \pi_2)\circ(j_1 \amalg j_2)\big)(\mathscr{B}_1 \sqcup \mathscr{B}_2)(\Omega_1 \otimes \Omega_2) + \mathbb{C}\Omega_1 \otimes \Omega_2}$$

$$= \Big(\mathsf{prep}\big(\pi_1 \bowtie \pi_2 \circ (j_1 \amalg j_2)\big)\Big). \tag{6.2.11}$$

(e) $\forall \big(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}\big)_{i\in[3]} \in \big(\mathsf{Obj}(\mathsf{Alg}_2)\big)^{\times 3}$, $\forall i \in [3]$, $\forall$ pointed rep. $(\pi_i, \Omega_i, \hat{V}_i)$ for $\mathscr{A}_i$, $\forall a \in (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3$:

$$\widetilde{\mathsf{can}} \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)(a) \circ \widetilde{\mathsf{can}}^{-1} = \Big(\big(\pi_1 \bowtie (\pi_2 \bowtie \pi_3)\big) \circ \mathsf{can}\Big)(a). \tag{6.2.12}$$

Herein, we denote

$$\widetilde{\mathsf{can}}\colon (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3), \tag{6.2.13}$$

$$\mathsf{can}\colon (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3 \longrightarrow \mathscr{A}_1 \sqcup (\mathscr{A}_2 \sqcup \mathscr{A}_3) \tag{6.2.14}$$

as canonical isomorphisms.

(f) $\forall \big(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}\big)_{i\in[2]} \in \big(\mathsf{Obj}(\mathsf{Alg}_2)\big)^{\times 2}$, $\forall i \in [2]$, $\forall$ pointed rep. $(\pi_i, \Omega_i, \hat{V}_i)$ for $\mathscr{A}_i$, $\forall a \in \mathscr{A}_1 \sqcup \mathscr{A}_2$:

$$\widetilde{\mathsf{can}} \circ (\pi_1 \bowtie \pi_2)(a) \circ \widetilde{\mathsf{can}}^{-1} = \big((\pi_2 \bowtie \pi_1) \circ \mathsf{can}\big)(a), \tag{6.2.15}$$

where

$$\widetilde{\mathsf{can}}\colon \operatorname{Lin}(V_1 \otimes V_2) \longrightarrow \operatorname{Lin}(V_2 \otimes V_1), \tag{6.2.16}$$

$$\mathsf{can}\colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathscr{A}_2 \sqcup \mathscr{A}_1 \tag{6.2.17}$$

are the canonical isomorphisms.

Proof: Ad **(a)**: The map $\pi_1 \bowtie \pi_2\colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \operatorname{Lin}(V_1 \otimes V_2)$ is indeed a homomorphism of algebras by the we have defined it in equation (6.2.6). The pointed representation property is obvious.
Ad **(b)**: Assume that $\mathscr{A}_1$ and $\mathscr{A}_2$ are $*$-algebras and $(\pi_i, \Omega_i, \hat{V}_i)$, $i \in [2]$ are pointed representations for these $*$-algebras (π_i now additionally are homomorphisms of $*$-algebras). To show that $\pi_1 \bowtie \pi_2$) homomorphism of $*$-algebras, it suffices to show that

$$\forall i \in [2], \forall j \in [2], \forall a \in \mathscr{A}_i^{(j)}\colon (\pi_1 \bowtie \pi_2)(a^*) = \big((\pi_1 \bowtie \pi_2)(a)\big)^*. \tag{I}$$

We write $\hat{H}_i := \hat{V}_i$ and $H_i := V_i$ to indicate that V_i is now a pre-Hilbert space (check Remark 6.1.19). We only show equation (I) for $a \in \mathscr{A}_1^{(1)}$, i. e., we show that $\gamma_{\pi_1}^{(1)}$ is a $*$-homomorphism of algebras. The other statements for $\gamma_{\pi_i}^{j}$ are done similarly. We calculate for $v_i \in H_i$

$$\begin{aligned}
&\langle v_1 \otimes v_2, (\pi_1 \bowtie \pi_2)(a)(v_1 \otimes v_2)\rangle \\
&= \langle v_1 \otimes v_2, \big(\pi_1(a) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}\big)(v_1 \otimes v_2)\rangle && [\![\text{ def. of } \pi_1 \bowtie \pi_2 \text{ in eq. (6.2.6) }]\!] \\
&= \langle v_1 \otimes v_2, \pi_1(a)v_1 \otimes \mathrm{P}_{\mathbb{C}\Omega_2} v_2\rangle \\
&= \langle v_1, \pi_1(a)v_1\rangle\langle v_2, \mathrm{P}_{\mathbb{C}\Omega_2} v_2\rangle && [\![\text{ inner product for } H_1 \otimes H_2]\!] \\
&= \langle \pi_1(a^*)v_1, v_1\rangle\langle \underbrace{(\mathrm{P}_{\mathbb{C}\Omega_2})^*}_{=\mathrm{P}_{\mathbb{C}\Omega_2}} v_2, v_2\rangle && [\![\text{ Rem. 6.1.19 \textbf{(d)}}, \pi \text{ is } *\text{-hom. }]\!] \\
&= \langle \big(\pi_1(a^*) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}\big)(v_1 \otimes v_2), v_1 \otimes v_2\rangle
\end{aligned}$$

This shows that $\pi_1 \bowtie \pi_2$ is a $*$-homomorphism if $\mathscr{A}_1$ and $\mathscr{A}_2$ are $*$-algebras.
Ad **(c)**: In the first step we claim that $\forall i \in [2]$, $\forall j \in [2]$, $\forall a \in \mathscr{A}_i^{(j)}$:

$$(\pi_1 \bowtie \pi_2)(a) \circ \mathsf{inc}_{V_i, V_1 \otimes V_2} = \mathsf{inc}_{V_i, V_1 \otimes V_2} \circ \pi_i(a). \tag{II}$$

Since, $V_i = \mathbb{C}\Omega_i \oplus \hat{V}_i$ an arbitrary element $v \in V_i$ has the form $v = \lambda\Omega_i + \hat{v}$ for some $\lambda \in \mathbb{C}$ and $\hat{v} \in \hat{V}_i$. Then,

$$(\mathsf{inc}_{V_1, V_1 \otimes V_2})(v) = \lambda\Omega_1 \otimes \Omega_2 \oplus \hat{v} \otimes \Omega_2 \text{ if } v \in V_2,$$

$$(\mathsf{inc}_{V_2,V_1\otimes V_2})(v) = \lambda\Omega_1 \otimes \Omega_2 \oplus \Omega_1 \otimes \hat{v} \text{ if } v \in V_2.$$

Going through all the four cases for $a \in \mathscr{A}_i^{(j)}$ and using equation (6.2.6) we can easily verify equation (II). Since, $\mathscr{A}_1 \sqcup \mathscr{A}_2$ is generated by the algebras $\mathscr{A}_i^{(j)}$ and $\pi_1 \bowtie \pi_2$ is a homomorphism of algebras the statement of equation (6.2.10) follows.
Ad **(d)**: The statement of equation (6.2.11) is well-posed by Convention 6.1.18, Lemma 6.1.17, the fact that the expression $\cdot \bowtie \cdot$ is defined for pointed representations and the fact

$$\begin{aligned}\big((\pi_1 \circ j_1)(\mathscr{B}_1)\Omega_1 + \mathbb{C}\Omega_1\big) \otimes \big((\pi_2 \circ j_2)(\mathscr{A}_2)\Omega_2 + \mathbb{C}\Omega_2\big)&\\ \supseteq \big((\pi_1 \bowtie \pi_2) \circ (j_1 \amalg j_2)\big)(\mathscr{A}_1 \sqcup \mathscr{A}_2)(\Omega_1 \otimes \Omega_2) + \mathbb{C}\Omega_1 \otimes \Omega_2.&\end{aligned}$$

Compare this to the discussion provided in Remark 6.2.3 **(a)**. Since, on the left hand side and of the right hand side of equation (6.2.11) only homomorphisms of algebras appear and $\mathscr{B}_1 \sqcup \mathscr{B}_2$ is generated by $\mathscr{B}_i^{(j)}$ for $i, j \in [2]$, it suffices to show equation (6.2.11) only for elements $b \in \mathscr{B}_i^{(j)}$. We perform the calculation for $b \in \mathscr{B}_1^{(1)}$ and note that similar calculations hold for the other cases. Let $v \in \big((\pi_1 \bowtie \pi_2) \circ (j_1 \amalg j_2)\big)(\mathscr{A}_1 \sqcup \mathscr{A}_2)(\Omega_1 \otimes \Omega_2) + \mathbb{C}\Omega_1 \otimes \Omega_2$, then we can calculate

$$\begin{aligned}
&\Big(\big(\mathsf{prep}(\pi_1 \circ j_1) \bowtie \mathsf{prep}(\pi_2 \circ j_2)\big)(b)\Big)v\\
&= \Big(\big(\mathsf{prep}(\pi_1 \circ j_1) \bowtie \mathsf{prep}(\pi_2 \circ j_2)\big)\restriction_{(\mathscr{B}_1\sqcup\mathscr{B}_2)^{(1)}}(b)\Big)v\\
&\qquad [\![\, (\mathscr{B}_1 \sqcup \mathscr{B}_2)^{(1)} = \mathscr{B}_1^{(1)} \sqcup \mathscr{B}_2^{(1)}, \text{UMP of free product of algebras} \,]\!]\\
&= \big(\gamma^{(1)}_{\mathsf{prep}(\pi_1\circ j_1)}(b)\big)v = \big(\mathsf{prep}(\pi_1 \circ j_1)(b) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}\big)v\\
&= \Big(\pi_1\big(\underbrace{j_1(b)}_{=:a\in\mathscr{A}_1^{(1)}}\big) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}\Big)v \quad [\![\, \text{Conv. 6.1.18, eq. (6.1.7), } j_1 \text{ is morphism in } \mathsf{Alg}_2 \,]\!]\\
&= \big(\gamma^{(1)}_{\pi_1}(a)\big)v = \Big((\pi_1 \bowtie \pi_2)(a)\Big)v = \Big((\pi_1 \bowtie \pi_2)\big(j_1(b)\big)\Big)v\\
&= \Big(\mathsf{prep}\big(\pi_1 \bowtie \pi_2 \circ (j_1 \amalg j_2)\big)(b)\Big)v \quad [\![\, \text{UMP of free product of algebras} \,]\!].
\end{aligned}$$

Ad **(e)**: By definition of the product for representations in equation (6.2.6) we have

$$\big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)(a) = \begin{cases} \big(\pi_1(a) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}\big) \otimes \mathrm{P}_{\mathbb{C}\Omega_3} & \text{for } a \in \mathscr{A}_1^{(1)}\\ \big(\mathrm{P}_{\mathbb{C}\Omega_1} \otimes \pi_2(a)\big) \otimes \mathrm{P}_{\mathbb{C}\Omega_3} & \text{for } a \in \mathscr{A}_2^{(1)}\\ \big(\mathrm{P}_{\mathbb{C}\Omega_1} \otimes \mathrm{P}_{\mathbb{C}\Omega_2}\big) \otimes \pi_3(a) & \text{for } a \in \mathscr{A}_3^{(1)}\\ \big(\pi_1(a) \otimes \mathsf{id}_{V_2}\big) \otimes \mathsf{id}_{V_3} & \text{for } a \in \mathscr{A}_1^{(2)}\\ \big(\mathsf{id}_{V_1} \otimes \pi_2(a)\big) \otimes \mathsf{id}_{V_3} & \text{for } a \in \mathscr{A}_2^{(2)}\\ \big(\mathsf{id}_{V_1} \otimes \mathsf{id}_{V_2}\big) \otimes \pi_3(a) & \text{for } a \in \mathscr{A}_3^{(2)}. \end{cases}$$

A similar result holds for $\big((\pi_1 \bowtie (\pi_2 \bowtie \pi_3)) \circ \mathsf{can}\big)(a)$ with shifted brackets to the right. If we now use the canonical isomorphism $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ we can see that equation (6.2.12) holds for generating elements of $(\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3$. Moreover, for an arbitrary element $a \in (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3$ there needs to exist $n \in \mathbb{N} \setminus \{2\}$, $\varepsilon = (\varepsilon_{i,1}, \varepsilon_{i,2}) \in ([3] \times [2])^{\times n}$, $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathscr{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$ such that

$$a = \iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1) \cdot \dots \cdot \iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n), \tag{III}$$

where $\iota_i^{(j)} \colon \mathscr{A}_i^{(j)} \hookrightarrow (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3$ is the canonical homomorphic insertion map. Then, we can calculate

$$\begin{aligned}
&\widetilde{\mathsf{can}} \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)(a) \circ \widetilde{\mathsf{can}}^{-1} \\
&\quad= \widetilde{\mathsf{can}} \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)\big(\iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1) \cdot \dots \cdot \iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n)\big) \circ \widetilde{\mathsf{can}}^{-1} \\
&\qquad [\![\text{ eq. (III) }]\!] \\
&\quad= \widetilde{\mathsf{can}} \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)\big(\iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1)\big) \\
&\qquad\quad \circ \dots \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)\big(\iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n)\big) \circ \widetilde{\mathsf{can}}^{-1} \\
&\qquad [\![\text{ } (\pi_1 \bowtie \pi_2) \bowtie \pi_3 \text{ is hom. of algebras }]\!] \\
&\quad= \widetilde{\mathsf{can}} \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)\big(\iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1)\big) \circ \widetilde{\mathsf{can}}^{-1} \circ \widetilde{\mathsf{can}} \circ \\
&\qquad\quad \circ \dots \circ \widetilde{\mathsf{can}}^{-1} \circ \widetilde{\mathsf{can}} \circ \big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)\big(\iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n)\big) \circ \widetilde{\mathsf{can}}^{-1} \\
&\qquad [\![\text{ } \widetilde{\mathsf{can}} \text{ is bijective }]\!] \\
&\quad= \Big(\big(\pi_1 \bowtie (\pi_2 \bowtie \pi_3)\big) \circ \mathsf{can}\Big)\big(\iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1)\big) \\
&\qquad\quad \circ \dots \circ \Big(\big(\pi_1 \bowtie (\pi_2 \bowtie \pi_3)\big) \circ \mathsf{can}\Big)\big(\iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n)\big) \\
&\quad= \Big(\big(\pi_1 \bowtie (\pi_2 \bowtie \pi_3)\big) \circ \mathsf{can}\Big)\big(\iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1) \cdot \dots \cdot \iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n)\big) \\
&\quad= \Big(\big(\pi_1 \bowtie (\pi_2 \bowtie \pi_3)\big) \circ \mathsf{can}\Big)(a).
\end{aligned}$$

Ad **(f)**: Since the canonical isomorphism has the property $\mathsf{can}((\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(j)}) = (\mathscr{A}_2 \sqcup \mathscr{A}_1)^{(j)}$ we can see that for each $j \in [2]$ and $a \in (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(j)}$

$$\widetilde{\mathsf{can}} \circ (\gamma_{\pi_1}^{(j)} \sqcup \gamma_{\pi_2}^{(j)})(a) \circ \widetilde{\mathsf{can}}^{-1} = \big((\gamma_{\pi_2}^{(j)} \sqcup \gamma_{\pi_1}^{(j)}) \circ \mathsf{can}\big)(a).$$

Now, the assertion follows from equation (6.2.6) and a similar argument used in **(e)** about generating elements of the free product of algebras. □

6.2.3 Remark. For each $i \in [2]$ let $(\pi_i, \Omega_i, \hat{V}_i)$ be a GNS-triple for the algebra $\mathscr{A}_i$.

(a) In the setting of the Lemma 6.2.2, we note that in general $\Omega_1 \otimes \Omega_2$ is not an algebraically cyclic vector for $\pi_1 \bowtie \pi_2$. It can be shown that

$$\hat{V}_1 \otimes \Omega_2 \subseteq +\mathbb{C}(\Omega_1 \otimes \Omega_2) + (\pi_1 \bowtie \pi_2)(\mathscr{A}_1 \sqcup \mathscr{A}_2)(\Omega_1 \otimes \Omega_2), \tag{6.2.18}$$

$$\Omega_1 \otimes \hat{V}_2 \subseteq +\mathbb{C}(\Omega_1 \otimes \Omega_2) + (\pi_1 \bowtie \pi_2)(\mathscr{A}_1 \sqcup \mathscr{A}_2)(\Omega_1 \otimes \Omega_2). \tag{6.2.19}$$

But in general it is not true that

$$\hat{V}_1 \otimes \hat{V}_2 \subseteq \mathbb{C}(\Omega_1 \otimes \Omega_2) + (\pi_1 \bowtie \pi_2)(\mathscr{A}_1 \sqcup \mathscr{A}_2)(\Omega_1 \otimes \Omega_2). \tag{6.2.20}$$

Because, once we have created an element $\Omega_1 \otimes v_2 \in \Omega_1 \otimes \hat{V}_2$ from the vacuum $\Omega_1 \otimes \Omega_2$ it is possible that $(\pi_1 \bowtie \pi_2)(a)$ for $a \in \mathscr{A}_1$ now acts as $\pi_1(a) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}$. Thus, the second argument would again be projected to the vacuum part Ω_2.

(b) Furthermore, we were not able to show that $\widehat{V_1 \otimes V_2} \subseteq V_1 \otimes V_2$ has no non-trivial $\pi_1 \bowtie \pi_2$-invariant subspaces. We do not look further into a proof for this claim. Instead, we just try to convince ourselves why a possible proof can be problematic. We let $K \subseteq \widehat{V_1 \otimes V_2}$ be a $\pi_1 \bowtie \pi_2$-invariant subspace. Assume there exists an element $v \in K$ which has the following expression. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and vectors $v_1 \otimes \Omega_2 \in \hat{V}_1 \otimes \mathbb{C}\Omega_2$, $\Omega_1 \otimes v_2 \in \mathbb{C}\Omega_1 \otimes \hat{V}_2$ and $v_3 \otimes v_4 \in \hat{V}_1 \otimes \hat{V}_2$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 (v_3 \otimes v_4). \tag{6.2.21}$$

Since $(\pi_i, \Omega_i, \hat{V}_i)$ is a GNS-triple for each $i \in [2]$ and therefore satisfies equation (6.1.15), we have $\forall\, i \in [2]$, $\exists\, a_i \in \mathscr{A}_i$:

$$\mathrm{P}_{\mathbb{C}\Omega_i}\, \pi_i(a_i) v_i \neq 0 \tag{6.2.22}$$

and there exist $a_3 \in \mathscr{A}_1$, $a_4 \in \mathscr{A}_2$ such that

$$\mathrm{P}_{\mathbb{C}\Omega_1}\, \pi_1(a_3) v_3 \neq 0 \text{ and } \mathrm{P}_{\mathbb{C}\Omega_2}\, \pi_1(a_4) v_4 \neq 0. \tag{6.2.23}$$

Now, we can calculate

$$(\pi_1 \bowtie \pi_2)(a_1)(\lambda_1 v_1 \otimes \Omega_2) \tag{6.2.24}$$

$$= \lambda_1 \big(\pi_1(a_1) v_1\big) \otimes \Omega_2 \quad [\![\text{ eq. (6.2.10) }]\!] \tag{6.2.25}$$

$$= \lambda_1 \underbrace{\Big(\mathsf{coord}_{\Omega_1}\big(\pi_1(a_1)v_1\big)\Big)}_{\neq 0 \quad [\![\text{ eq. (6.2.22) }]\!]} (\Omega_1 \otimes \Omega_2) + \lambda_1 \big(\mathrm{P}_{\hat{V}_1}\, \pi_1(a_1) v_1\big) \otimes \Omega_2. \tag{6.2.26}$$

Since we have assumed that $v \in K \subseteq \widehat{V_1 \otimes V_2}$ and K is $\pi_1 \bowtie \pi_2$-invariant, the above calculation implies that $\lambda_1 = 0$. Similarly, we can show that $\lambda_2 = 0$. But in general we doubt that λ_3 is equal to zero. It might be the case that $a_3 \in \mathscr{A}_1^{(1)}$ and is the only element such that equation (6.2.23) is satisfied. Then, $(\pi_1 \bowtie \pi_2)(a_3)$ would act by $\pi_1(a_3) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}$ and thus $(\pi_1 \bowtie \pi_2)(a_3)(v_3 \otimes v_4) = 0$, since $v_4 \in \hat{V}_2$. Therefore, we would not be able to create something in the subspace $\mathbb{C}\Omega_1 \otimes \Omega_2$ from where we could conclude that $\lambda_3 = 0$ and which prevents us to conclude that $K = \{0\}$. This is not a proper counterexample but it indicates that in general $(\pi_1 \bowtie \pi_2, \Omega_1 \otimes \Omega_2, \widehat{V_1 \otimes V_2})$ might not be a GNS-triple for $\mathscr{A}_1 \sqcup \mathscr{A}_2$.

6.3 Positivity of the boolean-tensor product

Let us call the partition induced universal product $\odot_{\mathsf{I}_\circ \mathsf{A}_\bullet}$ *boolean-tensor product*. We want to show that this symmetric two-faced u.a.u.-product is positive. We could be tempted to define the following, which resembles [Wil08, Chap. 4] only involving GNS-triples. Assume $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}, \varphi_i)_{i\in[2]} \in \mathsf{Obj}(\mathsf{AlgP}_2)$. Then, define

$$\varphi_1 \odot_{bt} \varphi_2 \colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathbb{C} \\ a \longmapsto \mathsf{vac}_{\Omega_{\varphi_1} \otimes \Omega_{\varphi_2}} \Big(\big(\mathsf{GNS}(\varphi_1) \bowtie \mathsf{GNS}(\varphi_2)\big)(a) \Big). \end{cases} \tag{6.3.1}$$

Actually, the above definition might work if $\big(\mathsf{GNS}(\varphi_1) \bowtie \mathsf{GNS}(\varphi_2), \Omega_{\varphi_1} \otimes \Omega_{\varphi_2}, \widehat{V_{\varphi_1} \otimes V_{\varphi_2}}\big)$ was a GNS-triple. But in Remark 6.2.3 we have discussed that we doubt that it is a GNS-triple for $\mathscr{A}_1 \sqcup \mathscr{A}_2$. At least, we were not able to show it is a GNS-triple for $\mathscr{A}_1 \sqcup \mathscr{A}_2$. So, we can expect that we might run into troubles with the definition in equation (6.3.1). We were able to show that equation (6.3.1) defines a positive and symmetric unital universal product in the category AlgP_2 but we could not prove associativity. Our proof of associativity for the product of equation (6.3.1) needs Lemma 6.1.15 but this would require that $\big(\mathsf{GNS}(\varphi_1) \bowtie \mathsf{GNS}(\varphi_2), \Omega_{\varphi_1} \otimes \Omega_{\varphi_2}, \widehat{V_{\varphi_1} \otimes V_{\varphi_2}}\big)$ is a GNS-triple. The problem discussed in Remark 6.2.3 **(a)** is not a big deal because we can restrict the representation $\mathsf{GNS}(\varphi_1) \bowtie \mathsf{GNS}(\varphi_2)$ to the subspace $\mathbb{C}(\Omega_1 \otimes \Omega_2) + (\pi_1 \bowtie \pi_2)(\mathscr{A}_1 \sqcup \mathscr{A}_2)(\Omega_1 \otimes \Omega_2)$ and work with this restricted representation, where $\Omega_1 \otimes \Omega_2$ is now an algebraically cyclic vector. The major problem, which prevents us to use Lemma 6.1.15, is the one discussed in Remark 6.2.3 **(b)** which we can not "fix". Lemma 6.1.15 is satisfied if we take the GNS-construction in the positive case. But then equation (6.3.1) would only lead to a positive and symmetric u.a.u.-product in the category of two-faced $*$-algebras with strongly positive linear functionals defined on them. This is insufficient for our approach to show positivity of the partition induced universal product $\odot_{\mathsf{I}_\circ \mathsf{A}_\bullet}$ because we need to be able to speak about universal products which are uniquely determined by their highest coefficients. Thus, for such a product we could not apply Proposition 2.3.7 and the whole Lachs-functor machinery collapses and we could not arrive at Theorem 2.5.13.

In spite of this possible trouble prone path, we do not gain any benefit when we restrict ourselves to only using the GNS-representation. So, proving universality resp. associativity of $\odot_{bt}$ would actually require some results about equivalent GNS-triples and homomorphic vacuum cyclic representations. But acquiring all these information is not necessary. Because, if we look at equation (6.3.1) once more we can see that we apply a product of certain pointed representations to the vacuum and project the image to the vacuum. So, we need to look at the boolean-tensor product of pointed representations and investigate how it does behave if we take their vacuum expectation value for any pointed representation, not only for the GNS-representation as a special case.

We can conclude that the definition of the $\odot_{bt}$ in equation (6.3.1) is too restrictive. Therefore, we follow now a similar path which has been described in [Ger17] for the bimonotone product. The product of representations $\bowtie$ has a remarkable property which is presented in the next statement. This property will finally allow us to prove universality and associativity very easily for a representation induced boolean-tensor product without any further knowledge about equivalent GNS-triples or homomorphic vacuum cyclic representations.

6.3.1 Proposition. Let $\bowtie$ be the boolean-tensor product of representations defined in Definition 6.2.1. For $i \in [2]$ let $(\mathcal{A}_i, (\mathcal{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg_m})$ and let $(\pi_i, \Omega_i, \hat{V}_i)$, $(\sigma_i, \Theta_i, \hat{W})$ be pointed representations for the algebra $\mathcal{A}_i$ such that

$$\forall a \in \mathcal{A}_i\colon\ \mathsf{vac}_{\Omega_i}(\pi_i(a)) = \mathsf{vac}_{\Theta_i}(\sigma_i(a)). \tag{6.3.2}$$

Then,

$$\forall a \in \mathcal{A}_1 \sqcup \mathcal{A}_2\colon\ \mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\pi_1 \bowtie \pi_2)(a)\big) = \mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\sigma_1 \bowtie \sigma_2)(a)\big). \tag{6.3.3}$$

Proof: Let $a \in \mathcal{A}_1 \sqcup \mathcal{A}_2$. Then, there exist $n \in \mathbb{N} \setminus \{1\}$, $\varepsilon = \big((\varepsilon_{i,1}, \varepsilon_{i,2})\big)_{i\in[n]} \in \mathbb{A}([2]\times[2])$ and $(a_i)_{i\in[n]} \in \prod_{i=1}^n \mathcal{A}_{\varepsilon_{i,1}}^{(\varepsilon_{i,2})}$ such that

$$a = \iota_{\varepsilon_{1,1}}^{(\varepsilon_{1,2})}(a_1)\cdot\dots\cdot\iota_{\varepsilon_{n,1}}^{(\varepsilon_{n,2})}(a_n), \tag{I}$$

where $\iota_i^{(j)}\colon \mathcal{A}_i^{(j)} \hookrightarrow \mathcal{A}_1 \sqcup \mathcal{A}_2$ denote canonical homomorphic insertion maps. Define

$$R_1(a) = \begin{cases} \pi_1(a) & \text{for } a \in \mathcal{A}_1^{(1)} \\ \mathrm{P}_{\mathbb{C}\Omega_1}(a) & \text{for } a \in \mathcal{A}_2^{(1)} \\ \pi_1(a) & \text{for } a \in \mathcal{A}_1^{(2)} \\ \mathsf{id}_{V_1}(a) & \text{for } a \in \mathcal{A}_2^{(2)} \end{cases}, \quad\text{and}\quad R_2(a) = \begin{cases} \pi_2(a) & \text{for } a \in \mathcal{A}_2^{(1)} \\ \mathrm{P}_{\mathbb{C}\Omega_2}(a) & \text{for } a \in \mathcal{A}_1^{(1)} \\ \pi_2(a) & \text{for } a \in \mathcal{A}_2^{(2)} \\ \mathsf{id}_{V_2}(a) & \text{for } a \in \mathcal{A}_1^{(2)}. \end{cases}$$

There exist natural numbers $r, s \in \mathbb{N}$ and tuples $(e_i)_{i\in r+1} \in \mathbb{N}^{\times r+s}$ and for all $u \in [r+s]$ there exist tuples $(i_v^u)_{v\in[e_u]} \in \mathbb{N}^{\times e_u}$ such that

$$\begin{aligned}
&\mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\pi_1 \bowtie \pi_2)(a)\big)\\
&= \mathsf{coord}_{\Omega_1\otimes\Omega_2}\Big((\pi_1 \bowtie \pi_2)(a)\Omega_1 \otimes \Omega_2\Big) \quad [\![\text{ eq. (6.1.4) }]\!]\\
&= \mathsf{coord}_{\Omega_1\otimes\Omega_2}\Big(\big((\pi_1 \bowtie \pi_2)(a_n)\circ\dots\circ(\pi_1 \bowtie \pi_2)(a_1)\big)\Omega_1 \otimes \Omega_2\Big)\\
&\qquad [\![\text{ eq. (I)}, \cdot \bowtie \cdot \text{ is hom. of algebras }]\!]\\
&= \mathsf{coord}_{\Omega_1\otimes\Omega_2}\Bigg(\Big(\big(R_1(a_1)\cdot\dots\cdot R_1(a_n)\big)\Omega_1\Big) \otimes \Big(\big(R_2(a_1)\cdot\dots\cdot R_2(a_n)\big)\Omega_2\Big)\Bigg)\\
&\qquad [\![\text{ Def. 6.2.1 }]\!]\\
&= \mathsf{vac}_{\Omega_1}\Big(\big(\pi_1(a_{i_1^1}\cdot\dots\cdot a_{i_{e_1}^1})\big)\Big)\cdot\dots\cdot\mathsf{vac}_{\Omega_1}\Big(\big(\pi_1(a_{i_1^r}\cdot\dots\cdot a_{i_{e_r}^r})\big)\Big)\\
&\quad\cdot\mathsf{vac}_{\Omega_2}\Big(\big(\pi_2(a_{i_1^{r+1}}\cdot\dots\cdot a_{i_{e_{r+1}}^{r+1}})\big)\Big)\cdot\dots\cdot\mathsf{vac}_{\Omega_2}\Big(\big(\pi_2(a_{i_1^{r+s}}\cdot\dots\cdot a_{i_{e_{r+s}}^{r+s}})\big)\Big)
\end{aligned} \tag{II}$$

The last step needs some justification. It can be shown rigorously by induction over the number of occurrences of $\mathrm{P}_{\mathbb{C}\Omega_1}$ and $\mathrm{P}_{\mathbb{C}\Omega_2}$ in $\big(R_1(a_1)\cdot\dots\cdot R_1(a_n)\big)$ and $\big(R_2(a_1)\cdot\dots\cdot R_2(a_n)\big)$. We only want to discuss this proof of induction. For the induction base we need to consider $\forall i \in [n]\colon a_i \notin \mathcal{A}_1^{(1)} \cup \mathcal{A}_2^{(1)}$. Then, for all $i \in [n]$ the expressions $R_1(a_i)$ resp. $R_2(a_i)$ is equal to either the identity map or $\pi_1(a_i)$ resp. $\pi_2(a_i)$ and the assertion follows. For the induction step, we just notice that once $R_1(a_i)$ resp. $R_2(a_i)$ is equal to $\mathrm{P}_{\mathbb{C}\Omega_1}$ resp. $\mathrm{P}_{\mathbb{C}\Omega_2}$, then everything which has been created from the vacuum is again projected to the vacuum and we obtain a complex number $\mathsf{vac}_{\Omega_1}\big((\pi_1(a_{i_1^j}\cdot\cdots\cdot a_{i_{e_j}^j}))\big)$ or $\mathsf{vac}_{\Omega_2}\big((\pi_2(a_{i_1^j}\cdot\cdots\cdot a_{i_{e_j}^j}))\big)$. We can do a similar calculation for the pointed representations σ_i and obtain the same result from equation (II), where we need to replace π_i by σ_i for each $i \in [2]$. Now, the claim follows from the assumption in equation (6.3.2). □

Now, let us forget the definition of $\odot_{bt}$ from equation (6.3.1). Here, comes a definition which works.

6.3.2 Definition. For any $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}, \varphi_i)_{i\in[2]} \in (\mathsf{Obj}(\mathsf{AlgP}_2))^{\times 2}$ and for any pointed representations $(\pi_i, \Omega_i, \hat{V}_i)$ for the algebra $\mathscr{A}_i$ with $i \in [2]$ with the property

$$\forall i \in [2],\ \forall a \in \mathscr{A}_i\colon \varphi_i(a) = \mathsf{vac}_{\Omega_i}\big(\pi_i(a)\big) \tag{6.3.4}$$

we put

$$\varphi_1 \odot_{bt} \varphi_2\colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathbb{C} \\ a \longmapsto \mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\pi_1 \bowtie \pi_2)(a)\big). \end{cases} \tag{6.3.5}$$

The above prescription is well-defined by Proposition 6.3.1. Furthermore, the prescription $\odot_{bt}$ is non-trivial because at least each pointed representation $(\pi_i, \Omega_i, \hat{V}_i)$ which satisfies equation (6.3.4) can be chosen to be the GNS-triple $(\mathsf{GNS}(\varphi_i), \Omega_{\varphi_i}, \hat{V}_{\varphi_i})$.

6.3.3 Theorem. The prescription $\odot_{bt}$ is a positive and symmetric u.a.u.-product in the category AlgP_2.

Proof: We show that $\odot_{bt}$ is unital. For each $i \in [2]$ let $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]}, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP}_2)$ and let $(\pi_i, \Omega_i, \hat{V}_i)$ be pointed representations which satisfy equation (6.3.4) for φ_i. Let $a \in \mathscr{A}_i$ then

$$\begin{aligned}(\varphi_1 \odot_{bt} \varphi_2)(a) &= \mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\pi_1 \bowtie \pi_2)(a)\big)\\ &= \mathsf{vac}_{\Omega_i}\big(\pi_i(a)\big) \quad [\![\text{ eq. (6.2.10) }]\!]\\ &= \varphi_i(a)\end{aligned}$$

Next, we show that $\odot_{bt}$ is universal. Let for each $i \in [2]$ be $(\mathscr{B}_i, (\mathscr{B}^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2)$, $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]}, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP}_2)$ and $j_i \in \mathsf{Morph}_{\mathsf{Alg}_2}\Big((\mathscr{B}_i, (\mathscr{B}^{(j)})_{j\in[2]}), (\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]})\Big)$. Furthermore, let $(\pi_i, \Omega_i, \hat{V}_i)$ be pointed representations which satisfy equation (6.3.4) for φ_i and $b \in \mathscr{B}_1 \sqcup \mathscr{B}_2$. Then, we can calculate

$$\begin{aligned}&\big((\varphi_1 \odot_{bt} \varphi_2) \circ (j_1 \amalg j_2)\big)(b)\\ &\quad= (\varphi_1 \odot_{bt} \varphi_2)(a) \quad [\![\ a := (j_1 \amalg j_2)(b)\ ]\!]\\ &\quad= \mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\pi_1 \bowtie \pi_2)(a)\big) \quad [\![\text{ Def. 6.3.2 }]\!]\\ &\quad= \mathsf{vac}_{\Omega_1\otimes\Omega_2}\Big(\mathsf{prep}\big(\pi_1 \bowtie \pi_2 \circ (j_1 \amalg j_2)\big)(b)\Big)\\ &\quad= \mathsf{vac}_{\Omega_1\otimes\Omega_2}\Big(\Big(\mathsf{prep}\big(\pi_1 \circ j_1\big) \bowtie \mathsf{prep}\big(\pi_2 \circ j_2\big)\Big)\restriction_{\big((\pi_1\bowtie\pi_2)\circ(j_1\amalg j_2)\big)(\mathscr{B}_1\sqcup\mathscr{B}_2)(\Omega_1\otimes\Omega_2)+\mathbb{C}\Omega_1\otimes\Omega_2}(b)\Big)\\ &\qquad [\![\text{ eq. (6.2.11) }]\!]\end{aligned}$$

$$= \mathsf{vac}_{\Omega_1 \otimes \Omega_2}\Big(\Big(\mathsf{prep}\big(\pi_1 \circ j_1\big) \bowtie \mathsf{prep}\big(\pi_2 \circ j_2\big)\Big)(b)\Big)$$

$$= \Big((\varphi_1 \circ j_1) \odot_{bt} (\varphi_2 \circ j_2)\Big)(b)$$

$$[\![\text{ Lem. 6.1.17, } \mathsf{vac}_{\Omega_i}\big(\mathsf{prep}\big(\pi_i \circ j_i\big)(\,\cdot\,)\big) = (\varphi_i \circ j_i)(\,\cdot\,), \text{ Def. 6.3.2 }]\!].$$

Next, we show associativity of $\odot_{bt}$. For each $i \in [3]$ let $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]}, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP}_2)$ and $(\pi_i, \Omega_i, \hat{V}_i)$ be pointed representations which satisfy equation (6.3.4) for φ_i. Let $a \in (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3$. Then, we can calculate

$$\big((\varphi_1 \odot_{bt} \varphi_2) \odot_{bt} \varphi_3\big)(a)$$

$$= \mathsf{vac}_{(\Omega_1\otimes\Omega_2)\otimes\Omega_3}\Big(\big((\pi_1 \bowtie \pi_2) \bowtie \pi_3\big)(a)\Big)$$

$$\left[\!\!\left[\begin{array}{l}(\pi_1 \bowtie \pi_2, \Omega_1 \otimes \Omega_2, \widehat{V_1 \otimes V_2}) \text{ is pointed rep. for } \mathscr{A}_1 \sqcup \mathscr{A}_2 \\ (\varphi_1 \odot_{bt} \varphi_2)(\,\cdot\,) = \mathsf{vac}_{\Omega_1\otimes\Omega_2}\big((\pi_1 \bowtie \pi_2)(\,\cdot\,)\big), \text{ Def. 6.3.2}\end{array}\right]\!\!\right]$$

$$= \mathsf{vac}_{\Omega_1\otimes(\Omega_2\otimes\Omega_3)}\Big(\big(\pi_1 \bowtie (\pi_2 \bowtie \pi_3\big)\big(\mathsf{can}(a)\big)\Big)$$

$$[\![\text{ eq. (6.2.12), } \mathsf{can}\colon (\mathscr{A}_1 \sqcup \mathscr{A}_2) \sqcup \mathscr{A}_3 \longrightarrow \mathscr{A}_1 \sqcup (\mathscr{A}_2 \sqcup \mathscr{A}_3)]\!]$$

$$= \big(\varphi_1 \odot_{bt} (\varphi_2 \odot_{bt} \varphi_3)\big)\big((\mathsf{can}(a))\big).$$

The symmetry of $\odot_{bt}$ is clear from equation (6.2.15).

In the last part of the proof we now show that $\odot_{bt}$ is positive. For each $i \in [2]$ let $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{i\in[2]}, \varphi_i) \in \mathsf{Obj}(\mathsf{AlgP}_2)$ and we additionally assume that $\mathscr{A}_i$ is a $*$-algebra and the linear functional $\varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ is strongly positive. Furthermore, for each $i \in [2]$ let $(\pi_i, \Omega_i, \hat{V}_i)$ be pointed $*$-representations which satisfy equation (6.3.4) for φ_i. Such representations need to exist by Remark 6.1.19 **(b)**. We also have a unital extension for $\pi_i \colon \mathscr{A}_i \longrightarrow \mathrm{Lin}(\mathbb{C}\Omega_i \oplus \hat{V}_i)$ denoted by $\pi_i^{\mathbb{1}} \colon \mathscr{A}_i^{\mathbb{1}} \longrightarrow \mathrm{Lin}(\mathbb{C}\Omega_i \oplus \hat{V}_i)$ such that $\pi_i^{\mathbb{1}}(\mathbb{1}) = \mathsf{id}_{\mathbb{C}\Omega_i\oplus\hat{V}_i}$. Let us denote the unit element of the algebra $(\mathscr{A}_i^{(j)})^{\mathbb{1}}$ by $\mathbb{1}_i^{(j)}$ for each $i, j \in [2]$. For each $i \in [2]$ we set

$$\tilde{\varphi}_i := \varphi_i^{\mathbb{1}} \circ \mathsf{can}_i \circ \mathsf{pr}_i,$$

$$\tilde{\pi}_i := \pi_i^{\mathbb{1}} \circ \mathsf{can}_i \circ \mathsf{pr}_i,$$

wherein we denote natural homomorphisms due to Remark 2.1.4 **(a)** and Lemma 2.1.5 by

$$\mathsf{pr}_i\colon \bigsqcup_{j=1}^{2} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \longrightarrow \bigsqcup_{j=1}^{2} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \Big/ \langle \mathbb{1}_i^1 - \mathbb{1}_i^2 \rangle,$$

$$\mathsf{can}_i\colon \bigsqcup_{j=1}^{2} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \Big/ \langle \mathbb{1}_i^1 - \mathbb{1}_i^2 \rangle \longrightarrow \bigsqcup_{j=1}^{2}{}_{\mathbb{1}} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \longrightarrow (\mathscr{A}_i)^{\mathbb{1}}.$$

For the above definitions we claim that the map $\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2 \colon \mathscr{A}_1^{\mathbb{1}} \sqcup \mathscr{A}_2^{\mathbb{1}} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2)$ satisfies

$$\forall i \in [2],\ \forall j \in [2] \colon (\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2)(\mathbb{1}_i^{(j)}) = 1. \tag{I}$$

For the proof we set

$$\forall\, i \in [2] \colon \tilde{\mathscr{A}}_i := \bigsqcup_{j=1}^{2} (\mathscr{A}_i^{(j)})^{\mathbb{1}}.$$

Then, we can see that for each $i \in [2]$ we have

$$\Big(\tilde{\mathscr{A}}_i, \big((\mathscr{A}_i^{(j)})^{\mathbb{1}}\big)_{j\in[2]}, \tilde{\varphi}_i\Big) \in \mathsf{AlgP}_2 \,.$$

Thus, the expression $\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2$ is well-defined. Now, we can calculate for any $i,\ j \in [2]$

$$\begin{aligned}
&(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2)(\mathbb{1}_i^{(j)}) \\
&\quad = \big\langle \Omega_1 \otimes \Omega_2, (\tilde{\pi}_1 \bowtie \tilde{\pi}_2)(\mathbb{1}_i^{(j)})(\Omega_1 \otimes \Omega_2)\big\rangle \\
&\qquad [\![\ \forall\, i \in [2],\ \forall a \in \tilde{\mathscr{A}}_i \colon \tilde{\varphi}_i(a) = \mathsf{vac}_{\Omega_i}\big(\tilde{\pi}_i(a)\big), \text{Def. 6.3.2, eq. (6.1.28)}\ ]\!] \\
&\quad = \big\langle \Omega_1 \otimes \Omega_2, (\tilde{\pi}_1 \bowtie \tilde{\pi}_2)(\mathbb{1}_i^{(j)})\, \mathsf{inc}_{V_i, V_1\otimes V_2}(\Omega_i)\big\rangle \\
&\quad = \big\langle \Omega_1 \otimes \Omega_2, \mathsf{inc}_{V_i, V_1\otimes V_2}(\tilde{\pi}_i(\mathbb{1}_i^{(j)})\Omega_i)\big\rangle \quad [\![\ \text{eq. (6.2.10)}\ ]\!] \\
&\quad = \big\langle \Omega_1 \otimes \Omega_2, \mathsf{inc}_{V_i, V_1\otimes V_2}(\Omega_i)\big\rangle \quad [\![\ \tilde{\pi}_i \text{ is unital alg. -hom.}\ ]\!] \\
&\quad = \big\langle \Omega_1 \otimes \Omega_2, \Omega_1 \otimes \Omega_2\big\rangle \\
&\quad = \underbrace{\big\langle \Omega_1, \Omega_1\big\rangle}_{=1} \underbrace{\big\langle \Omega_2, \Omega_2\big\rangle}_{=1} \quad [\![\ \text{inner product in } V_1 \otimes V_2\ ]\!] \\
&\quad = 1 \quad \left[\!\!\left[\begin{array}{l} \text{property of the vacuum vector } \Omega_i \text{ for the GNS-construction} \\ \text{in the positive case, follows from [Pal01, Thm. 9.4.7 (b)} \implies \text{(e)]} \end{array} \right]\!\!\right].
\end{aligned}$$

This shows equation (I). By equation (I) we may lift $\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2$ to the quotient algebra

$$\bigsqcup_{i=1}^{2} \bigsqcup_{j=1}^{2} (\mathscr{A}_i^{(j)})^{\mathbb{1}} \Big/ \underbrace{\big\langle \mathbb{1}_1^{(1)} - \mathbb{1}_1^{(2)},\ \mathbb{1}_2^{(1)} - \mathbb{1}_2^{(2)},\ \mathbb{1}_1^{(1)} - \mathbb{1}_2^{(1)}\big\rangle}_{=:I}$$

and we denote this unique quotient map by $\mathsf{lift}(\varphi_1^{\mathbb{1}} \odot_{bt} \varphi_2^{\mathbb{1}})$. Now, we claim that

$$(\varphi_1 \odot_{bt} \varphi_2)^{\mathbb{1}} = \mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{can}, \tag{II}$$

where can denotes a canonical isomorphism of unital algebras

$$\mathsf{can} \colon (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{\mathbb{1}} \longrightarrow \bigsqcup_{i=1}^{2} \bigsqcup_{j=1}^{2} (\mathscr{A}_i^{(j)})^{\mathbb{1}} / I.$$

This canonical isomorphism can be obtained by the help of Remark 2.1.4 **(a)**, Lemma 2.1.5 and is then a composition of the following canonical isomorphisms

$$(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{\mathbb{1}} \cong \mathscr{A}_1^{\mathbb{1}} \sqcup_{\mathbb{1}} \mathscr{A}_2^{\mathbb{1}} \cong \Big(\bigsqcup_{j=1}^{2} (\mathscr{A}_1^{(j)})^{\mathbb{1}}\Big) \sqcup_{\mathbb{1}} \Big(\bigsqcup_{j=1}^{2} (\mathscr{A}_2^{(j)})^{\mathbb{1}}\Big) \cong \bigsqcup_{i=1}^{2}{}_{\mathbb{1}} \bigsqcup_{j=1}^{2}{}_{\mathbb{1}} (\mathscr{A}_i^{(j)})^{\mathbb{1}}$$

$$\cong \Big(\bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}(\mathscr{A}_i^{(j)})^{\mathbb{1}}\Big)/I.$$

Since can as a homomorphism of unital algebras, preserves the unit element and by the result of equation (I), we have

$$\big(\mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{can}\big)(\mathbb{1}_{(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{\mathbb{1}}}) = 1.$$

For the proof of equation (II) it remains to show that

$$\varphi_1 \odot_{bt} \varphi_2 = \big(\mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{can}\big)\restriction_{\mathscr{A}_1 \sqcup \mathscr{A}_2}.$$

For this, we assume that $a \in \mathscr{A}_1 \sqcup \mathscr{A}_2$ and calculate

$$\begin{aligned}
&\big(\mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{can}\big)\restriction_{\mathscr{A}_1 \sqcup \mathscr{A}_2}(a)\\
&\quad = \big(\mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{pr}\big)\restriction_{\mathscr{A}_1 \sqcup \mathscr{A}_2}(a)\\
&\qquad [\![\, \mathsf{pr}\colon \bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}(\mathscr{A}_i^{(j)})^{\mathbb{1}} \longrightarrow \Big(\bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}(\mathscr{A}_i^{(j)})^{\mathbb{1}}\Big)/I \,]\!]\\
&\quad = \big(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2\big)\restriction_{\mathscr{A}_1 \sqcup \mathscr{A}_2}(a)\\
&\quad = \mathsf{vac}_{\Omega_1 \otimes \Omega_2}\big((\tilde{\pi}_1 \bowtie \tilde{\pi}_2)\restriction_{\mathscr{A}_1 \sqcup \mathscr{A}_2}(a)\big)\\
&\quad = \mathsf{vac}_{\Omega_1 \otimes \Omega_2}\Big(\Big((\tilde{\pi}_1 \bowtie \tilde{\pi}_2) \circ \Big(\bigsqcup_{j=1}^{2}\iota_1^{(j)} \sqcup \bigsqcup_{j=1}^{2}\iota_2^{(j)}\Big)\Big)(a)\Big) \qquad [\![\, \iota_i^{(j)} \colon \mathscr{A}_i^{(j)} \hookrightarrow \bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}\mathscr{A}_i^{(j)} \,]\!]\\
&\quad = \mathsf{vac}_{\Omega_1 \otimes \Omega_2}\Big(\Big((\tilde{\pi}_1 \circ \bigsqcup_{j=1}^{2}\iota_1^{(j)}) \bowtie (\tilde{\pi}_2 \circ \bigsqcup_{j=1}^{2}\iota_1^{(j)})\Big)(a)\Big) \qquad [\![\, \text{Lem. 6.2.2 } \mathbf{(d)} \,]\!]\\
&\quad = \mathsf{vac}_{\Omega_1 \otimes \Omega_2}\big((\pi_1 \bowtie \pi_2)(a)\big) \qquad [\![\, \pi_i^{\mathbb{1}}\restriction_{\mathscr{A}_i} = \pi_i \,]\!]\\
&\quad = (\varphi_1 \odot_{bt} \varphi_2)(a) \qquad [\![\, \text{eq. (6.3.5)} \,]\!].
\end{aligned}$$

From equation (II) we can deduce that $\odot_{bt}$ is positive. For this, let $a \in (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{\mathbb{1}}$, then there exist $\lambda \in \mathbb{C}$ and $a_0 \in \mathscr{A}_1 \sqcup \mathscr{A}_2$ such that $a = \lambda\mathbb{1}_{(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{\mathbb{1}}} + a_0$. Then, we can calculate

$$\begin{aligned}
&(\varphi_1 \odot_{bt} \varphi_2)^{\mathbb{1}}(a^*a)\\
&\quad = \big(\mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{can}\big)(a^*a) \qquad [\![\, \text{eq. (II)} \,]\!]\\
&\quad = \big(\mathsf{lift}(\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2) \circ \mathsf{pr}\big)\big((\lambda\mathbb{1}_1^{(1)} + a_0)^*\underbrace{(\lambda\mathbb{1}_1^{(1)} + a_0)}_{=:\tilde{a} \in (\mathscr{A}_1)^{\mathbb{1}} \sqcup (\mathscr{A}_2)^{\mathbb{1}} \quad [\![\, \text{Lem. 2.1.5} \,]\!]}\big)
\end{aligned}$$

[[because I is generated by self-adjoint elements
we have an involution on the quotient $\big(\bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}(\mathscr{A}_i^{(j)})^{\mathbb{1}}\big)/I$,
$\mathsf{pr}\colon \bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}(\mathscr{A}_i^{(j)})^{\mathbb{1}} \longrightarrow \big(\bigsqcup_{i=1}^{2}\bigsqcup_{j=1}^{2}(\mathscr{A}_i^{(j)})^{\mathbb{1}}\big)/I$
is homomorphism of $*$-algebras]]

$$= (\tilde{\varphi}_1 \odot_{bt} \tilde{\varphi}_2)(\tilde{a}^*\tilde{a})$$

$$= \big\langle \Omega_1 \otimes \Omega_2, (\tilde{\pi}_1 \bowtie \tilde{\pi}_2)(\tilde{a}^*\tilde{a})(\Omega_1 \otimes \Omega_2)\big\rangle$$

$$[\![\, \forall i \in [2],\ \forall a \in \tilde{\mathscr{A}}_i : \tilde{\varphi}_i(a) = \mathsf{vac}_{\Omega_i}\big(\tilde{\pi}_i(a)\big), \text{Def. 6.3.2, eq. (6.1.28)} \,]\!]$$

$$= \big\langle (\tilde{\pi}_1 \bowtie \tilde{\pi}_2)(\tilde{a})(\Omega_1 \otimes \Omega_2), (\tilde{\pi}_1 \bowtie \tilde{\pi}_2)(\tilde{a})(\Omega_1 \otimes \Omega_2)\big\rangle$$

$$[\![\, \tilde{\pi}_1 \bowtie \tilde{\pi}_2 \text{ is } *\text{-hom. by Lem. 6.2.2 } \mathbf{(b)} \,]\!]$$

$$= \|(\tilde{\pi}_1 \bowtie \tilde{\pi}_2)(\tilde{a})\|^2 \geq 0. \qquad \square$$

6.3.4 Lemma. Let $\odot_{bt}$ be the positive and symmetric two-faced u.a.u.-product, defined in equation (6.3.5), and $\odot_{\mathsf{I}_\circ \mathsf{A}_\bullet}$ be the partition induced u.a.u.-product for the two-colored universal class of partitions $\mathsf{I}_\circ\mathsf{A}_\bullet$. Using Convention 4.2.7 **(b)** we have

$$\odot_{bt} = \odot_{\mathsf{I}_\circ \mathsf{A}_\bullet}. \tag{6.3.6}$$

Proof: According to Lemma 5.2.24 and the classification result of the diagram of Figure 4.1, we are done if we can show

α  $= 1$ and α  $= 0$.

For this, we assume for each $i \in [2]$ that $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}, \varphi_i) \in (\mathsf{Obj}(\mathsf{AlgP}_2))^{\times 2}$ and $(\pi_i, \Omega_i, \hat{V}_i)$ are pointed representations for the algebra $\mathscr{A}_i$ which satisfy equation (6.3.4) for φ_i. Let us assume $a_1 \in \mathscr{A}_1^{(2)}$, $a_2 \in \mathscr{A}_2^{(2)}$, $a_3 \in \mathscr{A}_1^{(2)}$ and $a_4 \in \mathscr{A}_1^{(2)}$. Then, we can calculate

$$(\varphi_1 \odot_{bt} \varphi_2)(a_1 \cdot a_2 \cdot a_3 \cdot a_4)$$

$$= \mathsf{coord}_{\Omega_1\otimes\Omega_2}\Big(\big(\pi_1 \bowtie \pi_2\big)(a_1 \cdot a_2 \cdot a_3 \cdot a_4)(\Omega_1 \otimes \Omega_2)\Big)$$

$$= (\mathsf{can} \circ \mathrm{P}_{\mathbb{C}\Omega_1\otimes\Omega_2})\Big(\big(\pi_1 \bowtie \pi_2\big)(a_1 \cdot a_2 \cdot a_3 \cdot a_4)(\Omega_1 \otimes \Omega_2)\Big)$$

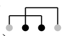
$$[\![\, \text{iso. of vector spaces } \mathsf{can}\colon \mathbb{C}\Omega_1 \otimes \Omega_2 \longrightarrow \mathbb{C}, \text{ def. of } \mathsf{coord}_\Omega \text{ in eq. (6.1.2)} \,]\!]$$

$$= \mathsf{can}\Bigg((\mathrm{P}_{\mathbb{C}\Omega_1} \otimes \mathrm{P}_{\mathbb{C}\Omega_2})\Bigg(\Big(\big(\pi_1(a_1 \cdot a_3)\Omega_1\big) \otimes \big(\pi_2(a_2 \cdot a_4)\Omega_2\big)\Big)\Bigg)\Bigg)$$

$$[\![\, \pi_1 \bowtie \pi_2 \text{ is morphism of algebras} \,]\!]$$

$$= \mathsf{coord}_{\Omega_1}\Big(\big(\pi_1(a_1 \cdot a_3)\Omega_1\big)\Big) \cdot \mathsf{coord}_{\Omega_2}\Big(\big(\pi_2(a_2 \cdot a_4)\Omega_2\big)\Big)$$

$$= 1 \cdot \varphi_1(a_1 \cdot a_3)\varphi_2(a_2 \cdot a_4) \quad [\![\, \text{eq. (6.1.14)} \,]\!].$$

If we compare this with equation (5.2.10), then we obtain α  $= 1$.

Now, we assume that $a_1 \in \mathscr{A}_1^{(2)}$, $a_2 \in \mathscr{A}_2^{(1)}$ and $a_3 \in \mathscr{A}_1^{(2)}$ and calculate

$$(\varphi_1 \odot_{bt} \varphi_2)(a_1 \cdot a_2 \cdot a_3 \cdot a_4)$$

$$= \mathsf{coord}_{\Omega_1\otimes\Omega_2}\Big(\big(\pi_1 \bowtie \pi_2\big)(a_1 \cdot a_2 \cdot a_3)(\Omega_1 \otimes \Omega_2)\Big)$$

$$= (\mathsf{can} \circ \mathrm{P}_{\mathbb{C}\Omega_1\otimes\Omega_2})\Big(\big(\pi_1 \bowtie \pi_2\big)(a_1 \cdot a_2 \cdot a_3)(\Omega_1 \otimes \Omega_2)\Big)$$

$$\llbracket \text{ iso. of vector spaces } \mathsf{can}\colon \mathbb{C}\Omega_1 \otimes \Omega_2 \longrightarrow \mathbb{C}, \text{ def. of } \mathsf{coord}_\Omega \text{ in eq. (6.1.2) } \rrbracket$$

$$= (\mathsf{can} \circ \mathsf{coord}_{\Omega_1 \otimes \Omega_2})\Bigg(\big(\pi_1 \bowtie \pi_2\big)(a_1)\Bigg(\Big(\mathrm{P}_{\mathbb{C}\Omega_1}\big(\pi_1(a_3)\Omega_1\big)\Big) \otimes \big(\pi_2(a_2)\Omega_2\big)\Bigg)\Bigg)$$

$$= \mathsf{coord}_{\Omega_1}\big(\pi_1(a_3)\Omega_1\big) \cdot \mathsf{can}\Bigg((\mathrm{P}_{\mathbb{C}\Omega_1} \otimes \mathrm{P}_{\mathbb{C}\Omega_2})\Bigg(\Big(\big(\pi_1(a_1)\Omega_1\big) \otimes \big(\pi_2(a_2)\Omega_{\varphi_2}\big)\Big)\Bigg)\Bigg)$$

$$= \varphi_1(a_1)\varphi_1(a_3)\varphi_2(a_2).$$

If we compare this with equation (5.2.10), then we obtain $\alpha_{\sqcap} = 0$. □

6.3.5 Remark. Let $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_\mathsf{m})$ and let $(\pi_i, \Omega_i, \hat{V}_i)$ be a pointed representation for the algebra $\mathscr{A}_i$ for each $i \in [2]$. Then, we define

$$\tilde{\gamma}_{\pi_1}^{(1)}\colon \begin{cases} \mathscr{A}_1^{(1)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \pi_1(a) \otimes \mathsf{id}_{V_2}, \end{cases} \tag{6.3.7}$$

$$\tilde{\gamma}_{\pi_2}^{(1)}\colon \begin{cases} \mathscr{A}_2^{(1)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \mathsf{id}_{V_1} \otimes \pi_2(a), \end{cases} \tag{6.3.8}$$

$$\tilde{\gamma}_{\pi_1}^{(2)}\colon \begin{cases} \mathscr{A}_1^{(2)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \pi_1(a) \otimes \mathrm{P}_{\mathbb{C}\Omega_2}, \end{cases} \tag{6.3.9}$$

$$\tilde{\gamma}_{\pi_2}^{(2)}\colon \begin{cases} \mathscr{A}_2^{(2)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2) \\ a \longmapsto \mathrm{P}_{\mathbb{C}\Omega_1} \otimes \pi_2(a). \end{cases} \tag{6.3.10}$$

By this we set

$$\pi_1 \tilde{\bowtie} \pi_2\colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \cong (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(1)} \sqcup (\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(2)} \longrightarrow \mathrm{Lin}(V_1 \otimes V_2), \\ a \longmapsto \Big(\underbrace{(\pi_1 \tilde{\bowtie} \pi_2)\restriction_{(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(1)}}}_{=:\tilde{\gamma}_{\pi_1}^{(1)} \sqcup \tilde{\gamma}_{\pi_2}^{(1)}} \sqcup \underbrace{(\pi_1 \tilde{\bowtie} \pi_2)\restriction_{(\mathscr{A}_1 \sqcup \mathscr{A}_2)^{(2)}}}_{=:\tilde{\gamma}_{\pi_1}^{(2)} \sqcup \tilde{\gamma}_{\pi_2}^{(2)}}\Big)(a). \end{cases} \tag{6.3.11}$$

It can be shown that we have a similar result as stated in Proposition 6.3.1 for $\pi_1 \tilde{\bowtie} \pi_2$, i. e., $\mathsf{vac}_{\Omega_1 \otimes \Omega_2}\big((\pi_1 \tilde{\bowtie} \pi_2)(a)\big)$ does not depend on the chosen representations whenever they satisfy equation (6.3.2). Furthermore, for each $i \in [2]$ we set for $\varphi_i \in \mathrm{Lin}(\mathscr{A}_i, \mathbb{C})$ and for any pointed representations $(\pi_i, \Omega_i, \hat{V}_i)$ for the algebra $\mathscr{A}_i$ which satisfy equation (6.3.4) for φ_i

$$\varphi_1 \odot_{tb} \varphi_2\colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathbb{C}, \\ a \longmapsto \mathsf{vac}_{\Omega_1 \otimes \Omega_2}\big((\pi_1 \tilde{\bowtie} \pi_2)(a)\big). \end{cases} \tag{6.3.12}$$

Analogously, it turns out that $\odot_{tb}$ is a positive and symmetric two-faced u.a.u.-product and $\odot_{tb} = \odot_{\mathsf{A}_\circ \mathsf{I}_\bullet}$.

6.3.6 Remark. We want to discuss the positivity of some partition induced universal products of the diagram of Figure 4.1. Throughout this discussion for each $i \in [2]$ we will use that $(\mathscr{A}_i, (\mathscr{A}_i^{(j)})_{j\in[2]}) \in \mathsf{Obj}(\mathsf{Alg}_2)$, $(\pi_i, \Omega_i, \hat{V}_i)$ is a pointed representation for the algebra $\mathscr{A}_i$ and furthermore use the definition $V_i := \mathbb{C}\Omega_i \oplus \hat{V}_i$ for each $i \in [2]$.

(a) It can be shown that $\odot_{\mathsf{Part}_{\{\circ,\bullet\}}}$ is positive, since we can define a representation product $\bowtie_t$, which uses the same prescription as the "tensor product" in the single-faced case ([Sch95, Sec. 1]) which is known to be positive. We can use the the same representation as in the single-faced case since we can ignore the faces of the algebra, i. e., we can define the algebra homomorphism $\pi_1 \bowtie_t \pi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Lin}(V_1 \otimes V_2)$ by

$$(\pi_1 \bowtie_t \pi_2)(a) = \begin{cases} \pi_1(a) \otimes \mathsf{id}_{V_1} & \text{for } a \in \mathscr{A}_1 \\ \mathsf{id}_{V_2} \otimes \pi_2(a) & \text{for } a \in \mathscr{A}_2. \end{cases} \tag{6.3.13}$$

Then, one can use a similar prescription as in equation (6.3.1) to define a product $\varphi_1 \odot_t \varphi_2$ induced by $\bowtie_t$. Then, one can show that $\odot_t$ is a positive and symmetric u.a.u.-product which is equal to the partition induced universal product $\odot_{\mathsf{Part}_{\{\circ,\bullet\}}}$.

(b) A similar construction holds for $\odot_{\mathsf{I}_{\{\circ,\bullet\}}}$. Here we take the representation of the boolean product in the single-faced case ([Sch95, Sec. 1]) which is also known to be positive. By neglecting faces of the algebras we define a representation $\pi_1 \bowtie_b \pi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Lin}(V_1 \oplus V_2)$

$$(\pi_1 \bowtie_b \pi_2)(a) = \begin{cases} \pi_1(a) \otimes \mathrm{P}_{\mathbb{C}\Omega_1} & \text{for } a \in \mathscr{A}_1 \\ \mathrm{P}_{\mathbb{C}\Omega_2} \otimes \pi_2(a) & \text{for } a \in \mathscr{A}_2, \end{cases} \tag{6.3.14}$$

which leads to a definition of a positive and symmetric u.a.u.-product in the sense of equation (6.3.1). This universal product equals the partition induced universal product $\odot_{\mathsf{I}_{\{\circ,\bullet\}}}$

(c) To show positivity for the partition induced universal product $\odot_{\mathsf{NC}_{\{\circ,\bullet\}}}$, we use the representation for the free product in the single-faced case . We set

$$\hat{V}_1 \sqcup \hat{V}_2 := \bigoplus_{\varepsilon \in \mathbb{A}([2])} \hat{V}_\varepsilon, \tag{6.3.15a}$$

$$\hat{V}_\varepsilon := \hat{V}_{\varepsilon_1} \otimes \cdots \otimes \hat{V}_{\varepsilon_n}. \tag{6.3.15b}$$

In preparation for the bi-free case, we list four vector space isomorphisms L_1, L_2, R_1, R_2 here ([MS17, Exa. 3])

$$(\mathbb{C}\Omega_1 \oplus \hat{V}_1) \otimes (\mathbb{C}\Omega_2 \oplus \bigoplus_{\substack{\varepsilon \in \mathbb{A}([2]),\\ \varepsilon_1 = 2}} \hat{V}_\varepsilon) \overset{L_1}{\cong} \mathbb{C} \oplus (\hat{V}_1 \sqcup \hat{V}_2), \tag{6.3.16a}$$

$$(\mathbb{C}\Omega_2 \oplus \hat{V}_2) \otimes (\mathbb{C}\Omega_1 \oplus \bigoplus_{\substack{\varepsilon \in \mathbb{A}([2]),\\ \varepsilon_1 = 1}} \hat{V}_\varepsilon) \overset{L_2}{\cong} \mathbb{C} \oplus (\hat{V}_1 \sqcup \hat{V}_2) \tag{6.3.16b}$$

and

$$(\mathbb{C}\Omega_2 \oplus \bigoplus_{\substack{\varepsilon \in \mathbb{A}([2]),\\ \varepsilon_n = 2}} \hat{V}_\varepsilon) \otimes (\mathbb{C}\Omega_1 \oplus \hat{V}_1) \overset{R_1}{\cong} \mathbb{C} \oplus (\hat{V}_1 \sqcup \hat{V}_2), \tag{6.3.17a}$$

$$(\mathbb{C}\Omega_1 \oplus \bigoplus_{\substack{\varepsilon\in\mathbb{A}([2]),\\ \varepsilon_n=1}} \hat{V}_\varepsilon) \otimes (\mathbb{C}\Omega_2 \oplus \hat{V}_2) \overset{R_2}{\cong} \mathbb{C} \oplus (\hat{V}_1 \sqcup \hat{V}_2). \tag{6.3.17b}$$

By this we define for each $i \in [2]$ representations of $\mathrm{Lin}(V_i)$ on $\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2$

$$\lambda_i \colon \begin{cases} \mathrm{Lin}(V_i) \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2), \\ T \longmapsto L_i(T \otimes \mathsf{id})L_i^{-1}, \end{cases} \tag{6.3.18}$$

$$\rho_i \colon \begin{cases} \mathrm{Lin}(W_i) \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2), \\ T \longmapsto R_i(\mathsf{id} \otimes T)R_i^{-1}. \end{cases} \tag{6.3.19}$$

Now, we can define a representation $\pi_1 \bowtie_f \pi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2)$ by

$$(\pi_1 \bowtie_f \pi_2)(a) = \begin{cases} \lambda_1\big(\pi_1(a)\big) & \text{for } a \in \mathscr{A}_1 \\ \lambda_2\big(\pi_2(a)\big) & \text{for } a \in \mathscr{A}_2 \end{cases} \tag{6.3.20}$$

which again leads to the definition of a positive and symmetric u.a.u.-product in the sense of equation (6.3.1) and equals the partition induced universal product $\odot_{\mathsf{NC}_{\{\circ,\bullet\}}}$.

(d) The bi-free product now incorporates the two representations λ_i and ρ_i from **(c)**. We can define $\pi_1 \bowtie_{\text{bi-}f} \pi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2)$ by

$$(\pi_1 \bowtie_{\text{bi-}f} \pi_2)(a) = \begin{cases} \lambda_1\big(\pi_1(a)\big) & \text{for } a \in \mathscr{A}_1^{(1)} \\ \rho_1\big(\pi_1(a)\big) & \text{for } a \in \mathscr{A}_1^{(2)} \\ \lambda_2\big(\pi_2(a)\big) & \text{for } a \in \mathscr{A}_2^{(1)} \\ \rho_2\big(\pi_2(a)\big) & \text{for } a \in \mathscr{A}_2^{(2)} \end{cases} \tag{6.3.21}$$

which is the representation of the well-known bi-free product (we have used the notation from [MS17, Ex. 3]). By taking care of all the involved vector space isomorphism it can be shown that this positive and symmetric u.a.u.-product satisfies α = 1 and α = 0 and by the Hasse diagram from Figure 4.1 we can conclude that the bi-free product equals the partition induced universal product $\odot_{\mathsf{biNC}}$.

(e) Next, we discuss the positivity of the partition induced universal product $\odot_{\mathsf{I}_\circ\mathsf{NC}_\bullet}$. We want to define a representation $\pi_1 \bowtie_{fb} \pi_2 \colon \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2)$. For this, we consider the remarks made in **(c)**. We have

$$\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2 = \underbrace{\mathbb{C} \oplus \bigoplus_{i\in[2]} \hat{V}_i}_{=:V_\uplus} \oplus \bigoplus_{\substack{\varepsilon\in\mathbb{A}([2]),\\ \mathsf{length}(\varepsilon)\geq 2}} \hat{V}_\varepsilon. \tag{6.3.22}$$

Then $\mathrm{P}_\uplus$ denotes the projection onto $V_\uplus$ with respect to the above direct sum and $\mathrm{P}_{\uplus,i} \colon V_\uplus \longrightarrow \mathbb{C}\Omega_i \oplus \hat{V}_i$ is the projection onto V_i for each $i \in [2]$. We define for each $i \in [2]$

$$\alpha_i \colon \begin{cases} \mathrm{Lin}(V_i) \longrightarrow \mathrm{Lin}(\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2), \\ T \longmapsto \mathsf{inc}_{V_i,\mathbb{C}\oplus\hat{V}_1\sqcup\hat{V}_2} \circ T \circ \mathrm{P}_{\uplus,i} \circ \mathrm{P}_\uplus \end{cases} \tag{6.3.23}$$

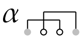
Now, we set

$$(\pi_1 \bowtie_{fb} \pi_2)(a) = \begin{cases} \lambda_1\big(\pi_1(a)\big) & \text{for } a \in \mathscr{A}_1^{(1)} \\ \alpha_1\big(\pi_1(a)\big) & \text{for } a \in \mathscr{A}_1^{(2)} \\ \lambda_2\big(\pi_2(a)\big) & \text{for } a \in \mathscr{A}_2^{(1)} \\ \alpha_2\big(\pi_2(a)\big) & \text{for } a \in \mathscr{A}_2^{(2)}. \end{cases} \tag{6.3.24}$$

This leads to the definition of the "free-boolean" product ([Liu19])

$$\varphi_1 \odot_{fb} \varphi_2 \colon \begin{cases} \mathscr{A}_1 \sqcup \mathscr{A}_2 \longrightarrow \mathbb{C}, \\ a \longmapsto \mathsf{coord}_{\mathbb{C}\Omega}\Big(\big(\pi_1 \bowtie_{fb} \pi_2\big)(a)\Omega\Big), \end{cases} \tag{6.3.25}$$

where $\Omega := (1,0)\mathbb{C} \oplus \hat{V}_1 \sqcup \hat{V}_2$ and $(\pi_i, \Omega_i, \hat{V}_i)$ are pointed representations which satisfy equation (6.3.4) for φ_i. Then, one can show that the highest coefficients for this positive and symmetric u.a.u.-product fulfill

α  $= 1,$ α  $= 0,$ α  $= 0,$ (6.3.26)

where we have used Convention 4.2.7 **(b)**. Thus, we can conclude from the Hasse diagram from Figure 4.1 and Lemma 5.2.24, that $\odot_{fb} = \odot_{\mathsf{NC}_\circ \mathsf{I}_\bullet}$. A similar construction holds for $\odot_{\mathsf{I}_\circ \mathsf{NC}_\bullet}$ by defining "boolean-free" product, where we swap the color indices in equation (6.3.24). One could also argue that the exponential $\exp_{\mathcal{P}}$ for $\mathcal{P} = \mathsf{NC}_\circ \mathsf{I}_\bullet$, defined in equation (3.4.49), matches the moment formula which is the inverse of the cumulant from [Liu19, Def. 7.1]. The two-sided inverse $\log_{\mathcal{P}}$ of $\exp_{\mathcal{P}}$ is unique and therefore matches the definition of the cumulant [Liu19, Def. 7.1]. From Theorem 2.4.8 we know that any u.a.u.-product is uniquely determined by its cumulants (logarithm) and by a Lie-bracket with respect to the operation $\boxplus$ which is trivial in this case since the universal products of consideration are symmetric. Positivity of the free-boolean product is proven in [LZ17, Sec. 7].

6.4 Tests for nonpositive universal products: first test

Since proving positivity for the remaining potential candidates of symmetric two-faced u.a.u.-products from the diagram of Figure 5.1 is not easy, in the first place it would be good to verify if the unclear candidates satisfy necessary conditions for positivity. Developing such a test method would allow us to exclude nonpositive products from the diagram of Figure 5.1. Thus, we would able to narrow down the list of candidates. In the next two sections we want to develop such test methods for the remaining candidates, whose positivity is unclear, and present our results.

In preparation for the next statements, we need the following definition.

6.4.1 Definition (Conditionally positive linear functional [BS05]). Let $\mathscr{D}$ be an algebra with involution. A linear functional $\varphi \in \mathrm{Lin}(\mathscr{D}, \mathbb{C})$ is said to be *conditionally positive* if and only if it is hermitian and

$$\forall b \in \mathscr{D} \colon \varphi(b^* b) \geq 0. \tag{6.4.1}$$

In this section we need a result about the so-called *Schoenberg correspondence* ([Sch93, Sec. 3.2], [Fra06, Cor. 4.13]). Theorem 2.3.14 is nearly the Schoenberg correspondence but we have left out any statements concerning positivity, although in its original source [BS05, Thm. 4.6] these statements can be found. In Theorem 2.3.14 we had a different focus in mind since we were seeking for a good definition of an exponential w. r. t. to a given universal product. Now, We want to catch up on the missing pieces for questions on positivity and therefore we define the following.

6.4.2 Definition (Schoenberg property for m-faced universal products [Ger21]). Let $m \in \mathbb{N}$. Let $\big((\mathcal{D}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta)\big)$ be an m-faced dual semigroup with involution $*$ such that Δ is a $*$-homomorphism and for each $i \in [m]$ the faces $\mathcal{D}^{(i)}$ are $*$-subalgebras. Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$. We say that $\odot$ is *Schoenberg* if and only if $\exp_\odot \psi$ is strongly positive for every conditionally positive linear functional $\psi \in \mathrm{Lin}(\mathcal{D}, \mathbb{C})$.

From [Ger21, Rem. 5.5] we can conclude that an m-faced u.a.u.-product is Schoenberg if and only if the *Schoenberg correspondence* holds, i. e.,

$$\psi \text{ is conditionally positive} \iff \forall t > 0\colon\ \exp_\odot(t\psi) \text{ is strongly positive.} \tag{6.4.2}$$

In [SV14] the Schoenberg correspondence has been proven for dual semigroups and u.a.u.-products in the single-faced setting. Using this result Gerhold could even extend this result to m-faced dual semigroups and more generally to unital associative (d, m)-universal products for $d \in \mathbb{N}$. We only need the case $d = 1$. Thus, we have:

6.4.3 Theorem ([Ger21, Thm. 5.8]). Let $m \in \mathbb{N}$. Let $\odot$ be a u.a.u.-product in the category $\mathsf{AlgP_m}$. If $\odot$ is positive, then $\odot$ is Schoenberg.

6.4.4 Remark. We notice that we did not assume any $\mathbb{N}_0$-grading for the m-faced dual semigroup $\mathcal{D}$ in Definition 6.4.2. Although, we have defined $\exp_\odot$ only for such $\mathbb{N}_0$-graded dual semigroups $\mathcal{D}$ with $\mathcal{D}_0 = \{0\}$ (Definition 2.4.1) we can extend the definition of $\exp_\odot$ in equation (2.4.1) and of $\log_\odot$ in equation (2.4.2) to arbitrary dual semigroups $\big((\mathcal{D}, (\mathcal{D}^{(i)})_{i\in[m]}), \Delta, 0\big)$ not necessarily $\mathbb{N}_0$-graded and with $\mathcal{D}_0 = \{0\}$. The reason for this possible generalization of the domain lies in the discussion provided in Remark 1.3.4. In our following considerations we will have a $\mathbb{N}_0$-graded dual semigroup, where the 0-th part of the $\mathbb{N}_0$-grading is $\{0\}$.

In preparation for the next lemma, we define the following. Assume we are given two one-dimensional vector spaces V_1 and V_2. Let us denote a basis vector of V_i by $x^{(i)}$ for each $i \in [2]$. Then, $\mathrm{T}(V_1 \oplus V_2)$ can be given the structure of a $\mathbb{N}_0$-graded two-faced dual semigroup with primitive comultiplication, where the 0-th part of the $\mathbb{N}_0$-grading is $\{0\}$ (Conv. 2.4.10). We want to equip $\mathbb{C}\langle x^{(1)}, x^{(2)}\rangle$ with a $*$-structure by assuming that $x^{(1)}$ and $x^{(2)}$ are self-adjoint. The primitive comultiplication then becomes a $*$-homomorphism of $*$-algebras. We identity $\mathbb{C}\langle x^{(1)}, x^{(2)}\rangle$ with $\mathrm{T}(V_1 \oplus V_2)$ and $\mathbb{C}_0\langle x^{(1)}, x^{(2)}\rangle$ (polynomial algebra with constant term) with $\mathrm{T}_0(V_1 \oplus V_2)$.

6.4.5 Lemma. Let $\odot$ be a positive u.a.u.-product in the category $\mathsf{AlgP_2}$. Let $\varphi \in \mathrm{Lin}\big(\mathrm{T}(V_1 \oplus V_2), \mathbb{C}\big)$ such that

$$\forall (\delta_1, \delta_2) \in ([2])^{\times 2}\colon\ \varphi(x^{(\delta_1)} \otimes x^{(\delta_2)}) = 1, \tag{6.4.3a}$$

$$\forall k \in \mathbb{N} \setminus \{2\},\ \forall (\delta_i)_{i\in[k]} \in ([2])^{\times k}\colon\ \varphi(x^{(\delta_1)} \otimes \cdots \otimes x^{(\delta_{i_k})}) = 0 \tag{6.4.3b}$$

and define the *standard $\odot$-Gaussian* as the unitization

$$G_\odot := \big(\exp_\odot \varphi\big)^{\mathbb{1}}\colon\ \mathrm{T}_0(V_1 \oplus V_2) \longrightarrow \mathbb{C}. \tag{6.4.4}$$

Then, $G_\odot$ is a positive linear functional.

Proof: Since $\varphi\colon \mathrm{T}(V_1 \oplus V_2) \longrightarrow \mathbb{C}$ is conditionally positive, the assertion follows from Theorem 6.4.3. □

Let $\odot$ be a positive u.a.u.-product in the category AlgP_2. Since the element $x^{(1)} + x^{(2)} \in \mathbb{C}\langle x^{(1)}, x^{(2)}\rangle$ is self-adjoint and the Gaussian $G_\odot$ is a positive linear functional, there needs to exist a unique probability measure μ such that

$$\forall\, k \in \mathbb{N}_0\colon\ \mathfrak{M}_k^{\odot} := G_\odot\big((x^{(1)} + x^{(2)})^k\big) = \int_{-\infty}^{\infty} t^k \,\mathrm{d}\mu(t). \tag{6.4.5}$$

This is basically a conclusion of the Riesz representation theorem and the Theorem of Stone-Weierstrass. Look at [NS06, Prop.3.13, Rem. 1.10] for further details on this topic. There is the following equivalent characterization for the existence of a probability measure, known as the Hamburger moment problem ([Akh65]).

6.4.6 Theorem (in this form stated in [Lac15, Thm. 7.3.1]). Let $(\mathfrak{M}_k)_{k\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$ be a sequence of real numbers.

Tfae: **(a)** There exists a probability measure μ such that

$$\forall\, k \in \mathbb{N}_0\colon\ \mathfrak{M}_k = \int_{\infty}^{\infty} t^k \,\mathrm{d}\mu(t). \tag{6.4.6}$$

(b) It holds

$$\forall\, k \in \mathbb{N}_0\colon \Delta_k > 0 \tag{6.4.7}$$

or

$$\exists\, k_0 \in \mathbb{N},\ \forall i \in \{0\} \cup [k_0 - 1],\ \forall j \in \mathbb{N}_0 \setminus (\{0\} \cup [k_0 - 1])\colon$$
$$(\Delta_i > 0) \wedge (\Delta_j = 0), \tag{6.4.8}$$

where Δ_k denotes the determinant of the $k+1$-dimensional Hankel Matrix

$$\forall\, k \in \mathbb{N}_0\colon H_k := \begin{pmatrix} \mathfrak{M}_0 & \mathfrak{M}_1 & \dots & \mathfrak{M}_k \\ \mathfrak{M}_1 & \mathfrak{M}_2 & \dots & \mathfrak{M}_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{M}_k & \mathfrak{M}_{k+1} & \dots & \mathfrak{M}_{2k}\,. \end{pmatrix} \tag{6.4.9}$$

Thus, we obtain the following test for potential positivity of a universal product. Assume that a given symmetric u.a.u.-product $\odot$ is positive, but does not satisfy the condition of Theorem 6.4.6 **(b)**, i.ė. the negation of Theorem 6.4.6 **(b)** is true. This implies that Theorem 6.4.6 **(a)** is also not satisfied. Since we have assumed that the universal product $\odot$ is positive, we obtain that equation (6.4.5) holds. But this is a contradiction to Theorem 6.4.6 **(a)**. Thus, $\odot$ can not be positive. In this way we may be in the position to find out which symmetric u.a.u.-products from the diagram of Figure 5.1 are not positive and maybe we can find further restrictions for the deformation parameters $0 < |q| \leq 1$ for instance for positive and symmetric u.a.u.-products $\odot$ with $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$ or $\mathfrak{P}_\odot = \mathsf{biNC}$.

In this sense, our task is to check the existence of a "small" natural number $n \in \mathbb{N}$ such that equations of Theorem 6.4.6 **(b)** are violated for $(\mathfrak{M}_k)_{k\in[n]} := (\mathfrak{M}_k^{\odot})_{k\in[n]}$ for various choices of deformation parameters of certain potential positive and symmetric two-faced u.a.u.-products and for various choices of partition induced universal products $\odot$ which appear in the list of the diagram of Figure 5.1. We shall discuss what we mean by a sufficient small natural number after equation (6.4.10). By equation (2.5.29), Convention 5.2.10 and equation (6.4.3) we obtain the following for any positive and symmetric two-faced u.a.u.-product

$$G_{\odot_{\mathcal{P}}}\big((x^{(1)} + x^{(2)})^n\big) = \begin{cases} \sum\limits_{\delta\in[2]^{\times n}} \sum\limits_{\pi\in\mathsf{Pair}(\mathcal{P}_\delta)} \alpha_\pi & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{6.4.10}$$

Herein, $\mathsf{Pair}(\mathcal{P}_\delta)$ denotes the so-called *pair partitions* of $\mathcal{P}_\delta$. A pair partition is a partition such that each block in this partition has 2 legs. We have performed calculations to test the conditions of Theorem 6.4.6 **(b)** for all $k \in [n]$ where $n \leq 5$ (this is what we mean by small). This involves the computation of the explicit values of equation (6.4.10), i. e., to "count" pair partitions and weight them with a possible occurring deformation parameter. We have done that using the software `Wolfram Mathematica 7.0`. Here we present the results of our calculations. The source code can be obtained from our `GitHub` repository (`https://github.com/varsop/positivity`). In all the following procedures the value for the k-th moment can be adjusted in the source code. So, if we execute one of the `Mathematica` files, then the program does not prompt for input but instead needs to be adjusted manually in the source code. This is not hard to do. By default it is set to $k = 5$.

First, we consider $\odot_{\mathsf{pureNC}}$. Then, by application of the procedure provided by the `Mathematica` file `test1_pureNC.m` we obtain

$$(\Delta_i)_{i\in\{0,\dots,5\}} = (1, 4, 32, 512, 16384, 1048576) \tag{6.4.11}$$

and for the moments

$$(\mathfrak{M}_i)_{i\in\{0,\dots,10\}} = (1, 0, 4, 0, 24, 0, 160, 0, 1120, 0, 8064). \tag{6.4.12}$$

So, at this point $\odot_{\mathsf{pureNC}}$ does not produce any contradictions to positivity and it still might be the case that $\odot_{\mathsf{pureNC}}$ is positive.

Next, we consider $\odot_{\mathsf{NC}_\circ\mathsf{A}_\bullet}$ and by symmetry of the colors likewise $\odot_{\mathsf{A}_\circ\mathsf{NC}_\bullet}$. By application of the procedure provided by the `Mathematica` file `test1_NCA.m` we obtain

$$(\Delta_i)_{i\in\{0,\dots,5\}} = (1, 4, 48, 1920, 320000, 278400000) \tag{6.4.13}$$

and for the moments

$$(\mathfrak{M}_i)_{i\in\{0,\dots,10\}} = (1, 0, 4, 0, 28, 0, 236, 0, 2232, 0, 23204). \tag{6.4.14}$$

With more patience or better computational power we could determine even higher moments. But at this point the universal products $\odot_{\mathsf{NC}_\circ\mathsf{A}_\bullet}$ and $\odot_{\mathsf{A}_\circ\mathsf{NC}_\bullet}$ do not lead to any contradiction of positivity. So, both products might be positive.

Next, we consider $\odot_{\mathsf{pureC}}$. By application of the procedure provided by the `Mathematica` file `test1_pureC.m` we obtain

$$(\Delta_i)_{i\in\{0,\dots,5\}} = (1, 4, 64, 4096, 1179648, 278400000) \tag{6.4.15}$$

and for the moments

$$(\mathfrak{M}_i)_{i\in\{0,\ldots,10\}} = (1, 0, 4, 0, 32, 0, 320, 0, 3616, 0, 44544). \tag{6.4.16}$$

Again, it still might be the case that higher moments $\mathfrak{M}_k$ of $\odot_{\mathsf{pureC}}$ may violate condition Theorem 6.4.6 **(b)** but at this point we do not observe any contradiction to positivity. Thus, $\odot_{\mathsf{pureC}}$ might be positive.

Next, we consider positive and symmetric two-faced u.a.u.-products $\odot$ such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$. Let $q := \alpha \in \mathbb{C}$. In Proposition 5.2.26 **(a)** we have already seen that $q \in \mathbb{C}$ needs to satisfy $0 < |q| \leq 1$ for any positive and symmetric two-faced u.a.u.-product such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$. We can look for further restrictions for the complex parameter q by our above described method. We emphasize that in this case any pair partition is a polynomial $p(q, q^*) \in \mathbb{C}[q, q^*]$. For instance we have

$$\alpha = |q|^2, \qquad \alpha = q. \tag{6.4.17}$$

We implemented a routine which takes account of this fact and determines the polynomial $p(q, q^*) \in \mathbb{C}[q, q^*]$ for a given pair partition of $\mathsf{NC}_{\{\circ,\bullet\}}$. Here are our results for this procedure provided by the `Mathematica` file `test1_NC2.m`

$$\begin{aligned}&(\Delta_i)_{i\in\{0,\ldots,5\}}\\ &\quad= \Big(1, 4, 32\big(1+\mathrm{Re}(q)\big), 512(1+\mathrm{Re}(q))^3, 16384\big(2+\mathrm{Re}(q)\big)^6, 1048576\big(1+\mathrm{Re}(q)\big)^{10}\Big)\end{aligned} \tag{6.4.18}$$

The necessary condition $0 < |q| \leq 1$ is stronger than the restriction we obtain in order to ensure that conditions from Theorem 6.4.6 **(b)** are fulfilled till $k = 5$. The moments $\mathfrak{M}_k$ are not very enlightening because these are somehow lengthy polynomials $p(q, q^*) \in \mathbb{C}[q, q^*]$. But again we can check if we get further restrictions for q by demanding that moments $\mathfrak{M}_k$ need to be real and $|q| \leq 1$ or if both conditions contradict each other for certain choices of q. But we did not obtain any further restrictions for q. As a special case we give here the values when q is the imaginary unit, i. .e., $q = \mathrm{i}$

$$(\Delta_i)_{i\in\{0,\ldots,5\}} = (1, 4, 32, 512, 16384, 1048576) \tag{6.4.19}$$

and for the moments

$$(\mathfrak{M}_i)_{i\in\{0,\ldots,10\}} = (1, 0, 4, 0, 24, 0, 160, 0, 1120, 0, 8064). \tag{6.4.20}$$

For the calculations of $(\Delta_i)_{i\in\{0,\ldots,k\}}$ and $(\mathfrak{M}_i)_{i\in\{0,\ldots,2k\}}$ up to $k \leq 5$ we can say that any positive and symmetric two-faced u.a.u.-product $\odot$ such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$ necessarily needs to satisfy $0 < |q| \leq 1$. We have discussed positivity for $q = 1$ in Remark 6.3.6 **(c)**.

At last, we look at positive and symmetric two-faced u.a.u.-products $\odot$ such that $\mathfrak{P}_\odot = \mathsf{biNC}$. In Proposition 5.2.26 **(b)** we have already seen that any such product needs to satisfy $0 < |q| \leq 1$, where $q := \alpha \in \mathbb{C}$. We have computed the Hankel Matrices Δ_k for $k \in \{0, \ldots, 5\}$. It is not worth to put them here because they are somehow unattractive lengthy polynomials $p(q, q^*) \in \mathbb{C}[q, q^*]$. The same holds for the moments. By our method we did not get further restrictions than $0 < |q| \leq 1$. As a special case we give here the values for $q = \mathrm{i}$ obtained from the procedure provided by the `Mathematica` file `test1_biNC.m`

$$(\Delta_i)_{i\in\{0,\ldots,5\}} = (1, 4, 32, 0, 0, 0) \tag{6.4.21}$$

and for the moments

$$(\mathfrak{M}_i)_{i\in\{0,\dots,10\}} = (1,0,4,0,24,0,144,0,864,0,5184). \tag{6.4.22}$$

For the calculations of $(\Delta_i)_{i\in\{0,\dots,k\}}$ and $(\mathfrak{M}_i)_{i\in\{0,\dots,2k\}}$ up to $k \leq 5$ we can say that any positive and symmetric two-faced u.a.u.-product $\odot$ such that $\mathfrak{P}_\odot = \mathsf{biNC}$ necessarily needs to satisfy $0 < |q| \leq 1$. We have discussed positivity for $q = 1$ in Remark 6.3.6 **(d)**.

We have not yet implemented any tests for positive and symmetric two-faced u.a.u.-products $\odot$ such that $\mathfrak{P}_\odot = \mathsf{Part}_{\{\circ,\bullet\}}$.

6.5 Tests for nonpositive universal products: second test

We build upon the setting introduced in Section 6.4. For a second test we again exploit the fact that the Gaussian functional $G_\odot$ is a positive linear functional on the two-faced algebra $\mathbb{C}_0\langle x^{(1)}, x^{(2)}\rangle$ whenever $\odot$ is assumed to be a positive (and symmetric) two-faced u.a.u.-product. Let $n = 3$. Then, we set

$$\mathfrak{D} := \left\{ (\delta_i)_{i\in[3]} \;\middle|\; \begin{array}{l} \forall\, i \in [3]\colon (\delta_i = \emptyset) \vee (\delta_i = 0) \\ \vee(\exists\, n_i \in [n]\colon \delta_i = (\delta_{i,j})_{j\in[n_i]} \in \{1,2\}^{\times n_i}) \end{array} \right\} \tag{6.5.1}$$

and

$$x^{\delta_i} = \begin{cases} 0 & \text{for } \delta_i = \emptyset \\ \mathbb{1} & \text{for } \delta_i = 0 \\ \displaystyle\prod_{j=1}^{n_i} x^{(\delta_{i,j})} & \text{for } \exists\, n_i \in [n]\colon \delta_i = (\delta_{i,j})_{j\in[n_i]} \in \{1,2\}^{\times n_i}. \end{cases} \tag{6.5.2}$$

Furthermore, for any $\delta \in \mathfrak{D}$ we set

$$y_\delta = x^{\delta_1} + x^{\delta_2} + x^{\delta_3} \in \mathbb{C}_0\langle x^{(1)}, x^{(2)}\rangle. \tag{6.5.3}$$

We want to describe the definition of y_δ less formal. We can think of the expression y_δ as a linear combination of maximal 3 "words" in $\mathbb{C}_0\langle x^{(1)}, x^{(2)}\rangle$, where each word can be the empty word, the unit $\mathbb{1}$ of $\mathbb{C}_0\langle x^{(1)}, x^{(2)}\rangle$ or a word of maximal length three in $\mathbb{C}\langle x^{(1)}, x^{(2)}\rangle$.

Moreover, for any given $\delta \in \mathfrak{D} \setminus \{(\emptyset,\emptyset,\emptyset)\}$ there exists $\tilde{n} \in [n]$ and $\tilde{n}$-tuple $(\tilde{\delta}_i)_{i\in[\tilde{n}]}$ such that

$$\sum_{i=1}^{n} x^{\delta_i} = y_\delta = \sum_{i}^{\tilde{n}} x^{\tilde{\delta}_i} \tag{6.5.4}$$

and $\forall\, i \in [\tilde{n}]\colon \tilde{\delta}_i \neq \emptyset$. If we assume that $\odot$ is a positive and symmetric u.a.u.-product, then the matrix

$$\mathcal{G}_\delta := \big((\mathcal{G}_\delta)_{ij}\big)_{i,j\in[\tilde{n}]} := \Big(G_\odot\Big((x^{\tilde{\delta}_i})^* x^{\tilde{\delta}_j}\Big)\Big)_{i,j\in[\tilde{n}]} \in M_{\tilde{n}}(\mathbb{C}) \tag{6.5.5}$$

is positive semidefinite. By equation (2.5.29), Convention 5.2.10 and equation (6.4.3) we obtain the following for any positive and symmetric two-faced u.a.u.-product or partition induced universal product $\odot_{\mathcal{P}}$

$$\left(G_{\odot}\left((x^{\tilde{\delta}_i})^* x^{\tilde{\delta}_j}\right)\right)_{i,j\in[\tilde{n}]} = \begin{cases} 1 & \text{for } (\tilde{\delta}_i)_{i\in[\tilde{n}]} = (\tilde{\delta}_j)_{j\in[\tilde{n}]} = 0 \\ \displaystyle\sum_{\pi\in\mathsf{Pair}(\mathcal{P}_{(\tilde{\delta}_j)_{j\in[\tilde{n}]}})} \alpha_\pi & \text{for } (\tilde{\delta}_i)_{i\in[\tilde{n}]} = 0, (\tilde{\delta}_j)_{j\in[\tilde{n}]} \neq 0 \\ \displaystyle\sum_{\pi\in\mathsf{Pair}(\mathcal{P}_{\mathsf{Rev}((\tilde{\delta}_i)_{i\in[\tilde{n}]})})} \alpha_\pi & \text{for } (\tilde{\delta}_i)_{i\in[\tilde{n}]} \neq 0, (\tilde{\delta}_j)_{j\in[\tilde{n}]} = 0 \\ \displaystyle\sum_{\pi\in\mathsf{Pair}(\mathcal{P}_\sigma)} \alpha_\pi & \text{for } (\tilde{\delta}_i)_{i\in[\tilde{n}]} \neq 0 \neq (\tilde{\delta}_j)_{j\in[\tilde{n}]}, \end{cases} \tag{6.5.6}$$

wherein $\mathsf{Rev}((\tilde{\delta}_i)_{i\in[\tilde{n}]})$ denotes the tuple $(\tilde{\delta}_i)_{i\in[\tilde{n}]}$ in reversed order and σ denotes the tuple we obtain if we append the tuple $(\tilde{\delta}_j)_{j\in[\tilde{n}]}$ after the reversed tuple $\mathsf{Rev}((\tilde{\delta}_i)_{i\in[\tilde{n}]})$. By equation (6.5.6), we can calculate the values of the matrix of equation (6.5.5). We have done that using the software `Wolfram Mathematica 7.0`. This provides another test for violation of positivity for a certain two-faced symmetric u.a.u.-product $\odot$ under the assumption that $\odot$ is positive. If we encounter any contradiction to the positive semidefiniteness of the matrix $\mathcal{G}_\delta$ for a given "decoration tuple" $\delta \in \mathfrak{D}$, then $\odot$ can not be positive. In the following we discuss our results for partition induced universal products and certain positive and symmetric u.a.u.-products depending on a deformation parameter $0 < |q| \leq 1$ from the diagram of Figure 5.1.

We consider the partition induced universal products $\{\odot_{\mathsf{pureNC}}, \odot_{\mathsf{NC_\circ A_\bullet}}, \odot_{\mathsf{A_\circ NC_\bullet}}, \odot_{\mathsf{pureC}}\}$. Our procedures provided by the `Mathematica` files

- `test2_pureNC.m` for $\odot_{\mathsf{pureNC}}$,
- `test2_NCA.m` for $\odot_{\mathsf{NC_\circ A_\bullet}}$ (and by symmetry of the colors likewise for $\odot_{\mathsf{A_\circ NC_\bullet}}$),
- `test2_pureC.m` for $\odot_{\mathsf{pureC}}$

calculates for each $\delta \in \mathfrak{D}$ the value of the matrix $\mathcal{G}_\delta$, determines its eigenvalues and checks if all eigenvalues are equal to or greater than zero. The result is that $\mathcal{G}_\delta$ is positive semidefinite for all $\delta \in \mathfrak{D}$. So, up to this point $\{\odot_{\mathsf{pureNC}}, \odot_{\mathsf{NC_\circ A_\bullet}}, \odot_{\mathsf{A_\circ NC_\bullet}}, \odot_{\mathsf{pureC}}\}$ do not produce any contradictions to positivity and it still might be the case that these partition induced universal products are positive.

Next, we consider positive and symmetric two-faced u.a.u.-products $\odot$ such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$. Let $q := \alpha_{\sqcap\!\sqcap} \in \mathbb{C}$. In Proposition 5.2.26 **(a)** we have already seen that $q \in \mathbb{C}$ needs to satisfy $0 < |q| \leq 1$ for any positive and symmetric two-faced u.a.u.-product such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$. We can look for further restrictions for the complex parameter q by our above described method. In this case there exist tuples $\delta \in \mathfrak{D}$ such that $\mathcal{G}_\delta$ depends on $q \in \mathbb{C}$. So in general, we were not able to retrieve all eigenvalues for such matrices, in particular for such matrices where the degree of the characteristic polynomial is greater than four. Instead we ask for an implication of a positive semidefinite matrix and check the value of its determinant. Our procedure provided by the `Mathematica` file `test2_NC2.m` calculates for each $\delta \in \mathfrak{D}$ the value of the matrix $\mathcal{G}_\delta$ and determines its determinant. We use built-in functions of `Wolfram Mathematica 7.0` to find the possible cases for $q \in \mathbb{C}$ such that $\forall\,\delta \in \mathfrak{D}\colon \det(\mathcal{G}_\delta) \geq 0$ and $0 < |q| \leq 1$. Our procedure returned $0 < |q| \leq 1$. Also from our second test we can say that any positive and symmetric two-faced u.a.u.-product $\odot$ such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$ necessarily needs to satisfy $0 < |q| \leq 1$. The value $q = 1$ is also sufficient (Remark 6.3.6 **(c)**).

At last, we look at positive and symmetric two-faced u.a.u.-products $\odot$ such that $\mathfrak{P}_\odot = \mathsf{biNC}$. In Proposition 5.2.26 **(b)** we have already seen that $q \in \mathbb{C}$ needs to satisfy $0 < |q| \leq 1$ for any positive and symmetric two-faced u.a.u.-product such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$. Like in the case $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$ we did not check an equivalent characterization for positive semidefiniteness. Instead, our procedure provided by the `Mathematica` file `test2_biNC.m` calculates for each $\delta \in \mathfrak{D}$ the value of the matrix $\mathcal{G}_\delta$ and determines its determinant. We use built-in functions of `Wolfram Mathematica 7.0` to find the possible cases for $q \in \mathbb{C}$ such that $\forall\,\delta \in \mathfrak{D}$: $\det(\mathcal{G}_\delta) \geq 0$ and $0 < |q| \leq 1$. Our procedure returned $0 < |q| \leq 1$. Also from our second test we can say that any positive and symmetric two-faced u.a.u.-product $\odot$ such that $\mathfrak{P}_\odot = \mathsf{NC}_{\{\circ,\bullet\}}$ necessarily needs to satisfy $0 < |q| \leq 1$. The value $q = 1$ is sufficient. (Remark 6.3.6 **(d)**).

The above used two test methods show that symmetric u.a.u.-products $\odot_{\mathcal{P}}$ for $\mathcal{P} \in \{\mathsf{pureNC}, \mathsf{NC}_\circ\mathsf{A}_\bullet, \mathsf{A}_\circ\mathsf{NC}_\bullet, \mathsf{pureC}\}$ and q-deformed u.a.u.-products $\odot$ such that $\mathfrak{P}_\odot \in \{\mathsf{NC}_{\{\circ,\bullet\}}, \mathsf{biNC}\}$ for $0 < |q| \leq 1$ satisfy certain necessary conditions for positivity. We think that it might be worth to try to develop techniques which are strong enough to prove positivity for the remaining candidates of the diagram of Figure 5.1, where positivity is unclear. It seems promisingly, that recent progress for proving positivity of the three q-deformed symmetric two-faced u.a.u.-products of the diagram of Figure 5.1 is contained in [HGU21].

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