On Several Problems in the Theory of Comonoidal Systems and Subproduct Systems

(*Über einige Fragestellungen in der Theorie der komonoidalen Systeme und Subproduktsysteme*)

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1 Introduction

In a very general sense, this thesis deals with quantum stochastic evolutions. One can say that the theory of such evolutions is well developed, but actually there are many different mathematical theories which were inspired by the questions, how quantum mechanical systems evolve in time, some of them can be thought of as physical models, others are difficult to find in nature, but turned out to have surprisingly fruitful applications inside mathematics. The kind of "evolutions" we will study look quite different at first sight, but they have in common a certain kind of *stationarity.* To formulate this similarity rigorously we need to deal with category theory, specifically *tensor categories* or *monoidal categories*. The categorial objects which have the desired stationarity are called *comonoidal systems*.

The prototype of a stationary stochastic evolution is a classical Lévy process, which consists of real valued random variables $X_t, t \geq 0$, with the following properties:

(LP1) $X_0 \sim \delta_0$.

(LP2) The *increments* $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for all $0 \leq t_0 < t_1 < \cdots < t_n$.

(LP3) The increments are *stationary*, that is $(X_t - X_s) \sim X_{t-s}$ for all $s \leq t$.

Lévy processes play a fundamental role in probability theory, as they are the building blocks of stochastic calculus. With the Brownian motion and the Poisson process as examples they are also highly important in physics. It is well known that a Lévy process can be reconstructed from its convolution semigroup of 1-dimensional distributions. Given any convolution semigroup of probability measures on $\mathbb{R}$, one can construct a projective system of probability spaces $(\mathbb{R}^n, \mathcal{B}^n, \mu_{t_1} \otimes \cdots \otimes \mu_{t_n})$, which has a projective limit by the Daniell-Kolmogoroff theorem. If $X_t \sim \mu_t$, the projective limit is stochastically equivalent to the Lévy process $(X_t)_{t \in \mathbb{R}_+}$.

The key idea of quantum probability is as follows: First, express a probabilistic notion in terms of a (commutative) algebra of functions on a probability space without an explicit reference to the elements of the probability space. Then allow for general, not necessarily commutative algebras. For example, $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is a commutative von Neumann algebra and $\Phi(f) := \mathbb{E}(f) = \int f \mathrm{d}\mathbb{P}$ is a normal state. This led to the definition of a von Neumann quantum probability space as a pair $(\mathcal{A}, \Phi)$ with $\mathcal{A}$ a (not necessarily commutative) von Neumann algebra and Φ a normal state. There are also other, more algebraic, notions of quantum probability spaces. The most general one means just an algebra with a linear functional. In fact, many interesting questions such as moments, cumulants, quantum stochastic independence, quantum Lévy processes, quantum stochastic integration, et cetera can be discussed at this level. We will also adopt this notion of a quantum probability space meaning simply a pair $(\mathcal{A}, \Phi)$ with $\mathcal{A}$ an algebra and Φ a linear functional. When we need additional structure, for example an involution or a unit, we will explicitly say so and speak of $*$-algebraic quantum probability spaces, unital quantum probability spaces, and so on.

The history of quantum Lévy processes starts with von Waldenfels' work on light emission and absorption, which he describes as a quantum stochastic process with stationary and independent multiplicative increments on a non-commutative version of the coefficient algebra of the unitary group $U(2)$ [vW84]. Later, Schürmann developed a comprehensive theory of quantum Lévy processes on $*$-bialgebras [Sch93]. A $*$-bialgebra is a unital $*$-algebra $\mathcal{A}$ together with a coassociative comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and a counit $\delta : \mathcal{A} \to \mathbb{C}$ which are unital $*$-algebra homomorphisms. In a $*$-bialgebra the convolution of linear functionals can be defined as $\varphi_1 \star \varphi_2 := (\varphi_1 \otimes \varphi_2) \circ \Delta$. A convolution semigroup on $\mathcal{A}$ is a family of linear functionals $\varphi_t : \mathcal{A} \to \mathbb{C}$ such that $\varphi_s \star \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}_+ := [0, \infty)$. Schürmann proves that quantum Lévy processes on $\mathcal{A}$ are in 1-1-correspondence with continuous convolution semigroups of states on $\mathcal{A}$. The construction of a quantum Lévy process from a convolution semigroup is in a sense dual to the classical construction mentioned above and involves inductive limits instead of projective limits. Note that when the φ_t form a convolution semigroup, then Δ can be seen as a functional preserving homomorphism from $(\mathcal{A}, \varphi_{s+t})$ to $(\mathcal{A} \otimes \mathcal{A}, \varphi_s \otimes \varphi_t)$. By iteration of Δ, we can define more general maps, which turn the family formed by the quantum probability spaces $(\mathcal{A}^{\otimes n}, \varphi_{t_1} \otimes \cdots \otimes \varphi_{t_n})$ into an inductive system. An analogue of the Daniell-Kolmogoroff theorem for quantum probability spaces guarantees the existence of an inductive limit. In case the φ_t form a continuous convolution semigroup of states, the inductive limit can be interpreted as a quantum Lévy process in the sense of [Sch93]. The sketched construction goes back to Accardi, Schürmann and von Waldenfels [ASvW88]. Up to now, quantum Lévy processes are an active area of research. Schürmann's theory of quantum Lévy processes on $*$-bialgebras can be applied in the study of compact quantum groups in the sense of Woronowicz, since a compact quantum group contains a dense sub-$*$-bialgebra [Wor87, Wor98]. Schürmann and Skeide were able to classify all quantum Lévy processes on Woronowicz' $SU_q(2)$, thus establishing a Lévy-Khintchine type result for a quantum group [SS98]. Recently, Franz, Kula, Lindsay and Skeide proved similar theorems for $SU_q(N)$ and $U_q(N)$ [FKLS14]. In the work of Lindsay and Skalski [LS05, LS08, LS12] a rich topological theory of quantum Lévy processes on C^*-bialgebras is developed, which can be used, for example, to study quantum Lévy processes on locally compact quantum groups. All quantum Lévy processes mentioned so far have independent increments with respect to the so-called *tensor independence*. There are other generalizations of stochastic independence in the quantum world, for example freeness or Boolean independence. These come from *universal products*, which are prescriptions to calculate mixed moments of pairs of random variables, and modelled on free products instead of tensor products. The reconstruction of processes that are Lévy processes with respect to one of these independences from their convolution semigroups works similar; see [BGS05] and [Vos13, Section2.2.2].

A *tensor category* (or *monoidal category*) is, roughly speaking, a category $\mathcal{C}$ with an essentially associative multiplication $\boxtimes$ (of objects as well as of morphisms) and an essentially neutral object E. We postpone a precise definition, in which we explain precisely what we mean by "essentially", to Section 2.2. Instead, we provide the examples that play a role in this thesis. The reader unfamiliar with tensor categories can simply view the term as standing for any of the examples. The name comes from the usual tensor product of linear algebra, which turns the category of complex vector spaces into a tensor category. The tensor product is associative in the sense that there are canonical isomorphisms $(\mathcal{V}_1 \otimes \mathcal{V}_2) \otimes \mathcal{V}_3 \cong \mathcal{V}_1 \otimes (\mathcal{V}_2 \otimes \mathcal{V}_3)$. These isomorphisms form a *natural isomor-*

phism in the sense of Section 2.1.1, that is they fulfill a certain compatibility condition with the corresponding morphisms, here linear maps. The neutral object is the ground field $\mathbb{C}$ viewed as a vector space over itself. Again, this means the existence of canonical isomorphisms $\mathbb{C} \otimes \mathcal{V} \cong \mathcal{V} \cong \mathcal{V} \otimes C$, which form natural transformations. The tensor product of two algebras $\mathcal{A}_1, \mathcal{A}_2$ is again an algebra with respect to the multiplication $(a_1 \otimes a_2)(b_1 \otimes b_2) := a_1 b_1 \otimes a_2 b_2$. The obtained tensor product of algebras turns the category of algebras into a tensor category. The tensor product of unital algebras with unit elements $\mathbb{1}_1$ and $\mathbb{1}_2$ respectively is again a unital algebra with unit element $\mathbb{1}_1 \otimes \mathbb{1}_2$. The unital algebras form a tensor category with respect to this product as well. Similarly, one can construct tensor categories of coalgebras, bialgebras and Hopf algebras. The tensor product of Hilbert spaces is another main example. The neutral object is the one dimensional Hilbert space $\mathbb{C}$. Examples quite different from the preceding ones are the tensor categories of (quantum) probability spaces. The tensor product of classical probability spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$, $(\Omega_2, \mathcal{F}_2, \mu_2)$ is defined as $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ and we get a tensor category. Tensor categories of quantum probability spaces are closely related to quantum stochastic independence. In contrast to classical probability, there are different notions of independence in quantum probability and, correspondingly, different structures of a tensor category. Recall that the *free product* $\mathcal{A}_1 \sqcup \mathcal{A}_2$ of two algebras $\mathcal{A}_i$ is determined by the universal property, that there are embeddings $\iota_i : \mathcal{A}_i \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ and for any algebra $\hat{\mathcal{A}}$ and algebra homomorphisms $f_i : \mathcal{A}_i \to \hat{\mathcal{A}}$ there exists a unique algebra homomorphism $f_1 \sqcup f_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \hat{\mathcal{A}}$ with $(f_1 \sqcup f_2) \circ \iota_i = f_i$, or in other words, the free product is the *coproduct* in the category of algebras. Of most interest are such tensor products on the category of quantum probability spaces for which the algebra of the product space as well as the product of morphisms are given by the free product. Five such products are well known in quantum probability: The tensor product, the free product (of quantum probability spaces), the Boolean product, the monotone and the anti-monotone product. In this case, the neutral object is always the zero algebra with zero functional. Of course it is possible to construct other tensor products of quantum probability spaces. But under mild assumptions one can show that these are always quotients of tensor products build on the free product of algebras. The categorial aspects of independence have been worked out by Franz [Fra06]. They also led him to the definition of independence in a purely categorial framework; see Section 2.2.1.

In Chapter 2, we introduce the abstract notion of a *comonoidal system*; see Definition 2.3.1. Let $(\mathcal{C}, \boxtimes, E)$ be a tensor category (or simply one of the examples of the preceding paragraph) and $\mathbb{S}$ a monoid. A comonoidal system consists of objects $(A_t)_{t\in\mathbb{S}}$ with embeddings of A_{s+t} into $A_s \boxtimes A_t$ that iterate coassociatively. The monoid $\mathbb{S}$ is usually $\mathbb{N}_0$ (discrete case) or $\mathbb{R}_+$ (continuous-time case). But sometimes other examples play a role, too. We consider the continuous-time case. Let J_t be the system of all finite subsets of $(0, t)$. Then J_t is a directed set with respect to the subset relation. For $I \in J_t$, $I = \{t_1, \ldots, t_n\}$ with $t_1 < t_2 < \cdots < t_n$ we put $A_I := A_{t_1} \boxtimes A_{t_2-t_1} \boxtimes \cdots \boxtimes A_{t-t_n}$. Then the $(A_I)_{I\in J_t}$ naturally form an inductive system. If the inductive limit $\mathcal{A}_t$ exists for every t, the obtained family $(\mathcal{A}_t)_{t\in\mathbb{R}_+}$ is again a comonoidal system referred to as *generated system*. Often, the generated system plays an important role. In the category of Hilbert spaces, one arrives at the famous *Arveson systems*, see below. In the case of quantum probability spaces, one can use the construction to build a quantum Lévy process from a convolution semigroup, see [ASvW88] for an early example. This involves taking a second inductive limit of the $\mathcal{A}_t$ over $\mathbb{R}_+$. The main aims of the chapter are

- ▶ To provide a general concept of stationarity encompassing ordinary convolution semigroups as well as the structures appearing in the following chapters, additive deformations, subproduct systems, and convolution semigroups for other noncommutative independences.
- ▶ To perform the typical inductive limit constructions at this general level.
- ▶ Find general conditions for the monoids which can be used instead of $\mathbb{R}_+$.
- ▶ Find general categorial conditions on the tensor category $\mathcal{C}$ for the inductive limits to have good properties known from the examples of Arveson systems and Lévy processes.

Let $\mathcal{C}$ be an inductively complete tensor category, that is all inductive systems possess inductive limits. Then we find the following results:

- ▶ If $\mathbb{S}$ is a *unique factorization (uf-) monoid* (see Definition 2.3.5), we can perform the first inductive limit $(A_t)_{t\in\mathbb{S}} \mapsto (\mathcal{A}_t)_{t\in\mathbb{S}}$ and $(\mathcal{A}_t)_{t\in\mathbb{S}}$ is again a comonoidal system;

see Theorem 2.3.14

- If the tensor product *preserves inductive limits*, then $\mathcal{A}_{st} \cong \mathcal{A}_s \boxtimes \mathcal{A}_t$; see Theorem 2.3.16.

- Suppose the uf-monoid $\mathbb{S}$ is abelian and has no nontrivial invertible elements. Furthermore, assume that $\mathcal{C}$ is equipped with *compatible inclusions* ι^1, ι^2 (see Section 2.3.4) and that the tensor product $\boxtimes$ perserves inductive limits. Then the second inductive limit yields an *abstract Lévy process* in the sense of Definition 2.3.22; see Theorem 2.3.23.

Chapter 3 deals with additive deformations. Sections 3.2 and 3.3 are based on [Ger11]. Section 3.4 contains generalizations of results obtained in [Ger11] and [Wir02] to braided additive deformations and is based on [GKL12] (joint work with Stefan Kietzmann and Stephanie Lachs). Let $\mathcal{B}$ be a bialgebra with multiplication μ, unit element $\mathbb{1}$, comultiplication Δ and counit δ. In [Wir02], Wirth defined an *additive deformation* of $\mathcal{B}$ as a family of linear maps $\mu_t : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$, such that the following four conditions hold:

(AD1) The map μ_0 coincides with the multiplication μ of $\mathcal{B}$.

(AD2) $(\mathcal{B}, \mu_t, \mathbb{1})$ is a unital algebra for all $t \geq 0$.

(AD3) $\delta \circ \mu_t : \mathcal{B} \to \mathbb{C}$ is pointwise continuous.

(AD4) For all $s, t \geq 0$ it holds that

$$\Delta \circ \mu_{s+t} = (\mu_s \otimes \mu_t) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \tag{1.0.1}$$

where $\tau : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ is the flip.

The tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ of unital algebras $\mathcal{A}_1, \mathcal{A}_2$ is again a unital algebra with multiplication

$$m_\otimes := (m_1 \otimes m_2) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$$

m_1, m_2 the multiplication maps of $\mathcal{A}_1, \mathcal{A}_2$. Therefore, writing $\mathcal{B}_t := (\mathcal{B}, \mu_t, \mathbb{1})$ for short, equation (1.0.1) is equivalent to

(AD4') For all $s, t \geq 0$, $\Delta_{s,t} : \mathcal{B}_{s+t} \to \mathcal{B}_s \otimes \mathcal{B}_t$, $\Delta_{s,t} = \Delta$ is an algebra homomorphism.

In other words, $(\mathcal{B}_t)_{t\in\mathbb{R}_+}$ is a comonoidal system in the category of unital algebras with all embeddings equal to the comultiplication.

A *Hochschild 2-cocycle* (or simply: *cocycle*) is a linear functional $L : \mathcal{B}\otimes\mathcal{B} \to \mathbb{C}$ such that

$$\delta(a)L(b \otimes c) - L(ab \otimes c) + L(a \otimes bc) - L(a \otimes b)\delta(c) = 0$$

for all $a, b, c \in \mathcal{B}$. A cocycle is called *normalized* if $L(1 \otimes 1) = 0$ and *commuting* if $L \star \mu = \mu \star L$, where $\star$ denotes the usual convolution on the $*$-bialgebra $\mathcal{B}\otimes\mathcal{B}$. A cocycle L is called *coboundary* if there exists a linear functional $\psi : \mathcal{B} \to \mathbb{C}$ such that $L(a\otimes b) = \delta(a)\psi(b)-\psi(ab)+\psi(a)\delta(b)$. Wirth's theorem states that additive deformations of a fixed bialgebra $\mathcal{B}$ are in 1-1 correspondence with normalized, commuting cocycles via the equations

$$L = \frac{\mathrm{d}}{\mathrm{d}t}\delta \circ \mu_t\Big|_{t=0}, \qquad \mu_t = \mu \star e_\star^{tL}$$

[Wir02]; see Theorem 3.2.2. There is an analogous theorem for $*$-bialgebras. One can ask, which deformations are generated by a coboundary. The answer was found in [Ger09] under the assumption that L is the coboundary of a *commuting* functional ψ, that is $\psi \star \mathrm{id} = \mathrm{id} \star \psi$; see also Theorem 3.2.5. An additive deformation is generated by the coboundary of a commuting functional if and only if there exists a semigroup of linear maps $\Phi_t : \mathcal{B} \to \mathcal{B}$ which are algebra isomomorphisms $\Phi_t : \mathcal{B}_s \to \mathcal{B}_{s+t}$ for all $s \geq 0$ and which fulfill some additional property.

It was shown in [Ger09] that one can construct quantum Lévy processes on additive deformations of $*$-bialgebras. Any convolution semigroup of states $\varphi_t : \mathcal{B}_t \to \mathbb{C}$ defines a comonoidal system of unital $*$-algebraic quantum probability spaces $(\mathcal{B}_t, \varphi_t)_{t\in\mathbb{R}_+}$ with respect to $\Delta_{s,t} = \Delta$ the comultiplication of $\mathcal{B}$. We know that Δ is an algebra homomorphism and the needed compatibility with the φ_t is exactly the semigroup property $\varphi_{s+t} = \varphi_s \star \varphi_t = (\varphi_s \otimes \varphi_t) \circ \Delta$. Thus, the construction of a corresponding quantum Lévy process is now a special case of the theory developed in Chapter 2. Another point is, how to get hold of convolution semigroups of states. To this end, Wirth proved a Schoenberg correspondence for additive deformations, which characterizes continuous convolution semigroups of states by their generators.

A key example comes from the algebra of the quantum harmonic oscillator, which is generated by two mutually adjoint elements x, x^* subject to the relation $[x, x^*] = xx^* - x^*x = 1$; see the introduction of Chapter 3 for details. One would also like to treat the Fermi harmonic oscillator, where the relation between x and x^* is $xx^* + x^*x = 1$. This only works by using braided algebras.

The aims of the chapter are:

- To improve the cohomological theory behind additive deformations
- To study additive deformations of Hopf algebras
- To generalize the theory to braided $*$-bialgebras.

We find the following results:

- By introducing the needed chain complexes to deal with the different cases (bialgebras and $*$-bialgebras) as subcomplexes of the standard Hochschild complex, we can reduce the calculations for the proof of Theorem 3.2.5 to a minimum.
- If $\mathcal{B}$ is a Hopf algebra we show that the identity map viewed as linear map from the coalgebra $\mathcal{B}$ to the algebra $\mathcal{B}_t$ has a convolution inverse S_t for every t and present an explicit formula, Theorem 3.3.10. In Theorem 3.3.12 we prove under an extra condition on the generator L that every additive deformation of $\mathcal{B}$ is equivalent to a deformation with constant antipodes. The equivalent generator can be explicitly calculated. The imposed condition is automatically fulfilled if $\mathcal{B}$ is cocommutative (see Lemma 3.3.13).
- We generalize Wirth's theorem on the generator of an additive deformation to the braided case, Theorem 3.4.7. We prove a general form of the Schoenberg correspondence, thus characterizing the generators of convolution semigroups of states on braided additive deformations, Theorem 3.4.14.

Chapter 4 deals with subproduct systems. Those parts of Chapter 4 that deal with the discrete case are based on joint work with Michael Skeide [GS14b]. A *product system* of Hilbert spaces is, roughly speaking, a family of Hilbert spaces H_t indexed by a monoid $\mathbb{S}$ with associative identifications

$$H_s \otimes H_t \;=\; H_{s+t}. \tag{1.0.2}$$

The interest in product systems mainly comes from quantum dynamics. Arveson [Arv89] gave the first formal definition of a product system (including also some technical conditions) of Hilbert spaces. He showed how to construct such *Arveson systems* from so-called normal E_0*–semigroups* (semigroups of normal unital endomorphisms) over $\mathbb{S} = \mathbb{R}_+$ on $\mathcal{B}(H)$. Bhat [Bha96] generalized this to normal *Markov semigroups* (semigroups of normal unital completely positive (CP-) maps) on $\mathcal{B}(H)$, by dilating the Markov semigroup in a unique minimal way to an E_0–semigroup and computing the Arveson system of the latter. Product systems of *correspondences* (that is, Hilbert bimodules) occur first in Bhat and Skeide [BS00]. They constructed directly from a Markov semigroup on a unital C^*-algebra or a von Neumann algebra $\mathcal{B}$ a product system of correspondences over $\mathcal{B}$, and used it to construct the minimal dilation. Muhly and Solel [MS02] constructed from a Markov semigroup on a von Neumann algebra $\mathcal{B}$ a product system over the commutant of $\mathcal{B}$. This product system turned out to be the *commutant* (see Skeide [Ske03, Ske09, Ske08]) of the product system constructed in [BS00].

In all these applications, the construction of a product system starts with a *subproduct system* (Shalit and Solel [SS09], and Bhat and Mukherjee [BM10]), where the condition (1.0.2) is weakened to

$$H_s \otimes H_t \;\supset\; H_{s+t}. \tag{1.0.3}$$

A subproduct system is nothing but a comonoidal system in the category of Hilbert spaces. The inductive limits $\mathcal{H}_t$ of $H_{t_1} \otimes \cdots \otimes H_{t_n}$ with $t_1 + \cdots + t_n = t$ using (1.0.3) are the generated comonoidal system in the sense of Chapter 2. It is always a product system, that is $\mathcal{H}_{s+t} \cong \mathcal{H}_s \otimes \mathcal{H}_t$. The construction for $\mathbb{S} = \mathbb{R}_+$ is described in detail by Bhat and Mukherjee [BM10] and their formulation actually generalizes without a problem to the general setting in Section 2.3.3.

The classification of product systems is a difficult subject. Product systems can roughly be distinguished into three types by their so-called *units*, that is sections $\omega = (\omega_t)_{t\in\mathbb{R}_+}$ with $\omega_s \otimes \omega_t = \omega_{s+t}$ under the identification $H_s \otimes H_t = H_{s+t}$. The linear spans $U_t := \operatorname{span}\{\omega_t \mid \omega \text{ is a unit of } (H_t)_{t\in\mathbb{R}_+}\}$ form a subproduct system. The product system is *type I* if it is generated by the U_t, it is *type III* if $U_t = \{0\}$ for all $t > 0$ and it is *type II* if it is neither type I nor type III, that is if there exists a nontrivial unit, but the units do not generate the system. The simplest examples of Arveson systems are the *Fock-systems*. The *symmetric Fock space* or *Bose Fock space* over a Hilbert space H is the direct sum over all symmetric tensor powers of H, $\Gamma(H) := \bigoplus_{n\in\mathbb{N}_0} H^{\otimes_s n}$. There is a natural unitary isomorphism $\Gamma(H_1 \oplus H_2) \cong \Gamma(H_1) \otimes \Gamma(H_2)$. In particular, for the Fock spaces $\Gamma\Big(L^2\big([0,t),K\big)\Big)$ over the square integrable functions with values in some Hilbert space K, one has the canonical unitary isomorphisms

$$\begin{aligned}\Gamma\Big(L^2\big([0,s+t),K\big)\Big) &\cong \Gamma\Big(L^2\big([0,s),K\big) \oplus L^2\big([s,s+t),K\big)\Big)\\ &\cong \Gamma\Big(L^2\big([0,s),K\big)\Big) \otimes \Gamma\Big(L^2\big([s,s+t),K\big)\Big) \cong \Gamma\Big(L^2\big([0,s),K\big)\Big) \otimes \Gamma\Big(L^2\big([0,t),K\big)\Big)\end{aligned}$$

with composition $u_{s,t} : \Gamma\Big(L^2\big([0,s+t),K\big)\Big) \to \Gamma\Big(L^2\big([0,s),K\big)\Big) \otimes \Gamma\Big(L^2\big([0,t),K\big)\Big)$. With respect to the identifications given by the $u_{s,t}$, the $\Gamma\Big(L^2\big([0,t),K\big)\Big)$ form a Type I Arveson system, the *Fock system over K*. It can be shown that every type I Arveson system is actually Fock. In principle, product systems of this type are known since Streater [Str69], Araki [Ara70], Guichardet [Gui72], or Parthasarathy and Schmidt [PS72]. Though more difficult to construct, there are many Arveson systems of type II and III; see, for instance, Tsirelson [Tsi00a, Tsi00b], Liebscher [Lie09], Powers [Pow04], Bhat and Srinivasan [RBS05], and Izumi and Srinivasan [IS08].

There is a 1-1 correspondence between units of a subproduct system und units of its generated product system; see [BM10, Theorem 10]. In view of this, the terminology of types can also be used for subproduct system. Since there are much more subproduct systems than product systems, a classification of all subproduct systems up to isomorphism is hopeless. In the 2009 Oberwolfach Mini-Workshop on “Product Systems and Independence in Quantum Dynamics” [BFS09], Bhat suggested to try to classify at least the finite-dimensional subproduct systems and the product systems they generate. Finite-dimensional subproduct systems occurred in several ways. For instance, every

CP-semigroup on the $n \times n$–matrices M_n gives rise to its finite-dimensional subproduct system of *Arveson-Stinespring correspondences*; see Shalit and Solel [SS09]. Moreover, every subproduct system (finite-dimensional or not) arises in this way from a normal CP-semigroup on $\mathcal{B}(H)$; see again [SS09]. Other examples arise from homogeneous relations on polynomials in several variables; see Davidson, Ramsey and Shalit [DRS11]. Also a subclass of *interacting Fock spaces* gives rise to finite-dimensional subproduct systems and further generalizes the notion of subproduct system; see Gerhold and Skeide [GS14a]. Tsirelson has determined the structure of two-dimensional discrete subproduct systems [Tsi09a] and of two-dimensional continuous time subproduct systems [Tsi09b] and the product systems they generate. He exploits that subproduct systems also may be viewed as graded algebras; see Remark 2.3.3 and Section 4.2.1.

It is obvious that the fibrewise dimension of a subproduct system is submultiplicative in the sense that $\dim H_{s+t} \leq \dim H_s \dim H_t$ for all $s, t \in \mathbb{S}$. Shalit and Solel [SS09] posed an interesting question: Is there a discrete subproduct system with $\dim H_n = d_n$ for all n for every submultiplicative sequence? We can show that the answer is "no", and ask instead, what are the sequences that arise as dimension sequences of discrete subproduct systems. Of course, similar questions make sense for other types of comonoidal systems. We will basically deal with two types, subproduct systems and *Cartesian systems* (see Definition 4.1.1), over the three monoids $\mathbb{N}_0, \mathbb{Q}_+$ and $\mathbb{R}_+$.

Our main results are:

- The dimension sequences of discrete subproduct systems are the same as the cardinality sequences of discrete Cartesian systems and the same as the *complexity sequences* of *factorial languages*; see Corollary 4.2.9. The proof uses a classical theorem about graded algebras. Complexity sequences of factorial languages are well studied by combinatorialists.

- We present a simple necessary and sufficient condition, (4.2.1), for a function to be the dimension function of a rational time subproduct system or a rational time Cartesian system; see Theorem 4.2.15.

- The same condition characterizes the dimension functions of continuous-time Cartesian systems; see Theorem 4.2.18. For continuous time subproduct systems we have to impose a continuity condition to show that the condition is necessary; see Theorem 4.2.19. It is left open, if the continuity condition is indeed needed.

The last chapter is based on [GL14] with Stephanie Lachs. As mentioned earlier, independence in quantum probability is often modelled by so-called *universal products*, which are basically tensor product structures on the category of algebraic quantum probability spaces such that the “algebra part” is simply the free product. It is customary to denote a universal product as a product of linear functionals, thus writing $\varphi_1 \boxdot \varphi_2$ instead of $(\mathcal{A}_1, \varphi_1) \boxdot (\mathcal{A}_2, \varphi_2)$. For any universal product, a rich quantum probabilistic theory can be developed, including independence, Lévy processes, central limit theorems, et cetera. Very successfull examples are Voiculescu's freeness and the Boolean independence. A universal product is called commutative if $\varphi_1 \boxdot \varphi_2 = \varphi_2 \boxdot \varphi_1$ holds under the canonical identification of $\mathcal{A}_1 \sqcup \mathcal{A}_2$ with $\mathcal{A}_2 \sqcup \mathcal{A}_1$. Commutative universal products were classified by Speicher [Spe97], and Ben Ghorbal and Schürmann [BGS02]. Without commutativity in general, but still assuming commutativity on elements of length 2 in the free product, the classification is due to Muraki [Mur03]. Giving up on this restricted commutativity also, Lachs found a new family of universal products, the (r, s)-products, depending on two complex parameters r and s, which coincides with the Boolean product for $r = s = 1$. These complete the classification of universal products except for degenerate cases; see [Lac14] and [GL14].

In the study of quantum Lévy processes, GNS-construction plays a crucial role, especially the question, how to calculate the GNS-construction for product functionals. Since the (r, s)-products do not preserve positivity, the usual GNS-construction can not be used.

The aims of this chapter are:

- Find a generalization of the GNS-construction for arbitrary, that is not necessarily positive, linear functionals on algebras.

- Calculate the GNS-construction for (r, s)-product functionals in terms of the GNS-construction for the factors.

Our results are:

- A general GNS-construction is presented in Section 5.2.2. The main difference to the usual construction is that instead of one representation, we get a pair consisting of two compatible representations. There was another approach to this question by Wilhelm [Wil08]. In Section 5.2.4 we prove that the two constructions are equivalent.

- The general form of the GNS-construction of the (r, s)-product functional is given as a quotient of a direct sum; see Theorem 5.2.13. It depends heavily on the parameters r and s. Even the dimension of the representation spaces can change. We present examples where the dimension is greater than in the Boolean case as well as examples where it is smaller than in the Boolean case.

2 Comonoidal Systems and Abstract Lévy-Processes

Suppose $(\mu_t)_{t\in\mathbb{R}_+}$ is a convolution semigroup of probability measures on the real line. Let us sketch, how to construct a Lévy process $X_t : \Omega \to \mathbb{R}$ with marginal distributions $\mathbb{P}_{X_t} = \mu_t$. First, for all finite subsets $J = \{t_1 < t_2 < \cdots < t_n\} \subset \mathbb{R}_+$ define probability measures $\mu_J := \mu_{t_1} \otimes \mu_{t_2-t_1} \otimes \cdots \otimes \mu_{t_n-t_{n-1}}$ on $\mathbb{R}^J$. Then show that these are coherent in the sense that

$$\mu_I = \mu_J \circ (p_I^J)^{-1}$$

for all $I \subset J$ and $p_I^J : \mathbb{R}^J \to \mathbb{R}^I$ the canonical projection. In this situation the probability spaces $(\mathbb{R}^J, \mathcal{B}(\mathbb{R}^J), \mu_J)$ with the projections (p_I^J) form a *projective system*. Now, the Daniell-Kolmogoroff theorem guarantees the existence of a *projective limit*, which is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with projections $p_J : \Omega \to \mathbb{R}^J$ such that $\mu_J = \mathbb{P} \circ (p_J)^{-1}$. The random variable $X_t := p_{\{t\}}$ has distribution μ_t and it is not difficult to prove that the X_t have independent and stationary increments $X_s - X_t \sim \mu_{s-t}$.

A similar construction allows to associate quantum Lévy processes with convolution semigroups of states on $*$-bialgebras. The formulation of quantum probability is dual to that of classical probability, so inductive limits appear instead of projective limits. Due to the fact that there are different notions of independence in quantum probability on the one hand and the interactions between quantum probability and operator algebras on the other hand, there are many different theorems of the same kind (construction of Lévy processes for other notions of independence) or similar kind (for example the construction of product systems from subproduct systems). The main aim of this section is to give a unified approach to these different situations. To this end we introduce the language of tensor categories and the concept of comonoidal systems, which generalizes that of convolution semigroups.

2.1 Basic Notions of Category Theory

The main point of this section is to fix notations and recall basic facts about inductive limits. We also give a list of those categories which appear as examples for the following sections.

2.1.1 Categories, Functors, Natural Transformations

A *category* $\mathcal{C}$ consists of

- a class of *objects* $\mathrm{Obj}(\mathcal{C})$
- a set of *morphisms* $\mathrm{Mor}(A,B)$ for each two objects A, B
- an *identity morphism* $\mathrm{id}_A \in \mathrm{Mor}(A,A)$ for each object A
- a *composition map* $(f,g) \mapsto f \circ g : \mathrm{Mor}(B,C) \times \mathrm{Mor}(A,B) \to \mathrm{Mor}(A,C)$ for each three objects A, B, C

such that, where defined, the composition is associative and the identity morphisms act neutrally under composition. We will frequently write $A \in \mathcal{C}$ instead of $A \in \mathrm{Obj}(\mathcal{C})$ and $f : A \to B$ instead of $f \in \mathrm{Mor}(A,B)$. A morphism $f : A \to B$ is called a *monomorphism* if it is left cancellative, that is if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. It is called an *epimorphism* if it is right cancellative, that is if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. It is called an *isomorphism* if it is invertible, that is if there exists a (necessarily unique) morphism $f^{-1} : B \to A$ with $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. Clearly, every isomorphism is both, a monomorphism and an epimorphism, but the converse does not hold in all categories. We sometimes write $f : A \cong B$ to indicate that f is an isomorphism. A category $\mathcal{D}$ is called a *subcategory* of $\mathcal{C}$ if all objects and morphisms of $\mathcal{D}$ are objects respectively morphisms of $\mathcal{C}$ and the identity morphisms and composition maps agree. A subcategory is called *full* if every morphism of $\mathcal{C}$ between objects of $\mathcal{D}$ is also a morphism of $\mathcal{D}$. For every subclass of objects there is a unique full subcategory with exactly these objects, called *the full subcategory* with the specified class of objects.

Equalities between morphisms will frequently be expressed in terms of commutative diagrams. A *diagram* is a directed graph with object-labeled vertices and morphism-labeled edges. We say that a diagram *commutes* if the composition of morphisms along any two directed paths with the same source and the same target vertex yield the same result. We will usually not explicitly write the inverse of an isomorphism with an extra edge, but it shall be included when we say that the diagram commutes.

We will now introduce categories which are of particular importance throughout this thesis. When two categories have the same objects, but different sets of morphisms, we will usually indicate this by a suggestive superscript which describes the type of morphisms.

Sets Define a category **Set** whose objects are all sets and whose morphisms between two sets A and B are the mappings from A to B. The composition is the usual composition of maps. The identity map on a set A serves as an identity morphism $\mathrm{id}_A : A \to A$. Since the composition of two injective maps is again injective and all identity maps are injective, we can define a subcategory $\mathbf{Set}^{inj}$ with the same objects as **Set** but only the injective maps as morphisms. Similarly we define $\mathbf{Set}^{surj}$ and $\mathbf{Set}^{bij}$ as the categories with objects all sets and morphisms all surjections respectively all bijections.

We can also restrict the class of objects. We define **FinSet** to be the full subcategory of **Set** with finite sets as objects.

Vector Spaces The category **Vect** has as objects all (complex) vector spaces and as morphisms all linear maps. Like for sets, we define subcategories $\mathbf{Vect}^{inj}$, $\mathbf{Vect}^{surj}$ and $\mathbf{Vect}^{bij}$ consisting of all vector spaces as objects, but only injective, surjective and bijective linear maps as morphisms respectively. We also define **FinVect** to be the full subcategory of **Vect** whose objects are the finite-dimensional vector spaces.

Hilbert Spaces The category **Hilb** has as objects all Hilbert spaces and as morphisms all bounded linear maps. We mainly consider the subcategory $\mathbf{Hilb}^{isom}$ with the same objects, but only isometries as morphisms. Other interesting subcategories are $\mathbf{Hilb}^{coisom}$

and $\mathbf{Hilb}^{pisom}$ whose morphisms are the coisometries and the partial isometries respectively. Also, **FinHilb** denotes the full subcategory of **Hilb** formed by the finite-dimensional Hilbert spaces.

Algebras By **Alg** we denote the category of all (associative, complex) algebras with algebra homomorphisms as morphisms. We use the same superscripts as for sets and vector spaces when we restrict to injective, surjective or bijective morphisms. The subcategory of unital algebras with unital algebra homomorphisms as morphisms is denoted by $\mathbf{Alg}_{\mathbb{1}}$. Similarly we define the categories $*$-**Alg** and $*$-$\mathbf{Alg}_{\mathbb{1}}$ with objects all (unital) $*$-algebras and morphisms all (unital) $*$-algebra homomorphisms.

Algebraic Quantum Probability Spaces A pair $(\mathcal{A}, \varphi)$ consisting of an algebra $\mathcal{A}$ and a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called an *(algebraic) quantum probability space.* A quantum probability space is called *unital* if $\mathcal{A}$ is unital and $\varphi(\mathbb{1}) = 1$, it is called *commutative* if $\mathcal{A}$ is commutative. A $*$-*algebraic quantum probability space* consists of a $*$-algebra $\mathcal{A}$ and a positive linear functional φ. The algebraic quantum probability spaces form a category **AlgQ** where the morphisms are the functional preserving algebra homomorphisms, that is a morphism from $(\mathcal{A}_1, \varphi_1)$ to $(\mathcal{A}_2, \varphi_2)$ is an algebra homomorphisms $f : \mathcal{A}_1 \to \mathcal{A}_2$ with $\varphi_2 \circ f = \varphi_1$. By $\mathbf{AlgQ}_{\mathbb{1}}$ we denote the subcategory formed by unital quantum probability spaces and unital functional preserving homomorphisms as morphisms. Of course, there are also categories $*$-**AlgQ** and $*$-$\mathbf{AlgQ}_{\mathbb{1}}$ of (unital) $*$-algebraic quantum probability spaces.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $\mathcal{F}$ is a prescription which assigns to each object $A \in \mathcal{C}$ an object $\mathcal{F}(A) \in \mathcal{D}$ and to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ such that $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ for all $A \in \mathcal{C}$ and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all $f : A \to B, g : B \to C$ with $A, B, C \in \mathcal{C}$. We write $F : \mathcal{C} \to \mathcal{D}$ to indicate that $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$. Given two functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{E}$, the composition $\mathcal{G} \circ \mathcal{F}$ can be defined in the obvious way and is a functor from $\mathcal{C}$ to $\mathcal{E}$.

The *Cartesian product* $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$ consists of ordered pairs (A, B) with $A \in \mathcal{C}, B \in \mathcal{D}$ as objects and ordered pairs (f, g) with $f : A \to A', g : B \to B'$ as morphisms from (A, B) to (A', B'). This becomes a category with $\mathrm{id}_{A,B} := (\mathrm{id}_A, \mathrm{id}_B)$

and entrywise composition. The projections on the first respectively second component for objects and morphisms yield functors $\mathcal{P}_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$, $\mathcal{P}_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$. A functor defined on a Cartesian product category is sometimes referred to as a *bifunctor*.

For two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ between the same categories, a *natural transformation* is a family $\alpha = (\alpha_A : \mathcal{F}(A) \to \mathcal{G}(A))_{A \in \mathcal{C}}$ of morphisms such that the diagram

$$\begin{CD} \mathcal{F}(A) @>{\alpha_A}>> \mathcal{G}(A) \\ @V{\mathcal{F}(f)}VV @VV{\mathcal{G}(f)}V \\ \mathcal{F}(B) @>{\alpha_B}>> \mathcal{G}(B) \end{CD}$$

commutes for every morphism $f : A \to B$ in $\mathcal{C}$. A natural transformation is usually denoted by $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$. Where it is convenient we simply write $\alpha : \mathcal{F}(A) \to \mathcal{G}(A)$ instead of α_A also for the single morphism. A natural transformation is called a *natural isomorphism* if all α_A are isomorphisms.

2.1.2 Inductive Limits

In category theory there are the general concepts of limits and colimits. Since in our applications only inductive limits play a role, we restrict to this special case. The general case can for example be found in the book of Adámek, Herrlich and Strecker [AHS04].

A partially ordered set I is called *directed* if any two Elements of I possess a common upper bound, that is if for all $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\gamma \geq \alpha, \beta$.

2.1.1 Definition. Let $\mathcal{C}$ be a category. An *inductive system* consists of

- a family of objects $(A_\alpha)_{\alpha \in I}$ indexed by a directed set I
- a family of morphisms $(f^\alpha_\beta : A_\alpha \to A_\beta)_{\alpha \leq \beta}$

such that

1. $f^\alpha_\alpha = \mathrm{id}_{A_\alpha}$ for all $\alpha \in I$
2. $f^\beta_\gamma \circ f^\alpha_\beta = f^\alpha_\gamma$ for all $\alpha \leq \beta \leq \gamma$

An object $\mathcal{A}$ together with morphisms $f^\alpha : A_\alpha \to \mathcal{A}$ for $\alpha \in I$ is called *inductive limit* of the inductive system $((A_\alpha)_{\alpha \in I}, (f^\alpha_\beta)_{\alpha \leq \beta})$ if

1. $f^\alpha = f^\beta \circ f^\alpha_\beta$ for all $\alpha \le \beta$

2. whenever $g^\alpha = g^\beta \circ f^\alpha_\beta$ holds for a family of morphisms $g^\alpha : A_\alpha \to B$ to some common object B, there exists a unique morphism $g : \mathcal{A} \to B$ such that $g \circ f^\alpha = g^\alpha$ for all $\alpha \in I$. This is referred to as the *universal property* of the inductive limit.

2.1.2 Example. The vector spaces $(\mathbb{C}^n)_{n\in\mathbb{N}}$ form an inductive system with respect to the linear maps $f^m_n : \mathbb{C}^m \to \mathbb{C}^n, (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. Let $l_0 := \{(x_n)_{n\in\mathbb{N}} \mid x_n = 0 \text{ except for finitly many } n\} \subset \mathbb{C}^\mathbb{N}$ denote the vector space consisting of terminating sequences. Then l_0 together with the linear maps $f^n : \mathbb{C}^n \to l_0$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, 0, \ldots)$ is an inductive limit.

If an inductive limit exists, it is essentially unique. More precisely, if $(\mathcal{A}, (f^\alpha)_{\alpha\in I})$ and $(\mathcal{B}, (g^\alpha)_{\alpha\in I})$ are two inductive limits of the same inductive system $(A_\alpha)_{\alpha\in I}$, then the uniquely determined morphisms $f : \mathcal{A} \to \mathcal{B}$ with $f \circ f^\alpha = g^\alpha$ and $g : \mathcal{B} \to \mathcal{A}$ with $g \circ g^\alpha = f^\alpha$ are mutually inverse isomorphisms.

In general, inductive limits may or may not exist. The inductive system of Example 2.1.2 has an inductive limit in **Vect** but the same inductive system viewed as an inductive system in **FinVect** has no inductive limit. We call a category in which all inductive systems have inductive limits *inductively complete*. For example, the categories **Set**, **Hilb**^{isom}, **Alg** and **AlgQ** are inductively complete; see [Bou04, § 7.5] for **Set**, [Bou89, § 10.3] for **Alg**, and [Vos13] or [Lac14] for **AlgQ**; see also [AHS04, Chapter 12] for general arguments showing that **Set**, **Hilb**^{isom} and **Alg** even fulfill the stronger property of *cocompleteness* .

A subset J of a directed set I is called *cofinal* if for every $\alpha \in I$ there exists a $\beta \in J$ with $\beta \ge \alpha$.

2.1.3 Example. For any fixed $\alpha_0 \in I$ the set $\{\beta \mid \beta \ge \alpha_0\}$ is cofinal. Indeed, since I is directed, there is a $\beta \ge \alpha, \alpha_0$ for all $\alpha \in I$.

Clearly, if $((A_\alpha)_{\alpha\in I}, (f^\alpha_\beta)_{\alpha\le\beta,\alpha,\beta\in I})$ is an inductive system and $J \subset I$ cofinal, then also $((A_\alpha)_{\alpha\in J}, (f^\alpha_\beta)_{\alpha\le\beta,\alpha,\beta\in J})$ is an inductive system. It is known that the inductive limits are canonically isomorphic if they exist. We will need the following generalization of this.

Let $(A_\alpha)_{\alpha\in I}$ be an inductive system with inductive limit $(\mathcal{A},(f^\alpha)_{\alpha\in I})$, K a directed set and $J_k\subset I$ for each $k\in K$ such that

- J_k is directed for all $k\in K$
- $J_k\subset J_{k'}$ for all $k\leq k'$
- $J=\bigcup_{k\in K}J_k$ cofinal in I.

Suppose the inductive systems $(A_\alpha)_{\alpha\in J_k}$ have inductive limits $(\mathcal{A}_k,(f^\alpha_{(k)})_{\alpha\in J_k})$. It holds that $f^\alpha=f^\beta\circ f^\alpha_\beta$ for all $\alpha\leq\beta\in J_k$, since $J_k\subset I$. Similarly, for $k\leq k'$ it holds that $f^\alpha_{(k')}=f^\beta_{(k')}\circ f^\alpha_\beta$ for all $\alpha\leq\beta\in J_k$, since $J_k\subset J_{k'}$. By the universal property of the inductive limit $\mathcal{A}_k$ there are unique morphisms $f^k_{k'}:\mathcal{A}_k\to\mathcal{A}_{k'}$ for $k\leq k'$ and $f^k:\mathcal{A}_k\to\mathcal{A}$ such that the diagrams

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f^\alpha} & \mathcal{A} \\ {\scriptstyle f^\alpha_{(k)}}\downarrow & \nearrow{\scriptstyle f^k} & \\ \mathcal{A}_k & & \end{array} \qquad\qquad \begin{array}{ccc} A_\alpha & \xrightarrow{f^\alpha_{(k')}} & \mathcal{A}_{k'} \\ {\scriptstyle f^\alpha_{(k)}}\downarrow & \nearrow{\scriptstyle f^k_{k'}} & \\ \mathcal{A}_k & & \end{array}$$

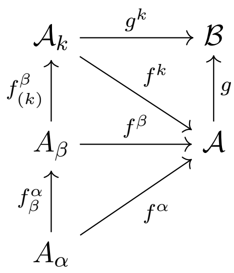
commute for all $\alpha\in J_k$.

2.1.4 Theorem. *In the described situation* $((\mathcal{A}_k)_{k\in K},(f^k_{k'})_{k\leq k'})$ *is an inductive system with inductive limit* $(\mathcal{A},(f^k)_{k\in K})$.

Proof. The diagrams

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f^\alpha} & \mathcal{A} \\ {\scriptstyle f^\alpha_{(k)}}\downarrow & \searrow{\scriptstyle f^\alpha_{(k')}} & \uparrow{\scriptstyle f^{k'}} \\ \mathcal{A}_k & \xrightarrow{f^k_{k'}} & \mathcal{A}_{k'} \end{array} \qquad\qquad \begin{array}{ccc} A_\alpha & \xrightarrow{f^\alpha_{(k'')}} & \mathcal{A}_{k''} \\ {\scriptstyle f^\alpha_{(k)}}\downarrow & \searrow{\scriptstyle f^\alpha_{(k')}} & \uparrow{\scriptstyle f^{k'}_{k''}} \\ \mathcal{A}_k & \xrightarrow{f^k_{k'}} & \mathcal{A}_{k'} \end{array}$$

commute, which implies $f^k=f^{k'}\circ f^k_{k'}$ and $f^k_{k''}=f^{k'}_{k''}\circ f^k_{k'}$ for all $k\leq k'\leq k''$. Now suppose there are $g^k:A_k\to B$ with $g^k=g^{k'}\circ f^k_{k'}$ for all $k\leq k'$. We put $g^\alpha:=g^k\circ f^\beta_{(k)}\circ f^\alpha_\beta$ for $\beta\in J_k$, $\alpha\leq\beta$. Since $J=\bigcup_{k\in K}J_k$ is cofinal in I, we can find such k and β for every α. One can check that the g^α do not depend on the choice and fulfill $g^\alpha=g^{\alpha'}\circ f^\alpha_{\alpha'}$ for all $\alpha \leq \alpha'$. This yields a morphism $g : \mathcal{A} \to \mathcal{B}$ which makes



commute. On the other hand, any morphism which makes the upper right triangle commute, automatically makes the whole diagram commute and will therefore equal g. □

2.1.5 Corollary. *Let $(A_\alpha)_{\alpha \in I}$ be an inductive system with inductive limit $(\mathcal{A}, (f^\alpha)_{\alpha \in I})$, $J \subset I$ cofinal. Then $(A_\alpha)_{\alpha \in J}$ is an inductive system with inductive limit $(\mathcal{A}, (f^\alpha)_{\alpha \in J})$.*

Proof. This is a special case of the previous theorem with $|K| = 1$, since the inductive system over the one point set K does not add anything. □

2.2 Tensor Categories

A *tensor category* is a category $\mathcal{C}$ together with a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which

- is associative under a natural isomorphism with components

$$\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \xrightarrow{\cong} (A \boxtimes B) \boxtimes C$$

  called *associativity constraint*,

- has a unit object $E \in \mathrm{Obj}(\mathcal{C})$ acting as left and right identity under natural isomorphisms with components

$$l_A : E \boxtimes A \xrightarrow{\cong} A, \quad r_A : A \boxtimes E \xrightarrow{\cong} A$$

  called *left unit constraint* and *right unit constraint* respectively such that the diagrams

$$\begin{array}{ccccc} & & (A \boxtimes B) \boxtimes (C \boxtimes D) & & \\ & \nearrow^{\alpha_{A,B,C\boxtimes D}} & & \searrow^{\alpha_{A\boxtimes B,C,D}} & \\ A \boxtimes \big(B \boxtimes (C \boxtimes D)\big) & & & & \big((A \boxtimes B) \boxtimes C\big) \boxtimes D \\ \downarrow{\scriptstyle \mathrm{id}_A \boxtimes \alpha_{B,C,D}} & & & & \uparrow{\scriptstyle \alpha_{A,B,C} \boxtimes \mathrm{id}_D} \\ A \boxtimes \big((B \boxtimes C) \boxtimes D\big) & & \xrightarrow{\alpha_{A,B\boxtimes C,D}} & & \big(A \boxtimes (B \boxtimes C)\big) \boxtimes D \end{array}$$

$$\begin{array}{ccc} A \boxtimes (E \boxtimes C) & \xrightarrow{\alpha_{A,E,C}} & (A \boxtimes E) \boxtimes C \\ & {\scriptstyle \mathrm{id}_A \boxtimes l_C} \searrow \quad \swarrow {\scriptstyle r_A \boxtimes \mathrm{id}_C} & \\ & A \boxtimes C & \end{array}$$

commute for all $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$. If the natural transformations α, l and r are all identities, we say the tensor category is *strict*.

The compatiblity conditions are called the pentagon and the triangle axioms. It is shown by Mac Lane [ML98, VII.2] that these imply commutativity of all diagrams which only contain α, l and r. This is called *coherence*.

Even for non-strict tensor categories, we will frequently suppress the associativity and unit constraints in the notation and write $(\mathcal{C}, \boxtimes, E)$, or even $(\mathcal{C}, \boxtimes)$ or $\mathcal{C}$. In the examples we treat, α, l and r are always canonical.

2.2.1 Categorial Independence

In order to unify the different notions of independence in quantum probability, Franz came up with a definition of independence in a tensor-categorial framework [Fra06, Section 3]. Recall that $\mathcal{P}_i : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for $i \in \{1, 2\}$ denotes the projection functor onto the first respectively second component.

2.2.1 Definition. Let $(\mathcal{C}, \boxtimes)$ be a tensor category. A natural transformation $\iota^1 : \mathcal{P}_1 \Rightarrow \boxtimes$ is called *left inclusion* and a natural transformations $\iota^2 : \mathcal{P}_2 \Rightarrow \boxtimes$ is called *right inclusion*. A tensor category together with a right and a left inclusion is referred to as *inclusive tensor category* or *tensor category with inclusions*.

In more detail, inclusions for a tensor category are two collections of morphisms

$\iota^i_{B_1,B_2} : B_i \to B_1 \boxtimes B_2$ for $B_1, B_2 \in \mathcal{C}$, $i \in \{1,2\}$ such that

$$\begin{array}{ccccc} A_1 & \xrightarrow{\iota^1} & A_1 \boxtimes A_2 & \xleftarrow{\iota^2} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1 \boxtimes f_2} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\iota^1} & B_1 \boxtimes B_2 & \xleftarrow{\iota^2} & B_2 \end{array}$$

commutes for all $f_i : A_i \to B_i$, $i \in \{1,2\}$.

2.2.2 Definition. Let $(\mathcal{C}, \boxtimes, \iota^1, \iota^2)$ be a tensor category with inclusions. Two morphisms $j_1, j_2 : B_i \to A$ are *independent* if there exists a morphism $h : B_1 \boxtimes B_2 \to A$ such that the diagram

$$\begin{array}{ccccc} & \overset{j_1}{\nearrow} & A & \overset{j_2}{\nwarrow} & \\ & & \uparrow{\scriptstyle h} & & \\ B_1 & \xrightarrow{\iota^1} & B_1 \boxtimes B_2 & \xleftarrow{\iota^2} & B_2 \end{array}$$

commutes. Such a morphism h is called *independence morphism* for j_1 and j_2.

2.2.3 Example. We begin with a trivial example. The direct sum of vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ has the following property: Given any two linear maps $f_i : \mathcal{V}_i \to \mathcal{W}$ to some third vector space $\mathcal{W}$, there exists a unique linear map $h : \mathcal{V}_1 \oplus \mathcal{V}_2 \to \mathcal{W}$ with $h(v_i) = f_i(v_i)$ for all $v_i \in \mathcal{V}_i$, namely $h = f_1 + f_2$ (here we identify V_i with the corresponding subspace of $\mathcal{V}_1 \oplus \mathcal{V}_2$). In particular, in the tensor category $(\mathbf{Vect}, \oplus)$ with the canonical inclusions $\mathcal{V}_i \hookrightarrow \mathcal{V}_1 \oplus \mathcal{V}_2$, all pairs of linear maps into a common vector space are independent.

In the tensor category $(\mathbf{Alg}, \sqcup)$ with $\sqcup$ the free product of algebras (see Section 5.1) the situation is similar. With respect to the canonical inclusions $\mathcal{A}_i \hookrightarrow \mathcal{A}_1 \sqcup \mathcal{A}_2$, any pair of algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{B}$ to a common algebra $\mathcal{B}$ is independent.

In the two mentioned cases the tensor product coincides with the so-called "coproduct" in the category; see [Fra06] or [McL92]. Coproducts exist in many categories. But we should not take the coproduct as tensor product if we are looking for interesting independences.

2.2.4 Example. Independence in quantum probability is usually implemented by a *universal product*, which is a prescription $\odot$ that assigns to two linear functionals on algebras $\mathcal{A}_1, \mathcal{A}_2$ a new linear functional $\varphi_1 \odot \varphi_2$ on the *free product* $\mathcal{A}_1 \sqcup \mathcal{A}_2$ such that the bifunctor $((\mathcal{A}_1,\Phi_1),(\mathcal{A}_2,\Phi_2)) \mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \Phi_1 \boxdot \mathcal{A}_2)$ turns the category **AlgQ** of quantum probability spaces into a tensor category with the canonical embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_1 \sqcup \mathcal{A}_2$ as inclusions (see Chapter 5 and in particular Definition 5.1.1). An example is the tensor product of linear functionals defined by

$$\Phi_1 \otimes \Phi_2(a_1 \cdots a_n) = \Phi_1\left(\overrightarrow{\prod_{a_i \in \mathcal{A}_1}} a_i\right)\Phi_2\left(\overrightarrow{\prod_{a_i \in \mathcal{A}_2}} a_i\right),$$

where $\overrightarrow{\prod}$ denotes the product of the algebra elements in the same order as they appear in $a_1 \cdots a_n$. In this case categorial independence reproduces the notion of *tensor-independence*. If $\boxdot$ is the *free product*, we get *freeness*.

Although Definition 2.2.2 was motivated by quantum probability, it encompasses also non-stochastic notions of independence, as the following examples show.

2.2.5 Example (Linear Independence)**.** Consider the category $\mathbf{Vect}^{inj}$ of vector spaces with injective linear maps. The direct sum turns this into a tensor category with inclusions with respect to the canonical embeddings $V_i \hookrightarrow V_1 \oplus V_2$. Two injections $f_i : V_i \to W$ are independent if and only if they have linearly independent ranges. The only choice for the independence morphism is the linear map $h := f_1 + f_2 : V_1 \oplus V_2 \to W$. If h is injective, then $f_1(v_1) + f_2(v_2) = 0$ implies $f_1(v_1) = f_2(v_2) = 0$, so the ranges are linearly independent. On the other hand, if the ranges are linearly independent and $h(v_1 \oplus v_2) = 0$, we can conclude that $f_i(v_i) = 0$ for $i \in \{1,2\}$. Since f_1 and f_2 are injections, it follows that $v_1 = 0$ and $v_2 = 0$, so h is injective.

2.2.6 Example (Orthogonality)**.** Similar to the previous example, $(\mathbf{Hilb}^{isom}, \oplus)$ is a tensor category with the canonical embeddings as inclusions. Two isometries $v_i : H_i \to G$ are independent if and only if they have orthogonal ranges. Indeed, the only choice for the independence morphism is the linear map $h = v_1 + v_2$. This is an isometry if and only if

$$\begin{aligned}
0 &= \left\langle v_1(x_1) + v_2(x_2), v_1(x_1) + v_2(x_2)\right\rangle - \left\langle x_1 \oplus x_2, x_1 \oplus x_2\right\rangle \\
&= \left\langle v_1(x_1), v_2(x_2)\right\rangle + \left\langle v_2(x_2), v_1(x_1)\right\rangle \\
&= 2\,\mathsf{Re}\left\langle v_1(x_1), v_2(x_2)\right\rangle
\end{aligned}$$

for all $x_1 \in H_1, x_2 \in H_2$, which is clearly equivalent to $v_1(H_1) \perp v_2(H_2)$.

In the previous examples the independence morphism h is always uniquely determined if it exists. The next example shows that this is not the case in general.

2.2.7 Example. Consider the category **Vect** with tensor product

$$V_1 \odot V_2 := V_1 \oplus V_2 \oplus V_1 \otimes V_2.$$

and the canonical inclusions $V_i \hookrightarrow V_1 \odot V_2$ which identify V_i with the summand V_i in $V_1 \oplus V_2$. Any two linear maps $f_i : V_i \to W$ are independent, but the independence morphism is not uniquely determined. Indeed, for an arbitrary linear map $f : V_1 \otimes V_2 \to W$, the linear map $h = f_1 + f_2 + f$ is an independence morphism for f_1 and f_2.

2.2.2 Cotensor Functors

Given tensor categories $(\mathcal{C}, \boxtimes)$ and $(\mathcal{C}', \boxtimes')$ with unit objects, associativity and unit constraints E, α, l, r and E', α', l', r' respectively, a *cotensor functor* is a triple $(\mathcal{F}, \delta, \Delta)$ consisting of

- a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$
- a morphism $\delta : \mathcal{F}(E) \to E'$
- a natural transformation $\Delta : \mathcal{F}(\cdot \boxtimes \cdot) \Rightarrow \mathcal{F}(\cdot) \boxtimes' \mathcal{F}(\cdot)$

such that the diagrams

$$\begin{array}{ccc}
\mathcal{F}\big(A \boxtimes (B \boxtimes C)\big) & \xrightarrow{\mathcal{F}(\alpha_{A,B,C})} & \mathcal{F}\big((A \boxtimes B) \boxtimes C\big) \\
\downarrow{\scriptstyle \Delta_{A,B\boxtimes C}} & & \downarrow{\scriptstyle \Delta_{A\boxtimes B,C}} \\
\mathcal{F}(A) \boxtimes' \mathcal{F}(B \boxtimes C) & & \mathcal{F}(A \boxtimes B) \boxtimes' \mathcal{F}(C) \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{F}(A)} \boxtimes' \Delta_{B,C}} & & \downarrow{\scriptstyle \Delta_{A,B} \boxtimes' \mathrm{id}_{\mathcal{F}(C)}} \\
\mathcal{F}(A) \boxtimes' \big(\mathcal{F}(B) \boxtimes' \mathcal{F}(C)\big) & \xrightarrow{\alpha'_{\mathcal{F}(A),\mathcal{F}(B),\mathcal{F}(C)}} & \big(\mathcal{F}(A) \boxtimes' \mathcal{F}(B)\big) \boxtimes' \mathcal{F}(C)
\end{array} \tag{2.2.1}$$

$$\begin{array}{ccc} \mathcal{F}(B \boxtimes E) & \xrightarrow{\Delta_{B,E}} & \mathcal{F}(B) \boxtimes' \mathcal{F}(E) \\ \Big\downarrow{\scriptstyle \mathcal{F}(r_B)} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{F}(B)} \boxtimes' \delta} \\ \mathcal{F}(B) & \xleftarrow{r'_{\mathcal{F}(B)}} & \mathcal{F}(B) \boxtimes' E' \end{array} \tag{2.2.2}$$

$$\begin{array}{ccc} \mathcal{F}(E \boxtimes B) & \xrightarrow{\Delta_{E,B}} & \mathcal{F}(E) \boxtimes' \mathcal{F}(B) \\ \Big\downarrow{\scriptstyle \mathcal{F}(l_B)} & & \Big\downarrow{\scriptstyle \delta \boxtimes' \mathrm{id}_{\mathcal{F}(B)}} \\ \mathcal{F}(B) & \xleftarrow{l'_{\mathcal{F}(B)}} & E' \boxtimes' \mathcal{F}(B) \end{array} \tag{2.2.3}$$

commute for all $A, B, C \in \mathrm{Obj}(\mathcal{C})$. A cotensor functor is called *strong* if Δ is a natural isomorphism and δ is an isomorphism.

2.2.8 Theorem. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ and $\mathcal{F}' : \mathcal{C}' \to \mathcal{C}''$ be cotensor functors with coproduct morphisms $\Delta_{A,B}, \Delta'_{A',B'}$ and counit morphisms δ, δ'. Then $\mathcal{F}' \circ \mathcal{F}$ is a cotensor functor with coproduct morphisms $\Delta'_{\mathcal{F}(A),\mathcal{F}(B)} \circ \mathcal{F}'(\Delta_{A,B})$ and counit morphism $\delta' \circ \mathcal{F}'(\delta)$.*

This is well known and can be shown by writing down the involved diagrams and check that they commute; see [Lac14] for an explicit proof.

Similarly, a *tensor functor* is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation $\mu : \mathcal{F}(\cdot) \boxtimes' \mathcal{F}(\cdot) \Rightarrow \mathcal{F}(\cdot \boxtimes \cdot)$ and a morphism $\mathbb{1} : E' \to \mathcal{F}(E)$ such that the diagrams one obtains from (2.2.1) -(2.2.3) by reversing the arrows and replacing Δ and δ with μ and $\mathbb{1}$ commute.

2.3 Comonoidal Systems

A *monoid* is a semigroup with a unit element. We identify a monoid $\mathbb{S}$ with the strict tensor category whose objects are the elements of $\mathbb{S}$ with only the identity morphisms and the tensor product given by the multiplication of $\mathbb{S}$.

2.3.1 Definition. Let $\mathbb{S}$ be a monoid and $(\mathcal{C}, \boxtimes)$ a tensor category. A *monoidal system* over $\mathbb{S}$ in $\mathcal{C}$ is a tensor functor from $\mathbb{S}$ to $\mathcal{C}$. A *comonoidal system* over $\mathbb{S}$ in $\mathcal{C}$ is a cotensor functor from $\mathbb{S}$ to $\mathcal{C}$. A comonoidal system is called *full* if the cotensor functor is strong. A monoidal system (respectively comonoidal system) over the trivial monoid $\{e\}$ is simply called a *monoid in* $\mathcal{C}$ (respectively *comonoid in* $\mathcal{C}$).

Since there are only identity morphisms in $\mathbb{S}$, any functor defined on $\mathbb{S}$ acts trivially on morphisms, so it is determined by the object assignment and can be identified with the family $(A_s)_{s\in\mathbb{S}}$ where A_s denotes the value of the functor at $s \in \mathbb{S}$. Thus, a monoidal system over $\mathbb{S}$ in $\mathcal{C}$ is the same as a family of objects $(A_s)_{s\in\mathbb{S}}$ together with *product morphisms* $\mu_{s,t} : A_s \boxtimes A_t \to A_{st}$ and a *unit morphism* $u : E \to A_e$ such that the natural associativity and unit properties

$$\begin{array}{ccc} A_r \boxtimes A_s \boxtimes A_t & \xrightarrow{\mu_{r,s}\boxtimes \mathrm{id}_t} & A_{rs} \boxtimes A_t \\ \downarrow{\scriptstyle \mathrm{id}_r \boxtimes \mu_{s,t}} & & \downarrow{\scriptstyle \mu_{rs,t}} \\ A_r \boxtimes A_{st} & \xrightarrow{\mu_{r,st}} & A_{rst} \end{array} \qquad \begin{array}{ccccc} E \boxtimes A_s & \xleftarrow{l_{A_s}^{-1}} & A_s & \xrightarrow{r_{A_s}^{-1}} & A_s \boxtimes E \\ \downarrow{\scriptstyle u\boxtimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\boxtimes u} \\ A_e \boxtimes A_s & \xrightarrow{\mu_{e,s}} & A_s & \xleftarrow{\mu_{s,e}} & A_s \boxtimes A_e \end{array}$$

are fulfilled. In particular, a monoid in **Set** is just a usual monoid. Similarly, a comonoidal system over $\mathbb{S}$ in $\mathcal{C}$ is a family of objects $(A_s)_{s\in\mathbb{S}}$ together with *coproduct morphisms* $\Delta_{s,t} : A_{st} \to A_s \boxtimes A_t$ and a *counit morphism* $\delta : A_e \to E$ such that coassociativity and the counit properties

$$\begin{array}{ccc} A_{rst} & \xrightarrow{\Delta_{rs,t}} & A_{rs} \boxtimes A_t \\ \downarrow{\scriptstyle \Delta_{r,st}} & & \downarrow{\scriptstyle \Delta_{r,s}\boxtimes \mathrm{id}_t} \\ A_r \boxtimes A_{st} & \xrightarrow{\mathrm{id}_r \boxtimes \Delta_{s,t}} & A_r \boxtimes A_s \boxtimes A_t \end{array} \qquad \begin{array}{ccccc} A_e \boxtimes A_s & \xleftarrow{\Delta_{e,s}} & A_s & \xrightarrow{\Delta_{s,e}} & A_s \boxtimes A_e \\ \downarrow{\scriptstyle \delta\boxtimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\boxtimes \delta} \\ E \boxtimes A_s & \xrightarrow{l_{A_s}} & A_s & \xleftarrow{r_{A_s}} & A_s \boxtimes E \end{array}$$

hold. The composition of two cotensor functors is again a cotensor functor in the sense of Theorem 2.2.8. This immediately implies that a cotensor functor $(\mathcal{F}, \mathcal{D}, d)$ maps a comonoidal system $(A_s)_{s\in\mathbb{S}}$ with coproduct morphisms $\Delta_{s,t}$ and counit morphism δ to a comonoidal systems $(\mathcal{F}(A_s))_{s\in\mathbb{S}}$ with coproduct morphisms $\mathcal{D}_{A_s,A_t} \circ \mathcal{F}(\Delta_{s,t})$ and counit morphism $d \circ \mathcal{F}(\delta)$. The analogous statements hold for monoidal systems.

In the following, we concentrate on comonoidal systems, because they are more important in the subsequent chapters. Most results have an obvious corresponding result for monoidal systems.

2.3.2 Theorem. *Let* $\mathbb{U} \subset \mathbb{S}$ *be a submonoid. If* $(A_s)_{s\in\mathbb{S}}$ *is a comonoidal system with* *coproduct morphisms* $(\Delta_{s,t})_{s,t\in\mathbb{S}}$ *and counit morphism* δ*, then* $(A_s)_{s\in\mathbb{U}}$ *is a comonoidal system with coproduct morphisms* $(\Delta_{s,t})_{s,t\in\mathbb{U}}$ *and counit morphism* δ*.*

Proof. The inclusion $\mathbb{U} \hookrightarrow \mathbb{S}$ is a monoid homomorphism, hence it is a cotensor functor with respect to the identity natural transformation and identity morphism. The theorem now follows from Theorem 2.2.8. □

2.3.1 The Main Examples

In this section we list all kinds of monoidal and comonoidal systems which will play a role later on.

Sets Monoids in $(\mathbf{Set}, \times)$ are just ordinary monoids. Consider the tensor category $(\mathbf{Set}^{inj}, \times)$ with the unit object E. Note that E is necessarily a one point set. For reasons that will become aparent later, we like to assume $E = \{\Lambda\}$ is the set which contains the *empty tuple* $\Lambda = ()$. If $(A_s)_{s\in\mathbb{S}}$ is a comonoidal system, injectivity of the counit morphism $\delta : A_e \to E$ implies that A_e is either empty, or also a one point set. A comonoidal system $(A_s)_{s\in\mathbb{S}}$ with $A_e = \{\Lambda\}$ is called a *Cartesian system*. Cartesian systems over $\mathbb{N}_0, \mathbb{Q}_+$ and $\mathbb{R}_+$ appear a lot throughout Chapter 4.

Vector Spaces Monoids in $(\mathbf{Vect}, \otimes)$ are unital algebras, comonoids are coalgebras. Let $(A_t)_{t\in\mathbb{S}}$ be a monoidal system in $(\mathbf{Vect}, \otimes)$. Then $A := \bigoplus_{t\in\mathbb{S}} A_t$ is an $\mathbb{S}$-graded algebra with respect to the multiplication given by

$$ab := \mu_{s,t}(a \otimes b)$$

for elements $a \in A_s, b \in A_t$. If we consider $(\mathbf{Vect}^{surj}, \otimes)$, a monoidal system over $\mathbb{N}_0$ yields a *standard graded algebra*, that is an $\mathbb{N}_0$-graded algebra $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ with $A_0 = \mathbb{C}\mathbb{1}$ and $A_m A_n = A_{m+n}$. Indeed, the two conditions are exactly the surjectivity of the unit morphism $\mathbb{1} : \mathbb{C} \to A_0$ and the product morphisms $\mu_{m,n} : A_m \otimes A_n \to A_{m+n}$.

Hilbert Spaces Comonoidal systems in $(\mathbf{Hilb}^{isom}, \otimes)$ with $H_e = \mathbb{C}$ and $\delta = \mathrm{id}_{\mathbb{C}}$ are called *subproduct systems*. Subproduct systems over $\mathbb{N}_0, \mathbb{Q}_+$ and $\mathbb{R}_+$ are the main subjects of Chapter 4. Full subproduct systems are called *product systems*. Actually, defining subproduct systems as monoidal systems in $(\mathbf{Hilb}^{coisom}, \otimes)$ gives an equivalent definition. More precisely $((H_s)_{s\in\mathbb{S}}, (\Delta_{s,t})_{s,t\in\mathbb{S}}, \delta)$ is a comonoidal system in $(\mathbf{Hilb}^{isom}, \otimes)$ if and only if $((H_s)_{s\in\mathbb{S}}, (\Delta^*_{s,t})_{s,t\in\mathbb{S}}, \delta^*)$ is a monoidal system in $(\mathbf{Hilb}^{coisom}, \otimes)$.

2.3.3 Remark. The forgetful functor $\mathcal{F} : (\mathbf{FinHilb}^{coisom}, \otimes) \to (\mathbf{FinVect}^{surj})$ is easily seen to be a tensor functor. Tensor functors map monoidal systems to monoidal systems (just as cotensor functors do with comonoidal systems). So it follows from the previous paragraph that $(\mathcal{F}(H_n))_{n\in\mathbb{N}_0}$ yields a standard graded algebra if $(H_n)_{n\in\mathbb{N}_0}$ is a subproduct system.

Algebras Comonoids in $(\mathbf{Alg}_{\mathbb{1}}, \otimes)$ are called *bialgebras*, those in $*$-$\mathbf{Alg}_{\mathbb{1}}$ are referred to as *$*$-bialgebras*. Comonoids in $(\mathbf{Alg}, \sqcup)$ are called *dual semigroups*. The additive deformations of Chapter 3 provide examples of comonoidal systems in $(\mathbf{Alg}_{\mathbb{1}}, \otimes)$ and $(*\text{-}\mathbf{Alg}_{\mathbb{1}}, \otimes)$.

Algebraic Quantum Probability Spaces The tensor categories of quantum probability spaces which will appear are $(\mathbf{AlgQ}_{\mathbb{1}}, \otimes)$ and $(\mathbf{AlgQ}, \boxdot)$ for a universal product $\boxdot$. The main interest lies in comonoidal systems coming from convolution semigroups. We explain this for $(\mathbf{AlgQ}_{\mathbb{1}}, \otimes)$. Suppose $\mathcal{B}$ is a bialgebra with comultiplication Δ and $(\varphi_t)_{t\in\mathbb{R}_+}$ a *convolution semigroup*, that is $\varphi_s \star \varphi_t := (\varphi_s \otimes \varphi_t) \circ \Delta = \varphi_{s+t}$ for all $s, t \in \mathbb{R}_+$ and $\varphi_0 = \delta$ (confer Section 3.1). Then Δ may be viewed as a morphism $\Delta_{s,t} : (\mathcal{B}, \varphi_{s+t}) \to (\mathcal{B} \otimes \mathcal{B}, \varphi_s \otimes \varphi_t)$ and the counit $\delta : \mathcal{B} \to \mathbb{C}$ as a morphism $\delta : (\mathcal{B}, \delta) \to (\mathbb{C}, \mathrm{id}_{\mathbb{C}})$. Coassociativity and the counit property are trivially fulfilled, so $((\mathcal{B}, \varphi_t)_{t\in\mathbb{R}_+}, (\Delta_{s,t})_{s,t\in\mathbb{R}_+}, \delta)$ is a comonoidal system.

2.3.2 Cancellative Monoids

A monoid $\mathbb{S}$ is called *cancellative* if $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$ for all $a \in \mathbb{S}$. Note that left invertibility, right invertibility and invertibility are all equivalent for elements of a cancellative monoid $\mathbb{S}$. Indeed, suppose $ab = e$ with $e \in \mathbb{S}$ the unit element. This implies $baba = bea = bae$. Since $\mathbb{S}$ is cancellative it follows that $ba = e$ and hence $a = b^{-1}$.

Denote by $\mathbb{S}^*$ the set of all tuples $(s_1, \ldots, s_n)$ over $\mathbb{S}$ of arbitrary length $n \in \mathbb{N}_0$. The concatenation of tuples is written in this section as

$$(s_1, \ldots, s_n) \smile (t_1, \ldots, t_m) := (s_1, \ldots, s_n, t_1, \ldots, t_m)$$

to clearly distinguish it from the monoid multiplication. The set of all invertible elements or *units* is denoted by $U(\mathbb{S})$.

A tuple $(s_1, \ldots, s_n) \in \mathbb{S}^*$ is called a *factorization* of $t \in \mathbb{S}$ if $t = s_1 \cdots s_n$ with $s_i \in \mathbb{S} \setminus U(\mathbb{S})$. The empty tuple is the unique factorization of e. The set of all factorizations of t is denoted by $\mathbb{F}_t$. A factorization $\sigma \in \mathbb{F}_t$ is said to be a *refinement* of a factorization $(t_1, \ldots, t_n) \in \mathbb{F}_t$ if $\sigma = \tau_1 \smile \cdots \smile \tau_n$ for some $\tau_k \in \mathbb{F}_{t_k}$. We write $\sigma \geq \tau$ if σ is a refinement of τ. This defines a partial order on $\mathbb{F}_t$.

2.3.4 Proposition. *Let $\mathbb{S}$ be a cancellative monoid. If $\tau_1 \smile \cdots \smile \tau_n = \tau'_1 \smile \cdots \smile \tau'_n$ with $\tau_k, \tau'_k \in \mathbb{F}_{t_k}$, then $\tau_k = \tau'_k$ for all k.*

Proof. For $n = 0$ or $n = 1$ there is nothing to prove. Suppose $\tau_1 \smile \cdots \smile \tau_n = \tau'_1 \smile \cdots \smile \tau'_n = (s_1, \ldots, s_\ell)$ with $\tau_n = (s_k, \ldots, s_\ell), \tau'_n = (s_{k'}, \ldots, s_\ell) \in \mathbb{F}_{t_n}$. We have $s_k \cdots s_\ell = s_{k'} \cdots s_\ell$. Suppose $k < k'$. Since $\mathbb{S}$ is cancellative, this implies $s_k \cdots s_{k'-1} = e$ and thus s_k is invertible which contradicts $\tau_n \in \mathbb{F}_t$. So $k \geq k'$. Analogously, we get $k' \geq k$ which shows $k = k'$ and thus $\tau_n = \tau'_n$. Now the proposition follows by induction. □

2.3.5 Definition. A cancellative monoid $\mathbb{S}$ is called a *unique factorization monoid* or *uf-monoid* for short if any two factorizations of the same element have a common refinement.

Equivalently, a uf-monoid is a cancellative monoid such that $\mathbb{F}_t$ is a directed set with respect to refinement for every $t \in \mathbb{S}$. The term uf-monoid is used by Johnson [Joh71] for this kind of monoids. In the paper he gives different characterizations of uf-monoids and presents constructions to find uf-monoids. The examples we will use later on are only $\mathbb{N}_0$, $\mathbb{Q}_+$ and $\mathbb{R}_+$, but it seems that uf-monoids provide the most general setting in which we can study the inductive limit constructions of the following sections.

2.3.6 Definition. A *positive monoid* is a cancellative monoid $\mathbb{S}$ with a partial order such that

1. $e \leq s$ for all $s \in \mathbb{S}$

2. $s \leq t$ implies $sr \leq tr$ and $rs \leq rt$ for all $r \in \mathbb{S}$.

Johnson requires a positive monoid to be linearly ordered. The next two propositions do not require a linear ordering, so we omit this condition.

2.3.7 Proposition. *Let* $(\mathbb{S}, \leq)$ *be a positive monoid. Then* $U(\mathbb{S}) = \{e\}$.

Proof. Suppose $s \in U(\mathbb{S})$. By the first condition, we have $e \leq s$ and $e \leq s^{-1}$. By the second condition, $e \leq s^{-1}$ implies $s \leq ss^{-1} = e$. Together this yields $s = e$. □

2.3.8 Proposition. *A positive monoid* $(\mathbb{S}, \leq)$ *is directed.*

Proof. From $e \leq s$ it follows that $t \leq st$ and from $e \leq t$ it follows that $s \leq st$. So st is a common upper bound for s and t. □

Let $\mathbb{S}$ be a cancellative abelian monoid with $U(\mathbb{S}) = \{0\}$. Put $s \leq t$ if $t = s + r$ for some $r \in \mathbb{S}$. Since $\mathbb{S}$ is cancellative, r is uniquely determined. So for $s \leq t$ we can define $t - s$ to be the unique r with $t = s + r$. Then it holds that:

2.3.9 Proposition. $(\mathbb{S}, \leq)$ *is a positive monoid.*

Proof. The relation $\leq$ is obviously reflexive and transitive. If $s = t + r$ and $t = s + r'$, we have $s = s + r' + r$ and $t = t + r + r'$, which implies $r' + r = 0$. By $U(\mathbb{S}) = \{0\}$ it follows that $r = r' = 0$, thus $s = t$. Hence, $\leq$ is a partial order on $\mathbb{S}$. The properties 1 and 2 of Definition 2.3.6 are obviously fulfilled. □

2.3.10 Proposition. *Let* $\mathbb{S}$ *be an abelian uf-monoid with* $U(\mathbb{S}) = \{0\}$. *Then* $\leq$ *is a linear order.*

Proof. Let $s, t \in \mathbb{S}$. Since $s + t = t + s$, the unique factorization property tells us that there are $r_1, \ldots, r_n \in \mathbb{S} \setminus \{0\}$ with $r_1 + \cdots + r_k = s$, $r_{k+1} + \cdots + r_n = t$ and $r_1 + \cdots + r_\ell = t$, $r_{\ell+1} + \cdots + r_n = s$. Now, $k \leq \ell$ implies $s \leq t$ and $\ell \leq k$ implies $t \leq s$. □

2.3.3 First Inductive Limit: The Generated Full Comonoidal System

Let $((A_s)_{s\in\mathbb{S}}, (\Delta_{s,t})_{s,t\in\mathbb{S}}, \delta)$ be a comonoidal system over a cancellative monoid $\mathbb{S}$ in a tensor category $(\mathcal{C}, \boxtimes)$ with unit object E. For a tuple $\sigma = (s_1, \dots, s_n) \in \mathbb{F}_t$ put $A_\sigma := A_{s_1} \boxtimes \cdots \boxtimes A_{s_n}$ for $n \geq 1$ and $A_{()} := E$. Define $\Delta_\sigma : A_t \to A_\sigma$ recursively by

$$\Delta_{()} := \delta$$
$$\Delta_{(t)} := \mathrm{id}_t$$
$$\Delta_{(s_1,\dots,s_{n+1})} := (\Delta_{(s_1,\dots,s_n)} \boxtimes \mathrm{id}_{s_{n+1}}) \circ \Delta_{(s_1\cdots s_n, s_{n+1})}.$$

Let $\tau = (t_1, \dots, t_n) \in \mathbb{F}_t$ and $\sigma \geq \tau$. Since $\mathbb{S}$ is cancellative, we can use Proposition 2.3.4 to write $\sigma = \tau_1 \smile \cdots \smile \tau_n$ for uniquely determined $\tau_k \in \mathbb{F}_{t_k}$. With this notation we put

$$\Delta^\tau_\sigma := \Delta_{\tau_1} \boxtimes \cdots \boxtimes \Delta_{\tau_n} : A_\tau \to A_\sigma$$

for all $\tau \leq \sigma \in \mathbb{F}_t$.

2.3.11 Lemma. *It holds that $\Delta^\sigma_\rho \circ \Delta^\tau_\sigma = \Delta^\tau_\rho$ for all $\tau \leq \sigma \leq \rho \in \mathbb{F}_t$ for every cancellative monoid.*

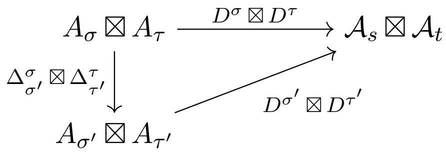
Proof. The proof of Bhat and Mukherjee for the case $\mathbb{S} = \mathbb{R}_+$ [BM10, Lemma 4] works without a change for general cancellative monoids. □

2.3.12 Corollary. *Let $\mathbb{S}$ be a uf-monoid and $((A_s)_{s\in\mathbb{S}}, (\Delta_{s,t})_{s,t\in\mathbb{S}}, \delta)$ a comonoidal system in a tensor category $\mathcal{C}$. Then for every $t \in \mathbb{S}$, $\big((A_\tau)_{\tau\in\mathbb{F}_t}, (\Delta^\tau_\sigma)_{\sigma\geq\tau\in\mathbb{F}_t}\big)$ is an inductive system.*

Proof. By Definition 2.3.5 of a uf-monoid, $\mathbb{F}_t$ is directed. The first condition of an inductive system, $\Delta^\tau_\tau = \mathrm{id}_\tau$, is obvious. The second condition, $\Delta^\sigma_\rho \circ \Delta^\tau_\sigma = \Delta^\tau_\rho$ for all $\tau \leq \sigma \leq \rho \in \mathbb{F}_t$, is the statement of Lemma 2.3.11. □

Suppose the inductive systems $(A_\tau)_{\tau\in\mathbb{F}_t}$ have inductive limits $\mathcal{A}_t$ with morphisms $D^\tau : A_\tau \to \mathcal{A}_t$. For $\tau \in \mathbb{F}_t$ denote by $\mathbb{F}_\tau$ the set of all refinements of τ. Then $\mathbb{F}_\tau$ is a cofinal subset of $\mathbb{F}_t$, see Example 2.1.3. Denote by $\mathcal{A}_\tau$ the inductive limit. Then there is a canonical isomorphism $\mathcal{A}_t \cong \mathcal{A}_\tau$ because of Corollary 2.1.5.

2.3.13 Lemma. *The diagram*



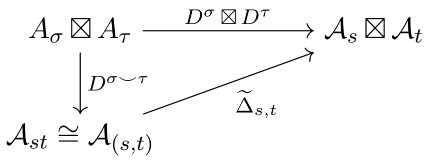
commutes for all $\sigma' \geq \sigma \in \mathbb{F}_s, \tau' \geq \tau \in \mathbb{F}_t$.

Proof. By functoriality of $\boxtimes$, we have

$$(D^{\sigma'} \boxtimes D^{\tau'}) \circ (\Delta^\sigma_{\sigma'} \boxtimes \Delta^\tau_{\tau'}) = (D^{\sigma'} \circ \Delta^\sigma_{\sigma'}) \boxtimes (D^{\tau'} \circ \Delta^\tau_{\tau'}) = D^\sigma \boxtimes D^\tau.$$

□

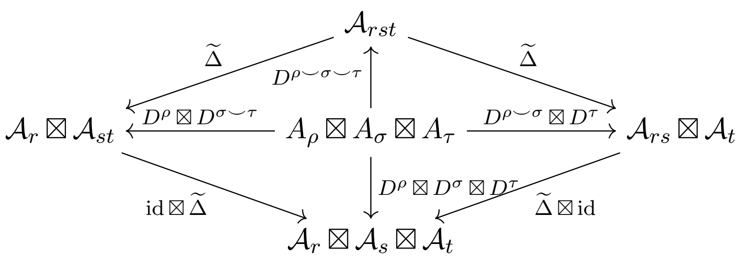
So, by the universal property of the inductive limit, there are unique morphisms $\widetilde{\Delta}_{s,t} : \mathcal{A}_{st} \to \mathcal{A}_s \boxtimes \mathcal{A}_t$ such that



commutes for every $\sigma \in \mathbb{F}_s, \tau \in \mathbb{F}_t$.

2.3.14 Theorem. *The* $\mathcal{A}_t$ *form a comonoidal system with respect to the coproduct morphisms* $\widetilde{\Delta}_{s,t}$ *and the counit morphism* id_E.

Proof. First note that $\mathcal{A}_e = E$, since $\mathbb{F}_e = \{()\}$ and $A_{()} = E$. The counit property is trivially fulfilled. In the diagram



the four corners commute by the definition of $\widetilde{\Delta}$ and this proves coassociativity. $\square$

Define $D_t : A_t \to \mathcal{A}_t$ by $D_t := D^{(t)}$ for $t \neq e$ and $D_e := \delta$.

2.3.15 Theorem. *The morphisms $(D_t)_{t\in\mathbb{S}}$ form a* morphism of comonoidal systems*, that is $\widetilde{\Delta}_{s,t} \circ D_{st} = (D_s \boxtimes D_t) \circ \Delta_{s,t}$ and $\mathrm{id}_E \circ D_e = \delta$.*

Proof. The counit is respected by definition of D_e. In the diagram

$$\begin{array}{ccc} A_{st} & \xrightarrow{\Delta_{s,t}} & A_s \boxtimes A_t \\ \big\downarrow{\scriptstyle D^{(st)}} & \swarrow{\scriptstyle D^{(s,t)}} & \big\downarrow{\scriptstyle D^{(s)} \boxtimes D^{(t)}} \\ \mathcal{A}_{st} \cong \mathcal{A}_{(s,t)} & \xrightarrow[\widetilde{\Delta}_{s,t}]{} & \mathcal{A}_s \boxtimes \mathcal{A}_t \end{array}$$

the lower right commutes by definition of $\widetilde{\Delta}$ and the upper left because $\mathcal{A}_{st}$ is the inductive limit. So the outside square commutes, which finishes the proof. $\square$

Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor. Then any inductive system $\big((A_\alpha)_\alpha, (f^\alpha_\beta)_{\alpha\leq\beta}\big)$ in $\mathcal{C}$ is mapped to an inductive system $\big((\mathcal{F}(A_\alpha))_\alpha, (\mathcal{F}(f^\alpha_\beta))_{\alpha\leq\beta}\big)$ in $\mathcal{D}$. We say that $\mathcal{F}$ *preserves inductive limits* if for every inductive limit $\big(\mathcal{A}, (f_\alpha)_\alpha\big)$ of an inductive system $\big((A_\alpha)_\alpha, (f^\alpha_\beta)_{\alpha\leq\beta}\big)$ it holds that $\big(\mathcal{F}(\mathcal{A}), (\mathcal{F}(f_\alpha))_\alpha\big)$ is an inductive limit of the inductive system $\big((\mathcal{F}(A_\alpha))_\alpha, (\mathcal{F}(f^\alpha_\beta))_{\alpha\leq\beta}\big)$.

2.3.16 Theorem. *If the tensor product preserves inductive limits, the morphisms $\widetilde{\Delta}_{s,t}$ are all isomorphisms. In other words $(\mathcal{A}_t)_{t\in\mathbb{S}}$ is a full comonoidal system.*

Proof. The tensor product is a bifunctor $\boxtimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$. Inductive systems in $\mathcal{C}\times\mathcal{C}$ are in bijection with pairs of inductive systems in $\mathcal{C}$ and an inductive limit in $\mathcal{C}\times\mathcal{C}$ is a pair of inductive limits for the inductive systems in $\mathcal{C}$. If $\boxtimes$ preserves inductive limits, $\mathcal{A}_s \boxtimes \mathcal{A}_t$ is an inductive limit of the inductive system formed by $(A_\sigma \boxtimes A_\tau)_{\sigma\in\mathbb{F}_s, \tau\in\mathbb{F}_t}$ with respect to the maps $D^\sigma \boxtimes D^\tau$. Since $\widetilde{\Delta}_{s,t}$ makes the diagram

$$\begin{array}{ccc} A_\sigma \boxtimes A_\tau & \xrightarrow{D^\sigma \boxtimes D^\tau} & \mathcal{A}_s \boxtimes \mathcal{A}_t \\ {\scriptstyle D^{\sigma\smile\tau}}\big\downarrow & \nearrow{\scriptstyle \widetilde{\Delta}_{s,t}} & \\ \mathcal{A}_{st} \cong \mathcal{A}_{(s,t)} & & \end{array}$$

commute, it is the canonical isomorphism between the two inductive limits. □

All tensor categories we are interested in have tensor products which do preserve inductive limits. The following example illustrates that this is not true in general.

2.3.17 Example. Consider the category **Vect** with the tensor product given by

$$V_1 \boxtimes V_2 := \begin{cases} V_1 \oplus V_2 & \text{if one of the two is finite-dimensional} \\ V_1 \oplus V_2 \oplus W & \text{if both are infinite-dimensional} \end{cases}$$

for some fixed vector space W. This is a tensor category with unit object $\{0\}$ and even has inclusions. Consider the inductive system $(\mathbb{C}^n)_{n\in\mathbb{N}}$ with morphisms $f_n^m(x_1,\ldots,x_m) := (x_1,\ldots,x_m,0,\ldots,0)$ with inductive limit l_0; see Example 2.1.2. Then the inductive limit of the inductive system $(\mathbb{C}^m \oplus \mathbb{C}^n)_{m,n\in\mathbb{N}}$ is $l_0 \oplus l_0$ and has a countable basis. But if W has no countable basis, then also the tensor product $l_0 \boxtimes l_0 = l_0 \oplus l_0 \oplus W$ of the seperate inductive limits has no countable basis, so there cannot be an isomorphism. In fact, even if we put $W = \mathbb{C}$ the tensor product defined above does not preserve inductive limits, but one would have to argue more carefully to prove this.

2.3.4 Compatible Inclusions

We can define independence of two morphisms in any tensor category with inclusions. But to get a good (associative) notion for more than two morphisms, we need a certain compatiblity between the inclusions and the structure of the tensor category. This is necessary in order to deal with Lévy processes in the categorial setting. As it turns out, the condition we need is equivalent to the unit object E being an initial object and ι^1, ι^2 being related to the unit constraints l and r in a nice way.

Let $\mathcal{C}$ be a tensor category with inclusions ι^1, ι^2. The inclusions are called *compatible* if $\iota^1_{A,E} = r_A^{-1}$, $\iota^2_{E,A} = l_A^{-1}$ and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\iota^1_{E,A}} & E \boxtimes A \\ \downarrow{\scriptstyle \iota^2_{A,E}} & & \downarrow{\scriptstyle l_A} \\ A \boxtimes E & \xrightarrow{r_A} & A \end{array} \tag{2.3.1}$$

commutes for all $A \in \mathcal{C}$

2.3.18 Theorem. *In a tensor category $(\mathcal{C}, \boxtimes, E)$ with inclusions ι^1, ι^2, the following are equivalent:*

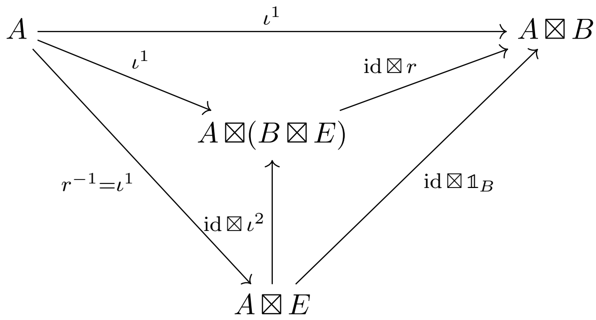
(*i*) *The inclusions are compatible.*

(*ii*) *There exists a natural transformation $\mathbb{1} : E \Rightarrow \mathrm{id}_\mathcal{C}$ such that $\mathbb{1}_E = \mathrm{id}_E$, $\iota^1_{A,B} = (\mathrm{id}_A \boxtimes \mathbb{1}_B) \circ r_A^{-1}$ and $\iota^2_{A,B} = (\mathbb{1}_A \boxtimes \mathrm{id}_B) \circ l_B^{-1}$ for all $A, B \in \mathcal{C}$.*

Furthermore, $\mathbb{1}$ is uniquely determined, if it exists.

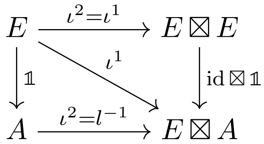
Proof. If $\mathbb{1}$ exists, $\iota^1_{E,A} = (\mathrm{id}_E \boxtimes \mathbb{1}_A) \circ r_E^{-1}$ implies $\mathrm{id} \boxtimes \mathbb{1}_A = \iota^1_{E,A} \circ r_E$ and thus $\mathbb{1}_A = l_A \circ \iota^1_{E,A} \circ r_E \circ l_E^{-1}$, so $\mathbb{1}$ is uniquely determined.

Suppose ι^1, ι^2 are compatible. Then define $\mathbb{1}_A := l_A \circ \iota^1_{E,A} = r_A \circ \iota^2_{A,E}$. From $\iota^1_{E,E} = r_E^{-1}$ we get $\mathbb{1}_E = \mathrm{id}_E$. In the diagram



the upper triangle and the left triangle commute by the naturality of ι^1. The right triangle commutes by definition of $\mathbb{1}_B$. Together this yields that the outside triangle commutes, hence $\iota^1_{A,B} = (\mathrm{id} \boxtimes \mathbb{1}_B) \circ r_A^{-1}$. Analogously, one shows $\iota^2_{A,B} = (\mathbb{1}_A \boxtimes \mathrm{id}) \circ l_B^{-1}$.

Now assume (*ii*) holds. Then $\iota^1_{A,E} = (\mathrm{id} \boxtimes \mathbb{1}_E) \circ r_A^{-1} = r_A^{-1}$ and in the same way we get $\iota^2_{E,A} = l_A^{-1}$. In particular $\iota^1_{E,E} = r_E^{-1} = l_E^{-1} = \iota^2_{E,E}$. Considering the diagram



the outside square commutes by naturality of ι^2. The upper right triangle commutes by naturality of ι^1. So the lower left triangle commutes and we find $l_A \circ \iota^1_{E,A} = \mathbb{1}_A$. Analogously, we get $r_A \circ \iota^2_{A,E} = \mathbb{1}_A$, which finishes the proof. □

Note that naturality of $\mathbb{1}$ just means that $f \circ \mathbb{1}_A = \mathbb{1}_B$ for all $f : A \to B$. In particular for $f : E \to B$ this yields $f = f \circ \mathbb{1}_E = \mathbb{1}_B$. So E is an *initial object* in $\mathcal{C}$, that is for every object $B \in \mathcal{C}$ there is a unique morphism from E to B, namely $\mathbb{1}_B$. Of course, given any tensor category such that E is initial, we can define compatible inclusions by the equations in (ii).

Compatible inclusions also respect the associativity constraint. For example it holds that:

2.3.19 Theorem. *For compatible inclusions ι^1, ι^2 the diagrams*

$$\begin{array}{ccccc} A \boxtimes B & \xleftarrow{\iota^2} & B & \xrightarrow{\iota^1} & B \boxtimes C \\ \downarrow{\scriptstyle \iota^1} & & & & \downarrow{\scriptstyle \iota^2} \\ (A \boxtimes B) \boxtimes C & & \xrightarrow{\quad\alpha\quad} & & A \boxtimes (B \boxtimes C) \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\quad\iota^2\quad} & B \boxtimes C \\ \downarrow{\scriptstyle \iota^2} & & \downarrow{\scriptstyle \iota^2} \\ (A \boxtimes B) \boxtimes C & \xrightarrow{\alpha} & A \boxtimes (B \boxtimes C) \end{array}$$

commute.

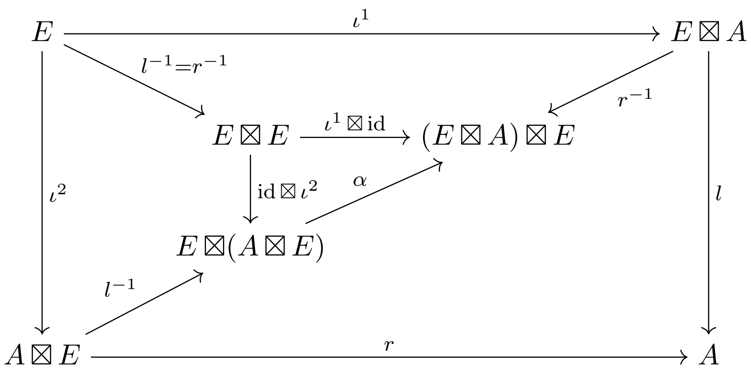
Proof. Since by Theorem 2.3.18 we have $\iota_1 = (\mathrm{id} \boxtimes \mathbb{1}) \circ r^{-1}$ and $\iota_2 = (\mathbb{1} \otimes \mathrm{id}) \circ l^{-1}$, the first diagram can be found on the outside border of

$$\begin{array}{ccccccccc} A \boxtimes B & \xleftarrow{\mathbb{1} \boxtimes \mathrm{id}} & E \boxtimes B & \xleftarrow{l^{-1}} & B & \xrightarrow{r^{-1}} & B \boxtimes E & \xrightarrow{\mathrm{id} \boxtimes \mathbb{1}} & B \boxtimes C \\ \downarrow{\scriptstyle r^{-1}} & & \downarrow{\scriptstyle r^{-1}} & & & & \downarrow{\scriptstyle l^{-1}} & & \downarrow{\scriptstyle l^{-1}} \\ (A \boxtimes B) \boxtimes E & \xleftarrow{(\mathbb{1} \boxtimes \mathrm{id}) \boxtimes \mathrm{id}} & (E \boxtimes B) \boxtimes E & & \xrightarrow{\quad\alpha\quad} & & E \boxtimes (B \boxtimes E) & \xrightarrow{\mathrm{id} \boxtimes (\mathrm{id} \boxtimes \mathbb{1})} & E \boxtimes (B \boxtimes C) \\ & \searrow{\scriptstyle \mathrm{id} \boxtimes \mathbb{1}} & \downarrow{\scriptstyle (\mathbb{1} \boxtimes \mathrm{id}) \boxtimes \mathbb{1}} & & & & \downarrow{\scriptstyle \mathbb{1} \boxtimes (\mathrm{id} \boxtimes \mathbb{1})} & \swarrow{\scriptstyle \mathbb{1} \boxtimes \mathrm{id}} & \\ & & (A \boxtimes B) \boxtimes C & & \xrightarrow{\quad\alpha\quad} & & A \boxtimes (B \boxtimes C) & & \end{array}$$

The upper squares commute due to the naturality of r^{-1}, coherence of $\mathcal{C}$ and the naturality of l^{-1}. The square in the lower line commutes due to the naturality of α. The remaining triangles commute because of the functoriality of $\boxtimes$. We conclude that the outside commutes. Commutativity of the second diagram in the theorem can be shown similarly. □

The two diagrams of the theorem above are the only compatibilities we need in the next section. But it is probably possible to formulate and prove a coherence theorem for tensor categories with compatible inclusions, similar to [ML98, VII.2].

2.3.20 Remark. As Stephanie Lachs observed, commutativity of (2.3.1) is equivalent to the somehow more natural condition that $\iota^2 \boxtimes \mathrm{id} = \alpha \circ (\mathrm{id} \boxtimes \iota^1)$. This can be read from the diagram



in which the outside square is (2.3.1) and the inside triangle is a special case of $\iota^2 \boxtimes \mathrm{id} = \alpha \circ (\mathrm{id} \boxtimes \iota^1)$. All other areas commute, the upper and the left side due to the naturality of l^{-1} and r^{-1} respectively, the lower right due to coherence. So commutativity of the inside and the outside are equivalent.

2.3.5 Second Inductive Limit: Lévy-Processes

Let $\mathcal{A}_t$ be a full comonoidal system over a cancellative abelian monoid $\mathbb{S}$ with coproduct isomorphisms $\widetilde{\Delta}_{s,t}$ in a tensor category with compatible inclusions ι^1, ι^2. Without loss of generality assume that $\mathcal{A}_e = E$ and that the counit morphism is $\delta = \mathrm{id}_E$.

For $s \leq t$ define $i_t^s : \mathcal{A}_s \to \mathcal{A}_t$ as the composition

$$\mathcal{A}_s \xrightarrow{\iota^1} \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} \xrightarrow{\widetilde{\Delta}^{-1}} \mathcal{A}_t.$$

2.3.21 Theorem. *$((\mathcal{A}_t)_{t \in \mathbb{S}}, (i_t^s)_{s \leq t})$ is an inductive system.*

Proof. In the diagram

$$\begin{array}{ccccc}
\mathcal{A}_r & \xrightarrow{\iota^1} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_s \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \mathrm{id}\boxtimes\iota^1}\Big\downarrow{\scriptstyle \iota^1} & & \Big\downarrow{\scriptstyle \iota^1} \\
\mathcal{A}_r & \xrightarrow{\iota^1} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\widetilde{\Delta}^{-1}\boxtimes\mathrm{id}} & \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}\boxtimes\widetilde{\Delta}^{-1}} & & \Big\downarrow{\scriptstyle \widetilde{\Delta}^{-1}} \\
\mathcal{A}_r & \xrightarrow{\iota^1} & \mathcal{A}_r \boxtimes \mathcal{A}_{t-r} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_t
\end{array}$$

the lower right corner commutes by coassociativity of $\widetilde{\Delta}$ and the other three corners commute by the naturality of ι^1. We suppressed the associativity constraint and identified $\mathcal{A}_r \boxtimes(\mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s})$ with $(\mathcal{A}_r \boxtimes \mathcal{A}_{s-r}) \boxtimes \mathcal{A}_{t-s}$, which leads to the two interpretations $\mathrm{id}_{\mathcal{A}_r} \boxtimes \iota^1_{\mathcal{A}_{s-r},\mathcal{A}_{t-s}}$ and $\iota^1_{\mathcal{A}_r \boxtimes \mathcal{A}_{s-t},\mathcal{A}_{t-s}}$ of the arrow $\mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \to \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s}$. □

For the rest of this section, we fix an abelian uf-monoid $\mathbb{S}$ with $U(\mathbb{S}) = \{0\}$ and an inductively complete tensor category $\mathcal{C}$ with compatible inclusions ι^1, ι^2, whose tensor product $\boxtimes$ perserves inductive limits.

2.3.22 Definition. Let $(A_t)_{t\in\mathbb{S}}$ be a comonoidal system in $\mathcal{C}$. An *abstract Lévy-process on* $(A_t)_{t\in\mathbb{S}}$ is a collection of morphisms $j_{s,t} : A_{t-s} \to B$ for $s \leq t$ to some common object $B \in \mathcal{C}$ such that

1. $j_{t,t} = \mathbb{1}_B \circ \delta$
2. $j_{s_1,t_1}, \ldots, j_{s_n,t_n}$ are independent if $s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_n \leq t_n$
3. $j_{r,s,t} \circ \Delta_{s-r,t-s} = j_{r,s}$ for some independence morphism $j_{r,s,t}$ of $j_{r,s}$ and $j_{s,t}$.

Given only the comonoidal system $\big((A_t)_{t\in\mathbb{S}}, (\Delta_{s,t})_{s,t\in\mathbb{S}}\big)$ one can construct a canonical abstract Lévy-process. Since $\mathbb{S}$ is a uf-monoid and $\boxtimes$ preserves inductive limits, $(A_t)_{t\in\mathbb{S}}$ generates a full comonoidal system $\big((\mathcal{A}_t)_{t\in\mathbb{S}}, \widetilde{\Delta}\big)$ by Theorem 2.3.16. Denote by $D^t : A_t \to \mathcal{A}_t$ the canonical morphisms. Let $\big(\mathcal{A}, (i^t : \mathcal{A}_t \to \mathcal{A})_{t\in\mathbb{S}}\big)$ the inductive limit of $(\mathcal{A}_t)_{t\in\mathbb{S}}$. Define $j_{s,t} : A_{t-s} \to \mathcal{A}$ as the composition

$$A_{t-s} \xrightarrow{D^{t-s}} \mathcal{A}_{t-s} \xrightarrow{\iota^2} \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} \xrightarrow{\widetilde{\Delta}^{-1}} \mathcal{A}_t \xrightarrow{i^t} \mathcal{A}.$$

2.3.23 Theorem. *The $j_{s,t}$ form an abstract Lévy process.*

Proof. We construct the independence morphism $j_{r,s,t}$ for $j_{r,s}$ and $j_{s,t}$ and show that $j_{r,s,t} \circ \Delta_{s-r,t-s} = j_{r,t}$. Define $j_{r,s,t}$ as the composition

$$A_{s-r} \boxtimes A_{t-s} \xrightarrow{D^{s-r} \boxtimes D^{t-s}} \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} \xrightarrow{\iota^2} \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} \xrightarrow{\widetilde{\Delta}^{-1}} \mathcal{A}_t \xrightarrow{i^t} \mathcal{A}.$$

Now, the diagram

$$\begin{array}{ccccccccc}
A_{t-s} & \xrightarrow{D^{t-s}} & \mathcal{A}_{t-s} & \xrightarrow{\iota^2} & \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_t & \overset{i^t}{\searrow} & \\
\downarrow{\scriptstyle \iota^2} & & \downarrow{\scriptstyle \iota^2} & & \downarrow{\scriptstyle \widetilde{\Delta}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
A_{s-r} \boxtimes A_{t-s} & \xrightarrow{D^{s-r} \boxtimes D^{t-s}} & \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_t & \xrightarrow{i^t} & \mathcal{A} \\
{\scriptstyle \iota^1}\uparrow & & {\scriptstyle \iota^1}\uparrow & & {\scriptstyle \iota^1}\uparrow & & {\scriptstyle i_t^s}\uparrow & & \\
A_{s-r} & \xrightarrow{D^{s-r}} & \mathcal{A}_{s-r} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_s & \underset{i^s}{\nearrow} &
\end{array}$$

commutes: The leftmost squares commute due to naturality of ι^1 and ι^2. The next squares commute by Theorem 2.3.19. The upper right square commutes by coassociativity of $\widetilde{\Delta}$. The triangles commute by definition of the inductive limit. It remains to show commutativity of the lower right square. In a bit more detail, this is

$$\begin{array}{ccccc}
\mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\widetilde{\Delta}^{-1} \boxtimes \mathrm{id}} & \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\widetilde{\Delta}} & \mathcal{A}_t \\
{\scriptstyle \iota^1}\uparrow & & {\scriptstyle \iota^1}\uparrow & \overset{i_t^s}{\nearrow} & \\
\mathcal{A}_r \boxtimes \mathcal{A}_{s-r} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_s & &
\end{array}$$

which commutes by naturality of ι^1 and the definition of i_t^s. This shows that $j_{r,s,t}$ is an independence morphism. Next we consider

$$\begin{array}{ccccccccc}
A_{t-r} & \xrightarrow{D^{t-r}} & \mathcal{A}_{t-r} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{t-r} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_t & \overset{i^t}{\searrow} & \\
\downarrow{\scriptstyle \Delta_{s-r,t-s}} & & \downarrow{\scriptstyle \widetilde{\Delta}} & & \downarrow{\scriptstyle \mathrm{id} \boxtimes \widetilde{\Delta}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
A_{s-r} \boxtimes A_{t-s} & \xrightarrow{D^{s-r} \boxtimes D^{t-s}} & \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\widetilde{\Delta}^{-1}} & \mathcal{A}_t & \xrightarrow{i^t} & \mathcal{A}
\end{array}$$

in which the first square commutes because the D^t form a morphism of comonoidal systems by Theorem 2.3.15, the second square commutes due to naturality of ι^2, and the last square and the triangle commute trivially. So the outside commutes, thus establishing $j_{r,s,t} \circ \Delta_{s-r,t-s} = j_{r,t}$.

The general construction of an independence morphism for $j_{t_1,t_2}, \ldots, j_{t_n,t_{n+1}}$ works similar to that for $j_{r,s}$ and $j_{s,t}$. □

There is also a direct way from the comonoidal system $(A_t)_{t\in\mathbb{S}}$ to the Lévy process. Put $\mathbb{F} := \{\sigma = (s_1, \ldots, s_n) \mid n \in \mathbb{N}_0, s_k \in \mathbb{S} \setminus \{0\}\} = \bigcup_{s\in\mathbb{S}} \mathbb{F}_s$ and define $\sigma \geq \tau = (t_1, \ldots, t_n)$ if there exist $\tau_1, \ldots, \tau_n, \tau_{n+1}$ with $\sigma = \tau_1 \smile \cdots \smile \tau_n \smile \tau_{n+1}$, $\tau_k \in \mathbb{F}_{t_k}$ for $k \in \{1, \ldots, n\}$ and $\tau_{n+1} \in \mathbb{F}$. One shows that $\mathbb{F}$ is directed analogously to $\mathbb{F}_t$. Then define an inductive system $(A_\sigma)_{\sigma\in\mathbb{F}}$ with respect to the morphisms $i_\sigma^\tau : A_\tau \to A_\sigma$ defined as the composition

$$A_\tau \xrightarrow{\Delta^\tau_{\tau_1\smile\cdots\smile\tau_n}} A_{\tau_1} \boxtimes \cdots \boxtimes A_{\tau_n} \xrightarrow{\iota^1} A_{\tau_1} \boxtimes \cdots \boxtimes A_{\tau_n} \boxtimes A_{\tau_{n+1}} = A_\sigma$$

for $\sigma = \tau_1 \smile \cdots \smile \tau_n \smile \tau_{n+1} \geq \tau$.

2.3.24 Theorem. *The inductive limits of $(A_\sigma)_{\sigma\in\mathbb{F}}$ and $(\mathcal{A}_t)_{t\in\mathbb{S}}$ are canonically isomorphic.*

Proof. This is exactly the situation of Theorem 2.1.4. □

3 Additive Deformations

The material presented in Sections 3.2 and 3.3 is taken from [Ger11], Section 3.4 is from [GKL12] with Stefan Kietzmann and Stephanie Lachs. Only minor changes have been made, mainly to avoid redundance, unify notations, or emphasize connections to other parts of this thesis.

The quantum harmonic oscillator is usually described in terms of the $*$-algebra $\mathcal{A}$ generated by a, a^* and a unit $\mathbb{1}$ with the relation $[a, a^*] = \mathbb{1}$. One wants to define a comultiplication on the generators by

$$\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a, \quad \Delta(a^*) = a^* \otimes \mathbb{1} + \mathbb{1} \otimes a^*,$$

but this cannot be extended as an algebra homomorphism from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$ since the relation is not respected. As a possible solution to this problem, one can define $\mathcal{A}_t$ as the algebra with the same generators as $\mathcal{A}$ but under relation $[a, a^*]_t = t\mathbb{1}$ for $t \in \mathbb{R}$. Then it is easily checked that Δ can be extended to an algebra homomorphism from $\mathcal{A}$ to $\mathcal{A}_s \otimes \mathcal{A}_t$ whenever $s + t = 1$. One observes that all $\mathcal{A}_t$ are defined on the same vector space and $\mathcal{A}_0$ is just the polynomial $*$-bialgebra in two commuting adjoint indeterminates. This way to look at the quantum harmonic oscillator is already mentioned by Majid in [Maj95, p. 71], where he just calls this a “bialgebra like structure”. Wirth gave a rigorous meaning to this phrase with the definition of additive deformations of bialgebras and $*$-bialgebras in [Wir02], see Definition 3.2.1. The terminology is slightly misleading: An additive deformation of a bialgebra is a deformation of the algebra structure of $\mathcal{B}$ which is compatible with the comultiplication in a certain way. But the deformed algebras $\mathcal{B}_t$ themselves are not bialgebras. Additive deformations are comonoidal systems in the category of unital $(*)$-algebras, see Section 2.3.1, Algebras. This special class of comonoidal systems is quite tractable and contains a lot of interesting examples.

Wirth developed a description of additive deformations in terms of infinitesimal generators. Furthermore he was able to prove a Schoenberg correspondence for additive deformations, which characterizes the continuous convolution semigroups of states on additive deformations by a condition on the generator. This was the starting point for the construction of quantum Lévy processes on additive deformations in [Sch05] and [Ger09]. The relationship between quantum Lévy processes and convolution semigroups in this case also follows from our general considerations in Chapter 2.

In Section 3.2 we will review Wirth's results and investigate the underlying cohomological theory. Most of the material of this section is already contained in my Diploma thesis [Ger09], but we will need it in the sequel and present it in a more systematic way, now also including the description of the cochain complex for the $*$-case which is new. In Section 3.3, we show that an additive deformation of a Hopf algebra automatically fulfills a compatibility with the antipode and find a natural decomposition of additive deformations of cocommutative Hopf algebras. Finally, Section 3.4 transfers the generator calculus, the result on compatibility with the antipode, and the Schoenberg correspondence to the braided case. This is necessary to treat, for example, the fermionic harmonic oscillator in the framework of additive deformations in Section 3.4.5.

3.1 Convolution

The following conventions and definitions will be used throughout this chapter. The algebraic dual of a vector space $\mathcal{V}$ is denoted $\mathcal{V}' := \{\varphi : \mathcal{V} \to \mathbb{C} \mid \varphi \text{ linear}\}$. The tensor product $\otimes$ is the usual tensor product of vector spaces. If $\mathcal{V}$ is a vector space, for $n \geq 1$ we write $\mathcal{V}^{\otimes n} := \mathcal{V} \otimes \cdots \otimes \mathcal{V}$ for the n-fold tensor product of $\mathcal{V}$ with itself and for $n = 0$ we put $\mathcal{V}^{\otimes 0} := \mathbb{C}$. For two vector spaces $\mathcal{V}$ and $\mathcal{W}$, we denote by τ the *flip* from $\mathcal{V} \otimes \mathcal{W}$ to $\mathcal{W} \otimes \mathcal{V}$ given on simple tensors by $\tau(v \otimes w) = w \otimes v$.

A *bialgebra* $(\mathcal{B}, \mu, \mathbb{1}, \Delta, \delta)$ is a complex unital associative algebra $(\mathcal{B}, \mu, \mathbb{1})$ for which the mappings $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and $\delta : \mathcal{B} \to \mathbb{C}$ are algebra homomorphisms and satisfy coassociativity and counit property respectively. In other words, a bialgebra is a comonoid in $\mathbf{Alg}_{\mathbb{1}}$ and, similarly, a $*$-*bialgebra* is a comonoid in $*$-$\mathbf{Alg}_{\mathbb{1}}$, see Section 2.3.1, Algebras.

A *Hopf algebra* is a bialgebra with an *antipode*, that is a linear mapping $S : \mathcal{B} \to \mathcal{B}$ with

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \mathbb{1}\delta = \mu \circ (S \otimes \mathrm{id}) \circ \Delta.$$

A *Hopf $*$-algebra* is a Hopf algebra which also is a $*$-bialgebra. For details on Hopf algebras and bialgebras see for example [Swe69] or [Abe80], for Hopf $*$-algebras [KS97].

We use Sweedler's notation, writing $\Delta a = \sum_{k=0}^{n} a_k^{(1)} \otimes a_k^{(2)} =: a_{(1)} \otimes a_{(2)}$ and the notations $\mu^{(n)} : \mathcal{B}^{\otimes n} \to \mathcal{B}$, $\Delta^{(n)} : \mathcal{B} \to \mathcal{B}^{\otimes n}$

$$\mu^{(0)}(\lambda) = \lambda \mathbb{1}, \qquad \Delta^{(0)} = \delta,$$
$$\mu^{(n+1)} = \mu \circ \left(\mathrm{id} \otimes \mu^{(n)}\right), \qquad \Delta^{(n+1)} = \left(\mathrm{id} \otimes \Delta^{(n)}\right) \circ \Delta.$$

The Sweedler notation for this is

$$\Delta^{(n)} a = a_{(1)} \otimes \cdots \otimes a_{(n)}.$$

With $\mathcal{B}$ also each $\mathcal{B}^{\otimes n}$ is a bialgebra in the natural way. We frequently use the comultiplication on $\mathcal{B} \otimes \mathcal{B}$, denoted by Λ, which is defined by

$$\Lambda(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)},$$

that is $\Lambda = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$. The counit of $\mathcal{B} \otimes \mathcal{B}$ is just $\delta \otimes \delta$. If $\mathcal{B}$ is a $*$-bialgebra, an involution on $\mathcal{B}^{\otimes n}$ is given by $(a_1 \otimes \cdots \otimes a_n)^* = a_1^* \otimes \cdots \otimes a_n^*$.

If $(\mathcal{C}, \Delta, \delta)$ is a coalgebra and $(\mathcal{A}, \mu, \mathbb{1})$ is a unital algebra, we define the *convolution* product for linear mappings $R, S : \mathcal{C} \to \mathcal{A}$ by

$$R \star S := \mu \circ (R \otimes S) \circ \Delta.$$

This turns the space of linear maps from $\mathcal{C}$ to $\mathcal{A}$ into a unital algebra with unit $\mathbb{1}\delta$. In our context $\mathcal{C}$ and $\mathcal{A}$ are usually tensor powers of the same bialgebra $\mathcal{B}$.

A *pointwise continuous convolution semigroup* is a family $(\varphi_t)_{t \geq 0}$ of linear functionals $\varphi_t : \mathcal{B} \to \mathbb{C}$ such that

- $\varphi_s \star \varphi_t = \varphi_{s+t}$

- $\varphi_t(b) \xrightarrow{t\to 0} \delta(b)$ for all $b \in \mathcal{B}$.

It follows from the fundamental theorem for coalgebras that for a pointwise continuous convolution semigroup there exists a *generator* ψ, which is the pointwise limit

$$\psi(b) = \left.\frac{\mathrm{d}\varphi_t(b)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0} \frac{\varphi_t(b) - \delta(b)}{t}$$

and for which we have

$$\varphi_t = e_\star^{t\psi} := \sum_{n=0}^{\infty} (t\psi)^{\star n},$$

also as a pointwise limit. Confer [ASvW88, Section 4] for details.

3.2 Additive Deformations of Bialgebras

Deformations of algebras are closely related to cohomology, as Gerstenhaber showed in his papers [Ger63, Ger64]. Suppose that $\mathcal{A}$ is an algebra and $(\mu_t)_{t\in\mathbb{R}_+}$ a family of associative multiplications on $\mathcal{A}$ which can in any sense be written in the form

$$\mu_t(a \otimes b) = \mu(a \otimes b) + tF(a \otimes b) + \mathcal{O}(t^2)$$

where $\mu_0 = \mu$ is the original multiplication of the algebra. Writing down the associativity condition for μ_t and comparing the terms of first order yields that

$$\mu\Big(F(a \otimes b) \otimes c\Big) + F\Big(\mu(a \otimes b) \otimes c\Big) = \mu\Big(a \otimes F(b \otimes c)\Big) + F\Big(a \otimes \mu(b \otimes c)\Big)$$

and after rearranging

$$aF(b \otimes c) - F(ab \otimes c) + F(a \otimes bc) - F(a \otimes b)c = 0,$$

so the infinitesimal deformation F is a cocycle in the Hochschild cohomology associated with the $\mathcal{A}$-bimodule structure on $\mathcal{A}$ given by multiplication.

3.2.1 Definition. An *additive deformation* of the bialgebra $\mathcal{B}$ is a family $(\mu_t)_{t\in\mathbb{R}_+}$ of mappings $\mu_t : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ such that

1. $(\mathcal{B}, \mu_t, \mathbb{1})$ is a unital algebra for each $t \in \mathbb{R}_+$
2. $\mu_0 = \mu$
3. $\Delta \circ \mu_{s+t} = (\mu_s \otimes \mu_t) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$
4. the mapping $t \mapsto \delta \circ \mu_t$ is pointwise continuous, that is $\delta \circ \mu_t \xrightarrow{t \to 0} \delta \circ \mu = \delta \otimes \delta$ pointwise
5. if $\mathcal{B}$ is a $*$-bialgebra and $(\mathcal{B}, \mu_t, \mathbb{1}, *)$ is a unital $*$-algebra for each $t \in \mathbb{R}_+$, we call the deformation an *additive deformation of a $*$-bialgebra.*

This definition implies that

$$\begin{aligned}(\delta \circ \mu_s) \star (\delta \circ \mu_t) &= (\delta \otimes \delta) \circ (\mu_s \otimes \mu_t) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \\ &= (\delta \otimes \delta) \circ \Delta \circ \mu_{s+t} = \delta \circ \mu_{s+t},\end{aligned}$$

hence $(\delta \circ \mu_t)_{t \in \mathbb{R}_+}$ is a pointwise continuous convolution semigroup of linear functionals on the coalgebra $\mathcal{B} \otimes \mathcal{B}$. As such it has a generator, which we will usually denote by L. The following theorem of Wirth was first proven in [Wir02], see also [Ger09]. We give a full proof of its generalization to the braided case in Section 3.4.

3.2.2 Theorem. *Let $(\mu_t)_{t \in \mathbb{R}_+}$ be an additive deformation of the bialgebra $\mathcal{B}$. Then $L = \frac{\mathrm{d}(\delta \circ \mu_t)}{\mathrm{d}t}\Big|_{t=0}$ exists pointwise and for $a, b, c \in \mathcal{B}, t \in \mathbb{R}_+$ the following statements hold:*

1. $\mu_t = \mu \star e_\star^{tL}$

2. $\mu \star L = L \star \mu$ *'L is commuting'*

3. $L(\mathbb{1} \otimes \mathbb{1}) = 0$ *'L is normalized'*

4. $\delta(a)L(b \otimes c) - L(ab \otimes c) + L(a \otimes bc) - L(a \otimes b)\delta(c) = 0.$
'L is a cocycle'.

If $(\mu_t)_{t \in \mathbb{R}_+}$ is a $*$-bialgebra deformation, then additionally

5. $L(b \otimes c) = \overline{L(c^* \otimes b^*)}$ *'L is hermitian'.*

Conversely, if $L : \mathcal{B} \otimes \mathcal{B} \to \mathbb{C}$ is a linear mapping, which fulfills conditions 2,3 and 4 (in case of $*$-bialgebra also 5), than the first equation defines an additive deformation on $\mathcal{B}$.

In this section we wish to introduce a cochain complex such that the generators of additive deformations are exactly the 2-cocycles. Once the complex is established the question is, what kind of deformations are generated by coboundaries. It is shown, that those deformations are of the form

$$\mu_t = \Phi_t \circ \mu \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right),$$

where the Φ_t constitute a pointwise continuous one parameter group of invertible linear operators on $\mathcal{B}$ that commute in the sense that

$$(\Phi_t \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Phi_t) \circ \Delta.$$

When the generator of the additive deformation is the coboundary of $\psi : \mathcal{B} \to \mathbb{C}$, then $\Phi_t = (\mathrm{id} \otimes e_\star^{-t\psi}) \circ \Delta$ is the one parameter group of operators.

3.2.1 Subcohomologies of the Hochschild Cohomology

A *cochain complex* consists of a sequence of vector spaces $C = (C_n)_{n\in\mathbb{N}}$ and linear mappings $\partial_n : C_n \to C_{n+1}$ such that $\partial_{n+1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$. The elements of $Z_n(C) = \operatorname{kern} \partial_n$ are called (n-)*cocycles*, the elements of $B_n(C) = \operatorname{im} \partial_{n-1}$ are called (n-)*coboundaries* and the vector space $H_n(C) = Z_n(C)/B_n(C)$ is called n-th *cohomology*. A sequence $D = (D_n)_{n\in\mathbb{N}}$ is called *subcomplex* if $D_n \subseteq C_n$ and $\partial_n D_n \subseteq D_{n+1}$ for all n. Then $\left((D_n)_{n\in\mathbb{N}}, (\partial_n|_{D_n})_{n\in\mathbb{N}}\right)$ is again a cochain complex and we have:

1. The cocycles of D are exactly the cocycles of C, belonging to D, that is

$$Z_n(D) = Z_n(C) \cap D_n.$$

2. Each coboundary of D is a coboundary of C, that is

$$B_n(D) \subseteq B_n(C) \cap D_n.$$

3. Equality holds in 2, if and only if the mapping

$$H_n(D) \to H_n(C), f + B_n(D) \mapsto f + B_n(C)$$

   is an injection.

4. If D, E are subcomplexes, then $(D_n \cap E_n)_{n \in \mathbb{N}}$ is a subcomplex.

Points 1,2 and 4 are obvious, while 3 follows from the observation, that the kernel of the given mapping is exactly $B_n(C) \cap D_n$.

For an algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule M we define

$$C_n := \operatorname{Lin}\left(\mathcal{A}^{\otimes n}, M\right) = \left\{f : \mathcal{A}^{\otimes n} \to M \mid f \text{ linear}\right\}.$$

One can show, that together with the coboundary operator

$$\begin{aligned}\partial_n f(a_1, \ldots, a_{n+a}) := a_1.f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \; f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ + (-1)^{n+1} \; f(a_1, \ldots, a_n).a_{n+1}\end{aligned}$$

the C_n form a cochain complex, the so-called *Hochschild complex*, see for example [Ger63]. Especially for $\mathcal{A} = \mathcal{B}$ a bialgebra and $M = \mathbb{C}$ the $\mathcal{B}$-bimodule given by $a.\lambda.b = \delta(a)\lambda\delta(b)$ for $\lambda \in \mathbb{C}$ and $a, b \in \mathcal{B}$ we have

$$\begin{aligned}\partial_n f(a_1, \ldots, a_{n+a}) := \delta(a_1) f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \; f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ + (-1)^{n+1} \; f(a_1, \ldots, a_n)\delta(a_{n+1}). \quad (3.2.1)\end{aligned}$$

The generators of additive deformations are normalized commuting cocycles, so it is natural to define

$$C_n^{(\mathbf{N})} = \left\{ f \in C_n \mid f(\mathbb{1}^{\otimes n}) = 0 \right\},$$
$$C_n^{(\mathbf{C})} = \left\{ f \in C_n \mid f \star \mu^{(n)} = \mu^{(n)} \star f \right\}.$$

If $\mathcal{B}$ is a $*$-bialgebra the generators are also hermitian. We define for $f \in C_n$

$$\widetilde{f}(a_1 \otimes \cdots \otimes a_n) := \overline{f(a_n^* \otimes \cdots \otimes a_1^*)}$$

and set

$$C_n^{(\mathbf{H})} = \begin{cases} \left\{ f \in C_n \,\middle|\, \widetilde{f} = f \right\}, & \text{if } \lceil \frac{n}{2} \rceil \text{ odd, that is } n = 1, 2, 5, 6, \ldots \\ \left\{ f \in C_n \,\middle|\, \widetilde{f} = -f \right\}, & \text{if } \lceil \frac{n}{2} \rceil \text{ even, that is } n = 0, 3, 4, 7, 8, \ldots \end{cases}$$

3.2.3 Proposition. *$C_n^{(\mathbf{N})}$, $C_n^{(\mathbf{C})}$ and $C_n^{(\mathbf{H})}$ are subcomplexes of C_n.*

Proof. We only need to show that $\partial C_n^{(*)} \subseteq C_{n+1}^{(*)}$ for $* = \mathbf{N}, \mathbf{C}, \mathbf{H}$.

N: Let $f \in C_n^{(\mathbf{N})}$. Then

$$\partial f(\mathbb{1}^{\otimes(n+1)}) = \delta(\mathbb{1}) f(\mathbb{1}^{\otimes n}) + \sum_{i=1}^{n} (-1)^i \; f(\mathbb{1}^{\otimes n}) + (-1)^{n+1} \; f(\mathbb{1}^{\otimes n})\delta(\mathbb{1}) = 0$$

proves $\partial f \in C_{n+1}^{(\mathbf{N})}$.

C: For $f \in C_n^{(\mathbf{C})}$ we get

$$\begin{aligned} &\partial f \star \mu^{(n+1)} \\ &= \left(\delta \otimes f + \sum_{k=1}^{n} (-1)^k f \circ (\mathrm{id}_{k-1} \otimes \mu \otimes \mathrm{id}_{n-k}) + (-1)^{n+1} f \otimes \delta \right) \star \mu^{(n+1)}. \end{aligned}$$

Next we show, that each summand commutes with μ under convolution:

$$\begin{aligned}
&(\delta \otimes f) \star \mu^{(n+1)}(a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = \delta\left(a_1^{(1)}\right) f\left(a_2^{(1)} \otimes \cdots \otimes a_{n+1}^{(1)}\right) a_1^{(2)} \ldots a_{n+1}^{(2)} \\
&\quad = a_1 f\left(a_2^{(1)} \otimes \cdots \otimes a_{n+1}^{(1)}\right) a_2^{(2)} \ldots a_{n+1}^{(2)} \\
&\quad = a_1 f\left(a_2^{(2)} \otimes \cdots \otimes a_{n+1}^{(2)}\right) a_2^{(1)} \ldots a_{n+1}^{(1)} \qquad (\text{as } f \star \mu^{(n)} = \mu^{(n)} \star f) \\
&\quad = a_1^{(1)} \ldots a_{n+1}^{(1)} \delta\left(a_1^{(2)}\right) f\left(a_2^{(2)} \otimes \cdots \otimes a_{n+1}^{(2)}\right) \\
&\quad = \mu^{(n+1)} \star (\delta \otimes f)(a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned}$$

Analogously, we see that $(f \otimes \delta) \star \mu^{(n+1)} = \mu^{(n+1)} \star (f \otimes \delta)$. For the remaining summands we calculate

$$\begin{aligned}
&\big(f \circ (\mathrm{id}_{k-1} \otimes \mu \otimes \mathrm{id}_{n-k})\big) \star \mu^{(n+1)}(a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = f\left(a_1^{(1)} \otimes \cdots \otimes (a_k^{(1)} a_{k+1}^{(1)}) \otimes \cdots \otimes a_{n+1}^{(1)}\right) a_1^{(2)} \ldots a_k^{(2)} a_{k+1}^{(2)} \ldots a_{n+1}^{(2)} \\
&\quad = f\left(a_1^{(1)} \otimes \cdots \otimes (a_k a_{k+1})^{(1)} \otimes \cdots \otimes a_{n+1}^{(1)}\right) a_1^{(2)} \ldots (a_k a_{k+1})^{(2)} \ldots a_{n+1}^{(2)} \\
&\qquad\qquad (\text{as } \Delta \text{ is an algebra homomorphism}) \\
&\quad = f\left(a_1^{(2)} \otimes \cdots \otimes (a_k a_{k+1})^{(2)} \otimes \cdots \otimes a_{n+1}^{(2)}\right) a_1^{(1)} \ldots (a_k a_{k+1})^{(1)} \ldots a_{n+1}^{(1)} \\
&\qquad\qquad (\text{as } f \star \mu^{(n)} = \mu^{(n)} \star f) \\
&\quad = \mu^{(n+1)} \star \big(f \circ (\mathrm{id}_{k-1} \otimes \mu \otimes \mathrm{id}_{n-k})\big)(a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned}$$

All in all, we conclude $\partial f \star \mu^{(n+1)} = \mu^{(n+1)} \star \partial f$, hence $\partial f \in C_{n+1}^{(\mathbf{C})}$.

H: Let $\widetilde{f} = \pm f$. For n odd we get

$$\begin{aligned}
&\widetilde{\partial f}(a_1, \ldots, a_{n+1}) = \overline{\partial f(a_{n+1}^*, \ldots, a_1^*)} \\
&\quad = \overline{\delta(a_{n+1}^*) f(a_n^*, \ldots, a_1^*)} + \sum_{i=1}^{n} (-1)^{n+1-i}\, \overline{f(a_{n+1}^*, \ldots, a_{i+1}^* a_i^*, \ldots, a_1^*)} \\
&\qquad + \overline{f(a_{n+1}^*, \ldots, a_2^*) \delta(a_1^*)}
\end{aligned}$$

$$= \delta(a_1)\widetilde{f}(a_2,\dots,a_{n+1}) + \sum_{i=1}^{n}(-1)^i\ \widetilde{f}(a_1,\dots,a_ia_{i+1},\dots,a_{n+1}) \\ + \widetilde{f}(a_1,\dots,a_n)\delta(a_{n+1})$$
$$= \pm\partial f(a_1,\dots,a_{n+1})$$

and for n even we calculate

$$\begin{aligned}\widetilde{\partial f}(a_1,\dots,a_{n+1}) &= \overline{\partial f(a_{n+1}^*,\dots,a_1^*)}\\ &= \overline{\delta(a_{n+1}^*)f(a_n^*,\dots,a_1^*)}\\ &\quad + \sum_{i=1}^{n}(-1)^{n+1-i}\ \overline{f(a_{n+1}^*,\dots,a_{i+1}^*a_i^*,\dots,a_1^*)} - \overline{f(a_{n+1}^*,\dots,a_2^*)\delta(a_1^*)}\\ &= -\delta(a_1)\widetilde{f}(a_2,\dots,a_{n+1})\\ &\quad - \sum_{i=1}^{n}(-1)^i\ \widetilde{f}(a_1,\dots,a_ia_{i+1},\dots,a_{n+1}) + \widetilde{f}(a_1,\dots,a_n)\delta(a_{n+1})\\ &= \mp\partial f(a_1,\dots,a_{n+1}).\end{aligned}$$

So we see that $f \in C_n^{(\mathbf{H})}$ implies $\partial f \in C_{n+1}^{(\mathbf{H})}$.

□

Since the intersection of subcomplexes is again a subcomplex we have

3.2.4 Corollary. *$C_n^{(\mathbf{NC})} := C_n^{(\mathbf{N})} \cap C_n^{(\mathbf{C})}$ and $C_n^{(\mathbf{NCH})} := C_n^{(\mathbf{NC})} \cap C_n^{(\mathbf{H})}$ are cochain complexes with the coboundary operator* (3.2.1).

3.2.2 Characterization of the Trivial Deformations

For an additive deformation of the bialgebra $\mathcal{B}$ the generator L of the convolution semigroup $(\delta \circ \mu_t)_{t\in\mathbb{R}_+}$ is an element of $Z_2^{(\mathbf{NC})}$ and conversely if $L \in Z_2^{(\mathbf{NC})}$ we can define an additive deformation via $\mu_t := \mu \star e_\star^{tL}$. In the case of a $*$-bialgebra the generators are exactly the elements of $Z_2^{(\mathbf{NCH})}$. We wish to answer the question, which deformations are generated by the coboundaries, that is the elements of $B_2^{(\mathbf{NC})}$ or $B_2^{(\mathbf{NCH})}$ respectively.

3.2.5 Theorem. *Let $\mathcal{B}$ be a bialgebra, $L \in B_2^{(\mathbf{NC})}$, $L = \partial\psi$ with $\psi \in C_1^{(\mathbf{NC})}$ and put $\Phi_t = \mathrm{id}\star e_\star^{-t\psi}$ and $\mu_t = \mu\star e_\star^{tL}$. Then $(\Phi_t)_{t\in\mathbb{R}_+}$ is a semigroup of linear automorphisms of*

$\mathcal{B}$. Furthermore, the Φ_t are unital algebra isomorphisms $\Phi_t : (\mathcal{B},\mu) \to (\mathcal{B},\mu_t)$, for which

$$(\Phi_t \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Phi_t) \circ \Delta \quad \textit{for all } t \in \mathbb{R}_+ \tag{3.2.2}$$

and $(\delta \circ \Phi_t)_{t\in\mathbb{R}_+}$ is a pointwise continuous convolution semigroup. If $\mathcal{B}$ is a $*$-bialgebra and $L \in B_2^{(\mathbf{NCH})}$, then we can choose $\psi \in C_1^{(\mathbf{NCH})}$ and the Φ_t are $*$-algebra isomorphisms.

Conversely, if $(\Phi_t)_{t\in\mathbb{R}_+}$ is semigroup of invertible linear mappings $\Phi_t : \mathcal{B} \to \mathcal{B}$ such that $(\delta \circ \Phi_t)_{t\in\mathbb{R}_+}$ pointwise continuous, (3.2.2) holds and $\Phi_t(\mathbb{1}) = \mathbb{1}$ for all $t \in \mathbb{R}_+$, then

$$\mu_t := \Phi_t \circ \mu \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right)$$

defines an additive deformation of $\mathcal{B}$ with generator $L \in B_2^{(\mathbf{NC})}$. If $\mathcal{B}$ is a $*$-algebra and the Φ_t are hermitian, then we get an additive deformation of a $*$-bialgebra and $L \in B_2^{(\mathbf{NCH})}$.

We postpone the proof a bit. When $\mathcal{B}$ is a bialgebra and $\varphi : \mathcal{B} \to \mathbb{C}$ a linear functional on $\mathcal{B}$ we define

$$R_\varphi : \mathcal{B} \to \mathcal{B}, \quad R_\varphi := \mathrm{id} \star \varphi = (\mathrm{id} \otimes \varphi) \circ \Delta.$$

3.2.6 Lemma. *For $\varphi, \psi \in \mathcal{B}'$ the following hold:*

(i) $R_\varphi \circ R_\psi = R_{\varphi\star\psi}$

(ii) $\delta \circ R_\varphi = \varphi$

(iii) $R_{\delta\otimes\varphi} = \mathrm{id} \otimes R_\varphi$

(iv) $R_{\varphi\otimes\delta} = R_\varphi \otimes \mathrm{id}$

(v) $\mu \circ R_{\varphi\circ\mu} = R_\varphi \circ \mu$

Note that the last three equations are between linear maps defined on the bialgebra $\mathcal{B}\otimes\mathcal{B}$.

Proof. This is all straightforward to verify. □

Proof of Theorem 3.2.5. Let $\mathcal{B}$ be a bialgebra and $L = \partial\psi \in B_2^{(\mathbf{NC})}$ a coboundary with $\psi \in C_1^{(\mathbf{NC})}$. We write $\varphi_t := e_\star^{-t\psi}$ and note that this is a pointwise continuous convolution semigroup and the φ_t are commuting (that is $\varphi_t \star \mathrm{id} = \mathrm{id} \star \varphi_t$) since ψ is. Then the mappings $\Phi_t = R_{\varphi_t}$ yield a semigroup of linear operators on $\mathcal{B}$ with $\delta \circ \Phi_t = \varphi_t$ and we only need to show that they are unital algebra isomorphisms. It is obvious, that $\Phi_t(\mathbb{1}) = \mathbb{1}$, since $\psi(\mathbb{1}) = 0$, and $\Phi_t \circ \Phi_{-t} = \mathrm{id}$, so Φ_t is invertible. We have to prove that $\Phi_t : (\mathcal{B}, \mu) \to (\mathcal{B}, \mu_t)$ is an algebra homomorphism, that is $\mu_t = \mu \star e_\star^{tL} = \Phi_t \circ \mu \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right)$. Using that $\delta \otimes \psi$, $\psi \circ \mu$ and $\psi \otimes \delta$ commute under convolution, we find

$$\begin{aligned} e_\star^{tL} = e_\star^{t\partial\psi} = e_\star^{t(\delta\otimes\psi - \psi\circ\mu + \psi\otimes\delta)} &= e_\star^{-t\psi\circ\mu} \star e_\star^{t\delta\otimes\psi} \star e_\star^{t\psi\otimes\delta} \\ &= (\varphi_t \circ \mu) \star (\delta \otimes \varphi_{-t}) \star (\varphi_{-t} \otimes \delta). \end{aligned}$$

From this we conclude

$$\begin{aligned} \mu_t = \mu \star e_\star^{tL} = \left(\mu \otimes e_\star^{t\partial\psi}\right) \circ \Lambda = \mu \circ R_{e_\star^{tL}} &= \mu \circ R_{(\varphi_t\circ\mu)\star(\delta\otimes\varphi_{-t})\star(\varphi_{-t}\otimes\delta)} \\ &= \mu \circ R_{\varphi_t\circ\mu} \circ \left(\mathrm{id} \otimes R_{\varphi_{-t}}\right) \circ \left(R_{\varphi_{-t}} \otimes \mathrm{id}\right) \\ &= R_{\varphi_t} \circ \mu \circ \left(R_{\varphi_{-t}} \otimes R_{\varphi_{-t}}\right) \\ &= \Phi_t \circ \mu \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right). \end{aligned}$$

It is clear that the Φ_t are $*$-homomorphisms in the $*$-bialgebra case.

Now let $(\Phi_t)_{t\in\mathbb{R}_+}$ be semigroup of invertible linear mappings with $\Phi_t(\mathbb{1}) = \mathbb{1}$ and $(\Phi_t \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Phi_t) \circ \Delta$ such that $(\varphi_t)_{t\in\mathbb{R}_+}$, $\varphi_t := \delta \circ \Phi_t$, is pointwise continuous. We observe that

1. The family $(\varphi_t)_{t\in\mathbb{R}_+}$ is a pointwise continuous convolution semigroup with generator $\psi \in C_1^{(\mathbf{NC})}$. Indeed,

$$\begin{aligned} \varphi_s \star \varphi_t = \Big((\delta \circ \Phi_s) \otimes (\delta \circ \Phi_t)\Big) \circ \Delta &= (\delta \otimes \delta) \circ (\Phi_s \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Phi_t) \circ \Delta \\ &= (\delta \otimes \delta) \circ (\Phi_s \otimes \mathrm{id}) \circ (\Phi_t \otimes \mathrm{id}) \circ \Delta \\ &= (\delta \otimes \delta) \circ (\Phi_{s+t} \otimes \mathrm{id}) \circ \Delta = \varphi_{s+t} \end{aligned}$$

   and $\psi(\mathbb{1}) = 0, \psi \star \mathrm{id} = \mathrm{id} \star \psi$ follow from $\varphi_t(\mathbb{1}) = 1$ and $\varphi_t \star \mathrm{id} = \mathrm{id} \star \varphi_t$ via differentiation.

2. It holds that $\Phi_t = R_{\varphi_t}$ for all t. This is shown by the simple calculation

$$R_{\varphi_t} = \big(\mathrm{id} \otimes (\delta \circ \Phi_t)\big) \circ \Delta = (\mathrm{id} \otimes \delta) \circ (\mathrm{id} \otimes \Phi_t) \circ \Delta$$
$$= (\mathrm{id} \otimes \delta) \circ (\Phi_t \otimes \mathrm{id}) \circ \Delta = \Phi_t.$$

So the first part of the theorem tells us, that $L = \partial\psi \in B_2^{(\mathbf{NC})}$ is the generator of an additive deformation, for which $\mu_t = \Phi_t \circ \mu \circ \left(\Phi_t^{-1} \otimes \Phi_t^{-1}\right)$. If $\mathcal{B}$ is a $*$-bialgebra and all the Φ_t are hermitian, then so are all the $\varphi_t = \delta \circ \Phi_t$ and via differentiation also ψ. This shows $\psi \in C_1^{(\mathbf{NCH})}$ and thus $L = \partial\psi \in B_2^{(\mathbf{NCH})}$. So the deformation is an additive deformation of a $*$-bialgebra. □

3.2.7 Remark. Let $L \in B_2$, that is $L = \partial\psi$ for an arbitrary linear functional ψ. Because of $L(\mathbb{1} \otimes \mathbb{1}) = \delta(\mathbb{1})\psi(\mathbb{1}) - \psi(\mathbb{1}) + \psi(\mathbb{1})\delta(\mathbb{1}) = \psi(\mathbb{1})$, $L \in C_2^{(\mathbf{N})}$ implies $\psi \in C_1^{(\mathbf{N})}$ and therefore $B_2^{(\mathbf{N})} = B_2 \cap C_2^{(\mathbf{N})}$. If L is hermitian, $\frac{1}{2}(\psi + \widetilde{\psi})$ is a hermitian linear functional, whose coboundary is also L. This shows $B_2^{(\mathbf{H})} = B_2 \cap C_2^{(\mathbf{H})}$. But it is not clear under which circumstances $B_2^{(\mathbf{C})} = B_2 \cap C_2^{(\mathbf{C})}$ holds, that is if there are 2-coboundaries that commute but are not coboundaries of a commuting linear functional. This possible difference is actually the main reason why we need the altered cochain complex to get a good notion of trivial deformations.

3.3 Additive Deformations of Hopf Algebras

Deforming the multiplication of a bialgebra $\mathcal{B}$ also gives a deformed convolution product $\star_t$ for linear maps from $\mathcal{B}$ to $\mathcal{B}$

$$A \star_t B := \mu_t \circ (A \otimes B) \circ \Delta,$$

where $(\mu_t)_{t\in\mathbb{R}}$ is a deformation of the multiplication map μ of $\mathcal{B}$. Recall that $\mathcal{B}$ is a Hopf algebra if the identity map on $\mathcal{B}$ has a convolution inverse S, called antipode. We ask, whether there are also *deformed antipodes* S_t which fulfill

$$\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta = \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta = \mathbb{1}\delta, \tag{3.3.1}$$

that is whether the identity map also has an inverse with respect to the deformed convolution $\star_t$. An additive deformation which admits deformed antipodes is called a *Hopf deformation* in this section. In a Hopf algebra, the antipode S is an algebra antihomomorphism and a coalgebra antihomomorphism, that is

$$S \circ \mu = \mu \circ (S \otimes S) \circ \tau,$$
$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta.$$

Similar properties hold for the deformed antipodes S_t of a Hopf deformation. We can prove (Theorem 3.3.4)

$$S_t \circ \mu_{-t} = \mu_t \circ \tau \circ (S_t \otimes S_t), \tag{3.3.2}$$
$$\Delta \circ S_{t+r} = (S_t \otimes S_r) \circ \tau \circ \Delta. \tag{3.3.3}$$

Applying $\delta \otimes \delta$ to (3.3.3) we get

$$\delta \circ S_{t+r} = \big((\delta \circ S_t) \otimes (\delta \circ S_r)\big) \circ \tau \circ \Delta = \big((\delta \circ S_r) \otimes (\delta \circ S_t)\big) \circ \Delta,$$

that is $\delta \circ S_t$ is a convolution semigroup with respect to $\star = (\cdot \otimes \cdot) \circ \Delta$. So one would like to prove that this semigroup has a generator, such that the S_t are of the form

$$S_t = S \star e_\star^{-t\sigma}. \tag{3.3.4}$$

To get a hint how to find σ, we assume for the moment that $\delta \circ S_t$ is differentiable in 0 and define

$$\sigma := -\frac{\mathrm{d}}{\mathrm{d}t} \delta \circ S_t \Big|_{t=0}.$$

Then we can apply δ to (3.3.1) and differentiate to get

$$L \circ (S \otimes \mathrm{id}) \circ \Delta - \sigma = L \circ (\mathrm{id} \otimes S) \circ \Delta - \sigma = 0$$

or after rearranging

$$\sigma = L \circ (S \otimes \mathrm{id}) \circ \Delta = L \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{3.3.5}$$

In fact, we will prove that every additive deformation of a Hopf algebra is a Hopf deformation and (3.3.4) and (3.3.5) give a formula for the deformed antipodes; see Theorem 3.3.10.

In two special cases the structure can even be better understood. In the case of a trivial deformation it is easy to see that

$$S_t = \Phi_t \circ S \circ \Phi_t$$

is another way to find the deformed antipodes. Differentiating this equation also gives a second formula for the generator

$$\sigma = \psi + \psi \circ S.$$

If the bialgebra $\mathcal{B}$ is cocommutative, we show that every additive deformation splits in a trivial part and a part with constant antipodes. Applying δ to (3.3.2) and differentiating yields

$$-\sigma \circ \mu - L = L \circ (S \otimes S) \circ \tau - \sigma \otimes \delta - \delta \otimes \sigma$$

or after rearranging

$$L + L \circ (S \otimes S) \circ \tau = \delta \otimes \sigma - \sigma \circ \mu + \sigma \otimes \delta = \partial\sigma.$$

So L can be written as

$$L = \underbrace{\frac{1}{2}\partial\sigma}_{:=L_1} + \underbrace{\frac{1}{2}\big(L - L \circ (S \otimes S) \circ \tau\big)}_{:=L_2}.$$

If $\mathcal{B}$ is cocommutative, the second part corresponds to constant antipodes, see Theorem 3.3.12 and Lemma 3.3.13.

3.3.1 Existence of Deformed Antipodes

We start with extending an additive deformation from the half line to the line.

3.3.1 Lemma. *Let $\mathcal{B}$ be a bialgebra and L the generator of an additive deformation. Then we can define $\mu_t := e_\star^{tL} \star \mu$ for all $t \in \mathbb{R}$ (that is not only for $t \geq 0$). Furthermore, $\Delta : \mathcal{B}_{s+t} \to \mathcal{B}_s \otimes \mathcal{B}_t$ is an algebra homomorphism for all $s, t \in \mathbb{R}$.*

Proof. It follows from Theorem 3.2.2 that $-L$ is the generator of an additive deformation, so for $t < 0$ the definition of μ_t yields a multiplication on $\mathcal{B}$. We calculate

$$\begin{aligned}
\Delta \circ \mu_{s+t} = \Delta \circ \left(\mu \otimes e_\star^{(s+t)L}\right) \circ \Lambda &= \left((\Delta \circ \mu) \otimes e_\star^{(s+t)L}\right) \circ \Lambda \\
&= \left(\mu \otimes \mu \otimes e_\star^{sL} \otimes e_\star^{tL}\right) \circ \Lambda^{(4)} \\
&= \left(\mu \otimes e_\star^{sL} \otimes \mu \otimes e_\star^{tL}\right) \circ \Lambda^{(4)} \\
&= \left(\left(\mu \star e_\star^{sL}\right) \otimes \left(\mu \star e_\star^{tL}\right)\right) \circ \Lambda \\
&= (\mu_s \otimes \mu_t) \circ \Lambda
\end{aligned}$$

for all $s, t \in \mathbb{R}$. □

From now on we always view an additive deformation as a family of multiplications indexed by all real numbers.

3.3.2 Definition. An additive deformation is called a *Hopf deformation* if for all $t \in \mathbb{R}$ there exists a linear mapping $S_t : \mathcal{B} \to \mathcal{B}$ such that

$$\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta = \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta = \mathbb{1}\delta. \tag{3.3.6}$$

The S_t are referred to as *deformed antipodes.*

For $t = 0$ the above defintion of course implies that $\mathcal{B}$ is a Hopf algebra with antipode $S = S_0$.

3.3.3 Remark. Many proofs in this section follow a common path. To show an identity $a = b$, we find an element c and a convolution product $\diamond$ such that $a \diamond c = c \diamond b = \delta$ where

δ is the neutral element for $\diamond$. Then we can conclude

$$a = a \diamond \delta = a \diamond c \diamond b = \delta \diamond b = b.$$

This is the usual argument to prove that an element in a semigroup is invertible if and only if it has a left and a right inverse.

3.3.4 Theorem. *Let $(\mu_t)_{t\in\mathbb{R}}$ be a Hopf deformation of $\mathcal{B}$. Then the deformed antipodes S_t are uniquely determined by* (3.3.6)*. Furthermore, the following statements hold:*

(i) $S_t(\mathbb{1}) = \mathbb{1}$

(ii) $S_t \circ \mu_{-t} = \mu_t \circ (S_t \otimes S_t) \circ \tau$

(iii) $\Delta \circ S_{t+r} = (S_t \otimes S_r) \circ \tau \circ \Delta$

(iv) *If $\mathcal{B}$ is cocommutative, that is $\Delta = \tau \circ \Delta$, then S_t is invertible for all $t \in \mathbb{R}$ and $(S_t)^{-1} = S_{-t}$.*

Proof. (Uniqueness) : By (3.3.6), S_t is the two-sided convolution inverse of the identity map on $\mathcal{B}$ with respect to $\star_t$.

(i): Follows from $\mathbb{1} = \mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta(\mathbb{1}) = S_t(\mathbb{1})$.

(ii): We prove that both sides are convolution inverses of μ_t with respect to $\star_t$. For the left hand side we calculate

$$\begin{aligned}(S_t \circ \mu_{-t}) \star_t \mu_t &= \mu_t \circ (S_t \otimes \mathrm{id}) \circ (\mu_{-t} \otimes \mu_t) \circ \Lambda \\ &= \mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta \circ \mu = \mathbb{1}\delta \circ \mu = \mathbb{1}\delta \otimes \delta,\end{aligned}$$

and for the right hand side we get

$$\begin{aligned}&\mu_t \star_t \Big(\mu_t \circ (S_t \otimes S_t) \circ \tau\Big)(a \otimes b) \\ &\quad = \mu_t \circ (\mu_t \otimes \mu_t) \circ \Big(\mathrm{id}_2 \otimes \Big((S_t \otimes S_t) \circ \tau\Big)\Big) \circ \Lambda\,(a \otimes b) \\ &\quad = \mu_t^{(4)}\Big(a_{(1)} \otimes b_{(1)} \otimes S_t\left(b_{(2)}\right) \otimes S_t\left(a_{(2)}\right)\Big) \\ &\quad = \delta(b)\mu_t\Big(a_{(1)} \otimes S_t\left(a_{(1)}\right)\Big) = \delta(a)\delta(b)\mathbb{1}.\end{aligned}$$

(iii): For linear maps A, B from the coalgebra $(\mathcal{B}, \Delta)$ to the algebra $(\mathcal{B}_t \otimes \mathcal{B}_r)$ we have a convolution $\diamond$ defined as

$$A \diamond B = (\mu_t \otimes \mu_r) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (A \otimes B) \circ \Delta.$$

We show that both sides of (iii) are inverses of Δ with respect to $\diamond$. For the left hand side we get

$$\begin{aligned}(\Delta \circ S_{t+r}) \diamond \Delta &= (\mu_t \otimes \mu_r) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (S_{t+r} \otimes \mathrm{id}) \circ \Delta \\ &= \Delta \circ \mu_{t+r} \circ (S_{t+r} \otimes \mathrm{id}) \circ \Delta = \Delta(\mathbb{1})\delta = \mathbb{1} \otimes \mathbb{1}\delta,\end{aligned}$$

and for the right hand side

$$\begin{aligned}&\Delta \diamond \big((S_t \otimes S_r) \circ \tau \circ \Delta\big)(a) \\ &\quad = (\mu_t \otimes \mu_r) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id}_2 \otimes S_t \otimes S_r) \circ (\mathrm{id}_2 \otimes \tau) \circ \Delta^{(4)}(a) \\ &\quad = (\mu_t \otimes \mu_r)\big(a_{(1)} \otimes S_t(a_{(4)}) \otimes a_{(2)} \otimes S_r(a_{(3)})\big) \\ &\quad = \mu_t\big(a_{(1)} \otimes S_t(a_{(2)})\big) \otimes \mathbb{1} = \delta(a)\mathbb{1} \otimes \mathbb{1}.\end{aligned}$$

(iv): Suppose $\Delta = \tau \circ \Delta$. We prove $S_t \circ S_{-t} = \mathrm{id}$ by showing that $S_t \circ S_{-t}$ is convolution inverse to S_t. Therefore we calculate

$$\begin{aligned}(S_t \circ S_{-t}) \star_t S_t &= \mu_t \circ (S_t \otimes S_t) \circ (S_{-t} \otimes \mathrm{id}) \circ \Delta \\ &= S_t \circ \mu_{-t} \circ \tau \circ (S_{-t} \otimes \mathrm{id}) \circ \Delta \\ &= S_t \circ \mu_{-t} \circ (\mathrm{id} \otimes S_{-t}) \circ \tau \circ \Delta = \delta S_t(\mathbb{1}) = \mathbb{1}\delta.\end{aligned}$$

Since S_t is invertible with respect to $\star_t$, the left inverse $S_t \circ S_{-t}$ is automatically a two sided inverse. Thus $S_t \circ S_{-t} = \mathrm{id}$.

□

The Deformed Antipodes for Trivial Deformations We start with showing the existence of deformed antipodes for all trivial deformations. We find simple descriptions of the deformed antipodes. From these descriptions we can easily read, when the deformed antipodes of a trivial deformation are constant.

3.3.5 Theorem. *Let $\mathcal{B}$ be a Hopf algebra and $(\mu_t)_{t\in\mathbb{R}}$ a trivial deformation with $\mu_t = \Phi_t \circ \mu \circ (\Phi_t^{-1} \otimes \Phi_t^{-1})$ and $\Phi_t = \mathrm{id} \star e_\star^{-t\psi}$ for a commuting, normalized linear functional ψ. Then $(\mu_t)_{t\in\mathbb{R}}$ is a Hopf deformation with deformed antipodes*

$$S_t = \Phi_t \circ S \circ \Phi_t = S \star e_\star^{-t(\psi\circ S+\psi)}$$

for all $t \in \mathbb{R}$.

Proof. For $S_t = \Phi_t \circ S \circ \Phi_t$ we calculate

$$\begin{aligned}
\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta &= \Phi_t \circ \mu \circ (\Phi_t^{-1} \otimes \Phi_t^{-1}) \circ \big((\Phi_t \circ S \circ \Phi_t) \otimes \mathrm{id}\big) \circ \Delta \\
&= \Phi_t \circ \mu \circ (S \circ \Phi_t \otimes \Phi_t^{-1}) \circ \Delta \\
&= \Phi_t \circ \mu \circ (S \otimes \Phi_t^{-1}) \circ (\Phi_t \otimes \mathrm{id}) \circ \Delta \\
&= \Phi_t \circ \mu \circ (S \otimes \Phi_t^{-1}) \circ (\mathrm{id} \otimes \Phi_t) \circ \Delta \\
&= \Phi_t \circ \mu \circ (S \otimes \mathrm{id}) \circ \Delta \\
&= \Phi_t(\mathbb{1})\delta = \mathbb{1}\delta
\end{aligned}$$

which proves that S_t is a left inverse of the identity on $\mathcal{B}$ with respect to $\star_t$. In the same way one proves that S_t is a right inverse.

Now we consider the case $S_t = S \star e_\star^{-t(\psi\circ S+\psi)}$. We first recall that ψ is commuting and $L = \partial\psi$ is the generator of the additive deformation. Next we observe that

$$\begin{aligned}
(\psi \circ S) \star S &= (\psi \otimes \mathrm{id}) \circ (S \otimes S) \circ \Delta \\
&= (\psi \otimes \mathrm{id}) \circ \tau \circ \Delta \circ S \\
&= (\psi \otimes \mathrm{id}) \circ \Delta \circ S \\
&= (\mathrm{id} \otimes \psi) \circ (S \otimes S) \circ \Delta = S \star (\psi \circ S).
\end{aligned}$$

With this in mind we calculate

$$\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta\,(a) = (\mu \otimes e_\star^{tL}) \circ \Lambda\Big(e_\star^{-t\psi}\big(S(a_{(1)})\big)e_\star^{-t\psi}(a_{(2)}) \cdot S(a_{(3)}) \otimes a_{(4)}\Big)$$

$$= e_\star^{-t\psi}\big(S(a_{(1)})\big)e_\star^{-t\psi}(a_{(2)})e_\star^{tL}\big(S(a_{(3)})\otimes a_{(4)}\big)\mathbb{1}$$
$$= \delta(a)\mathbb{1},$$

since

$$\begin{aligned} &e_\star^{tL}(S(a_{(1)})\otimes a_{(2)}) \\ &\quad = e_\star^{t\delta\otimes\psi}(S(a_{(1)})\otimes a_{(2)})e_\star^{-t\psi\circ\mu}(S(a_{(3)})\otimes a_{(4)})e_\star^{t\psi\otimes\delta}(S(a_{(5)})\otimes a_{(6)}) \\ &\quad = e_\star^{t\psi}(a_{(1)})e_\star^{t\psi}(S(a_{(2)})). \end{aligned}$$

Again, the proof that S_t is also a right inverse is similar.

An alternative way to prove the second equality in the theorem is to write $\Phi_t = (e_\star^{-t\psi}\otimes \mathrm{id})\circ\Delta$ in $S_t = \Phi_t\circ S\circ\Phi_t$ and make use of the fact that S, ψ and $\psi\circ S$ all commute with each other under convolution. □

It is still possible that the deformed antipodes are constant. We have

3.3.6 Theorem. *Let $(S_t)_{t\in\mathbb{R}}$ be the deformed antipodes of a trivial deformation with $\mu_t = \Phi_t\circ\mu\circ(\Phi_t^{-1}\otimes\Phi_t^{-1})$. Then $S_t = S$ for all $t\in\mathbb{R}$ if and only if*

$$\Phi_t\circ S = S\circ\Phi_{-t}.$$

for all $t\in\mathbb{R}$.

Proof. This follow directly from $S_t = \Phi_t\circ S\circ\Phi_t$ and $\Phi_t^{-1} = \Phi_{-t}$. □

The Deformed Antipodes in the General Case We want to show that every additive deformation of a Hopf algebra is a Hopf deformation and give a formula for the deformed antipodes. Let $\mathcal{B}$ be a Hopf algebra and L the generator of an additive deformation $(\mu_t)_{t\in\mathbb{R}}$.

3.3.7 Lemma. *We have*

$$L\circ(S\otimes\mathrm{id})\circ\Delta = L\circ(\mathrm{id}\otimes S)\circ\Delta.$$

Proof. From $\partial L = 0$ it follows easily that $L(a \otimes \mathbb{1}) = L(\mathbb{1} \otimes a) = 0$ for all $a \in \mathcal{B}$. Hence,

$$\begin{aligned} 0 &= \partial L\big(a_{(1)} \otimes S(a_{(2)}) \otimes a_{(3)}\big) \\ &= \delta(a_{(1)}) L\big(S(a_{(2)}) \otimes a_{(3)}\big) - L\big(a_{(1)} S(a_{(2)}) \otimes a_{(3)}\big) \\ &\quad + L\big(a_{(1)} \otimes S(a_{(2)}) a_{(3)}\big) - L\big(a_{(1)} \otimes S(a_{(2)})\big)\delta(a_{(3)}) \\ &= L\big(S(a_{(1)}) \otimes a_{(2)}\big) - L\big(a_{(1)} \otimes S(a_{(2)})\big). \end{aligned}$$

□

We define

$$\sigma := L \circ (\mathrm{id} \otimes S) \circ \Delta = L \circ (S \otimes \mathrm{id}) \circ \Delta. \tag{3.3.7}$$

We freely choose between the two possibilities to write σ, but Lemma 3.3.7 will only be essential in the proofs of Theorem 3.3.12 and Lemma 3.3.13 in Section 3.3.2.

3.3.8 Lemma. *The linear functional σ defined by* (3.3.7) *is commuting, that is*

$$(\sigma \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \sigma) \circ \Delta.$$

Proof. Since L is commuting, we have

$$\begin{aligned} L(a_{(1)} \otimes S(a_{(2)})) = L\big(a_{(1)} \otimes S(a_{(4)})\big) a_{(2)} S(a_{(3)}) &= (L \star \mu)\big(a_{(1)} \otimes S(a_{(2)})\big) \\ &= (\mu \star L)\big(a_{(1)} \otimes S(a_{(2)})\big) \\ &= L\big(a_{(2)} \otimes S(a_{(3)})\big) a_{(1)} S(a_{(4)}) \\ &= a_{(1)} L\big(a_{(2)} \otimes S(a_{(3)})\big) S(a_{(4)}). \end{aligned}$$

This helps us to see

$$\begin{aligned} (\sigma \otimes \mathrm{id}) \circ \Delta(a) = \sigma(a_{(1)}) a_{(2)} &= \big(L \circ (\mathrm{id} \otimes S) \circ \Delta\big)(a_{(1)})\, a_{(2)} \\ &= L\big(a_{(1)} \otimes S(a_{(2)})\big) a_{(3)} \\ &= a_{(1)} L\big(a_{(2)} \otimes S(a_{(3)})\big) S(a_{(4)}) a_{(5)} \end{aligned}$$

$$= a_{(1)}L\big(a_{(2)} \otimes S(a_{(3)})\big)$$
$$= (\mathrm{id} \otimes \sigma) \circ \Delta(a).$$

□

3.3.9 Lemma. *The following equations hold:*

(i) $L^{\star n} \circ (\mathrm{id} \otimes S) \circ \Delta = \sigma^{\star n}$

(ii) $e_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta = e_\star^{t\sigma}$.

Proof. We prove (i) by induction over n. For $n = 0, 1$ the proposition is clear. Using that σ commutes by Lemma 3.3.8, we calculate

$$\begin{aligned} L^{\star n+1}\big(a_{(1)} \otimes S(a_{(2)})\big) &= L \star L^{\star n}\big(a_{(1)} \otimes S(a_{(2)})\big) \\ &= L\big(a_{(1)} \otimes S(a_{(4)})\big) L^{\star n}\big(a_{(2)} \otimes S(a_{(3)})\big) \\ &= L\big(a_{(1)} \otimes S(a_{(3)})\big)\sigma^{\star n}(a_{(2)}) \\ &= L\big(a_{(1)}\sigma^{\star n}(a_{(2)}) \otimes S(a_{(3)})\big) \\ &= L\big(\sigma^{\star n}(a_{(1)})a_{(2)} \otimes S(a_{(3)})\big) \\ &= \sigma^{\star n}(a_{(1)})\sigma(a_{(2)}) = \sigma^{\star n+1}(a). \end{aligned}$$

The equation (ii) follows easily from (i) by

$$e_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^{\star n} \circ (\mathrm{id} \otimes S) \circ \Delta = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sigma^{\star n} = e_\star^{t\sigma}.$$

□

3.3.10 Theorem. *Every additive deformation of a Hopf algebra is a Hopf deformation. The deformed antipodes are given by the formula*

$$S_t = S \star e_\star^{-t\sigma}$$

with σ defined by (3.3.7).

Proof. Making use Lemma 3.3.9, we calculate

$$\begin{aligned}\mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta(a) &= e_\star^{tL} \star \mu\big(a_{(1)} \otimes S(a_{(2)})\big) e_\star^{-t\sigma}(a_{(3)}) \\ &= e_\star^{tL}\big(a_{(1)} \otimes S(a_{(4)})\big) a_{(2)} S(a_{(3)}) e_\star^{-t\sigma}(a_{(5)}) \\ &= e_\star^{tL}\big(a_{(1)} \otimes S(a_{(2)})\big) e_\star^{-t\sigma}(a_{(3)}) \mathbb{1} \\ &= e_\star^{t\sigma}(a_{(1)}) e_\star^{-t\sigma}(a_{(2)}) \mathbb{1} = \delta(a)\mathbb{1},\end{aligned}$$

so S_t is a left inverse of the identity map with respect to $\star_t$. In the same manner one can show that S_t is a right inverse. □

3.3.2 Constant Antipodes in the Cocommutative Case

We will show in this section that an additive deformation of a cocommutative Hopf algebra always splits into a trivial part and a part with constant antipodes. Let σ be the linear functional defined by (3.3.7).

3.3.11 Lemma. *We have $\partial\sigma = L + L \circ (S \otimes S) \circ \tau$.*

Proof. From

$$\begin{aligned}0 &= \partial L\big(S(b_{(1)}) \otimes S(a_{(1)}) \otimes a_{(2)} b_{(2)}\big) \\ &= \delta(b_{(1)}) L\big(S(a_{(1)}) \otimes a_{(2)} b_{(2)}\big) - L\big(S(b_{(1)}) S(a_{(1)}) \otimes a_{(2)} b_{(2)}\big) \\ &\quad + L\big(S(b_{(1)}) \otimes S(a_{(1)}) a_{(2)} b_{(2)}\big) - L\big(S(b_{(1)}) \otimes S(a_{(1)})\big) \delta(a_{(2)} b_{(2)}) \\ &= L\big(S(a_{(1)}) \otimes a_{(2)} b\big) - L\big(S(b_{(1)}) S(a_{(1)}) \otimes a_{(2)} b_{(2)}\big) \\ &\quad + \delta(a) L\big(S(b_{(1)}) \otimes b_{(2)}\big) - L\big(S(b) \otimes S(a)\big)\end{aligned}$$

we conclude

$$\begin{aligned}\delta(a) L\big(S(b_{(1)}) \otimes b_{(2)}\big) &- L\big(S(b_{(1)}) S(a_{(1)}) \otimes a_{(2)} b_{(2)}\big) \\ &= L\big(S(b) \otimes S(a)\big) - L\big(S(a_{(1)}) \otimes a_{(2)} b\big). \qquad (3.3.8)\end{aligned}$$

From

$$\begin{aligned}0 = \partial L\big(S(a_{(1)}) \otimes a_{(2)} \otimes b\big) &= \delta(a_{(1)})L(a_{(2)} \otimes b) - L\big(S(a_{(1)})a_{(2)} \otimes b\big) \\ &\quad + L\big(S(a_{(1)}) \otimes a_{(2)}b\big) - L\big(S(a_{(1)}) \otimes a_{(2)}\big)\delta(b) \\ &= L(a \otimes b) - \delta(a)L(\mathbb{1} \otimes b) \\ &\quad + L\big(S(a_{(1)}) \otimes a_{(2)}b\big) - L\big(S(a_{(1)}) \otimes a_{(2)}\big)\delta(b)\end{aligned}$$

we conclude

$$-L\big(S(a_{(1)}) \otimes a_{(2)}b\big) + L\big(S(a_{(1)}) \otimes a_{(2)}\big)\delta(b) = L(a \otimes b), \tag{3.3.9}$$

since $L(\mathbb{1} \otimes b) = 0$. Using the (3.3.8) and (3.3.9), we calculate

$$\begin{aligned}\partial\sigma(a \otimes b) &= \delta(a)\sigma(b) - \sigma(ab) + \sigma(a)\delta(b) \\ &= \delta(a)L\big(S(b_{(1)}) \otimes b_{(2)}\big) - L\big(S(a_{(1)}b_{(1)}) \otimes a_{(2)}b_{(2)}\big) \\ &\quad + L\big(S(a_{(1)}) \otimes a_{(2)}\big)\delta(b) \\ &= \delta(a)L\big(S(b_{(1)}) \otimes b_{(2)}\big) - L\big(S(b_{(1)})S(a_{(1)}) \otimes a_{(2)}b_{(2)}\big) \\ &\quad + L\big(S(a_{(1)}) \otimes a_{(2)}\big)\delta(b) \\ &= L\big(S(b) \otimes S(a)\big) - L\big(S(a_{(1)}) \otimes a_{(2)}b\big) + L\big(S(a_{(1)}) \otimes a_{(2)}\big)\delta(b) \\ &= L\big(S(b) \otimes S(a)\big) + L(a \otimes b).\end{aligned}$$

□

3.3.12 Theorem. *Let $\mathcal{B}$ be a Hopf algebra and L the generator of an additive deformation. Furthermore assume $\sigma = \sigma \circ S$. Then*

$$\widetilde{L} = L - \frac{1}{2}\partial\sigma$$

is the generator of a Hopf deformation $(\widetilde{\mu}_t)_{t\in\mathbb{R}}$, which has constant antipodes, that is $\widetilde{\mu}_t \circ (S \otimes \mathrm{id}) \circ \Delta = \mathbb{1}\delta = \widetilde{\mu}_t \circ (\mathrm{id} \otimes S) \circ \Delta$ for all $t \in \mathbb{R}$.

Proof. We can write

$$L = \underbrace{\frac{1}{2}\big(L + L \circ (S \otimes S) \circ \tau\big)}_{:=L_1} + \underbrace{\frac{1}{2}\big(L - L \circ (S \otimes S) \circ \tau\big)}_{:=L_2}$$

Then we have $L_1 = \partial \frac{\sigma}{2}$ and $\sigma_2 := L_2 \circ (S \otimes \mathrm{id}) \circ \Delta = 0$, since

$$\begin{aligned} L \circ (S \otimes S) \circ \tau \circ (S \otimes \mathrm{id}) \circ \Delta &= L \circ (\mathrm{id} \otimes S) \circ (S \otimes S) \circ \tau \circ \Delta \\ &= L \circ (S \otimes \mathrm{id}) \circ \Delta \circ S \\ &= \sigma \circ S = \sigma, \end{aligned}$$

where we made essential use of Lemma 3.3.7. □

3.3.13 Lemma. *If $\mathcal{B}$ is cocommutative, we have $\sigma = \sigma \circ S$ for every additive deformation.*

Proof. With the use Lemma 3.3.7 we calculate

$$\begin{aligned} \sigma \circ S = L \circ (S \otimes \mathrm{id}) \circ \Delta \circ S &= L \circ (S \otimes \mathrm{id}) \circ (S \otimes S) \circ \tau \circ \Delta \\ &= L \circ (S^2 \otimes S) \circ \tau \circ \Delta = L \circ (\mathrm{id} \otimes S) \circ \Delta = \sigma. \end{aligned}$$

□

So when deforming a cocommutative Hopf algebra one can always find an equivalent deformation such that $S_t = S$ for all $t \in \mathbb{R}$.

3.3.3 Examples

We will use Theorem 3.3.12 to decompose additive deformations in several examples. For the first example we need some preparatory results. Let $\mathcal{B}$ be a bialgebra. An element $a \in \mathcal{B}$ is called *primitive*, if

$$\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a.$$

It follows directly that $\delta(a) = 0$ for every primitive element a.

3.3.14 Proposition. *Let $\mathcal{B}$ be a bialgebra with additive deformation $\mu_t = \mu \star e_\star^{tL}$ and $a, b \in \mathcal{B}$. If a and b are primitive, we have*

$$\mu_t(a \otimes b) = ab + tL(a \otimes b)\mathbb{1}.$$

Proof. First let us calculate the coproduct,

$$\begin{aligned}
\Lambda(a \otimes b) &= (\mathrm{id} \otimes \tau \otimes \mathrm{id})\big(\Delta(a) \otimes \Delta(b)\big) \\
&= (\mathrm{id} \otimes \tau \otimes \mathrm{id})\big((a \otimes \mathbb{1} + \mathbb{1} \otimes a) \otimes (b \otimes \mathbb{1} + \mathbb{1} \otimes b)\big) \\
&= (\mathrm{id} \otimes \tau \otimes \mathrm{id})(a \otimes \mathbb{1} \otimes b \otimes \mathbb{1} + a \otimes \mathbb{1} \otimes \mathbb{1} \otimes b + \mathbb{1} \otimes a \otimes b \otimes \mathbb{1} + \mathbb{1} \otimes a \otimes \mathbb{1} \otimes b) \\
&= a \otimes b \otimes \mathbb{1} \otimes \mathbb{1} + a \otimes \mathbb{1} \otimes \mathbb{1} \otimes b + \mathbb{1} \otimes \tau(a \otimes b) \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes a \otimes b.
\end{aligned}$$

Since $L(\mathbb{1} \otimes c) = L(c \otimes \mathbb{1}) = 0$ for all $c \in \mathcal{B}$ and $\delta(b) = \delta(a) = 0$, we get

$$\begin{aligned}
\mathrm{e}_\star^{tL}(a \otimes b) &= \left(\delta \otimes \delta + tL + \frac{t^2}{2} L \star L + \cdots\right)(a \otimes b) = tL(a \otimes b) \\
\mathrm{e}_\star^{tL}(a \otimes \mathbb{1}) &= \mathrm{e}_\star^{tL}(\mathbb{1} \otimes b) = 0 \\
\mathrm{e}_\star^{tL}(\mathbb{1} \otimes \mathbb{1}) &= (\delta \otimes \delta)(\mathbb{1} \otimes \mathbb{1}) = 1.
\end{aligned}$$

It follows that

$$\mu_t(a \otimes b) = \left(\mu \star \mathrm{e}_\star^{tL}\right)(a \otimes b) = ab + tL(a \otimes b)\mathbb{1}.$$

□

Let $\mathcal{L}$ be an abelian Lie algebra, that is $[a, b] = 0$ for all $a, b \in \mathcal{L}$. Consider the universal enveloping Hopf algebra $U(\mathcal{L})$, that is the symmetric tensor algebra over $\mathcal{L}$ with the unique comultiplication such that all elements of $\mathcal{L}$ are primitive. This is indeed a Hopf algebra with antipode given by $S(a) := -a$ on elements of $\mathcal{L}$. In the case where $\mathcal{L}$ is of finite dimension n this is just the polynomial Hopf algebra in n commuting indeterminates.

3.3.15 Proposition. *For two additive deformations $\mu_t^{(1)}, \mu_t^{(2)}$ of $U(\mathcal{L})$ with generators L_1, L_2 the following statements are equivalent:*

(*i*) $L_1 - L_2$ *is a coboundary that is the two deformations differ by a trivial deformation*

(*ii*) $\mu_t^{(1)}(a \otimes b - b \otimes a) = \mu_t^{(2)}(a \otimes b - b \otimes a)$ *for all* $a, b \in \mathcal{L}, t \in \mathbb{R}$

(*iii*) $L_1(a \otimes b - b \otimes a) = L_2(a \otimes b - b \otimes a)$ *for all* $a, b \in \mathcal{L}, t \in \mathbb{R}$

Proof. For any additive deformation of $U(\mathcal{L})$ we have

$$\mu_t(a \otimes b) = ab + tL(a \otimes b)\mathbb{1}$$

by Proposition 3.3.14. This implies the equivalence of (*ii*) and (*iii*).

To prove that (*i*) is equivalent to (*iii*), it suffices to show that L is a coboundary if and only if $L(a \otimes b - b \otimes a) = 0$ for all $a, b \in \mathcal{L}$. Indeed, we can put $L = L_1 - L_2$. So let $L = \partial\psi$ be a coboundary. Since the counit vanishes on elements of $\mathcal{L}$ and $\mathcal{L}$ is abelian, we conclude

$$L(a \otimes b - b \otimes a) = -\psi(ab - ba) = 0.$$

Now let $L(a \otimes b - b \otimes a) = 0$ for all $a, b \in \mathcal{L}$. Choose a basis of $\mathcal{L}$ and introduce any ordering on this bases. Then define

$$\psi(a_1 \cdots a_n) := \begin{cases} L(a_1 \cdots a_{n-1} \otimes a_n) & \text{for } n \geq 2 \text{ and } a_1 \leq \ldots \leq a_n \\ 0 & \text{otherwise.} \end{cases}$$

We write $\widetilde{L} = L + \partial\psi$ and $\widetilde{\mu}_t = \mu \star e_\star^{t\widetilde{L}}$. Now an easy induction on n shows that $\widetilde{\mu}_t^{(n)}(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_n$ for $a_1 \leq \cdots \leq a_n$. But from the equivalence of (*ii*) and (*iii*) we know that $\widetilde{\mu}_t$ is commutative so we get $\widetilde{\mu}_t = \mu$ for all $t \in \mathbb{R}$. Finally, $\widetilde{L} = L + \partial\psi = 0$ shows that L is a coboundary. □

3.3.16 Example. In this example we realize the algebra of the quantum harmonic oscillator as the essentially only non-trivial additive deformation of the Hopf $*$-algebra of polynomials in adjoint commuting variables $\mathbb{C}[x, x^*]$ with comultiplication and counit defined via

$$\Delta(x^\epsilon) = x^\epsilon \otimes \mathbb{1} + \mathbb{1} \otimes x^\epsilon \qquad \text{and} \qquad \delta(x^\epsilon) = 0,$$

where $\epsilon \in \{1, *\}$.

It follows from Proposition 3.3.15 that a deformation of $\mathbb{C}[x, x^*]$ is determined up to a trivial deformation by the value of $L(x \otimes x^* - x^* \otimes y) = \mu_1(x \otimes x^* - x^* \otimes x)$. In case of a $*$-deformation L must be hermitian, so this is a real number. Choosing different constants here corresponds to a rescaling of the deformation parameter t. We assume $L(x \otimes x^* - x^* \otimes x) = 1$. There is also a canonical representative for the cohomology class of the generator for which the antipodes are constant. Choosing $L(x \otimes x^*) = -L(x^* \otimes x) = \frac{1}{2}$ yields $\sigma = 0$.

One gets a well defined $*$-algebra isomorphism from the algebra generated by $a, a^\dagger$ and $\mathbb{1}$ with the relation $aa^\dagger - a^\dagger a = \mathbb{1}$ to the deformation of the polynomial algebra $(\mathbb{C}[x, x^*], \mu_1)$ by setting $\Phi(a) = x$ and $\Phi(a^\dagger) = x^*$. In this sense the quantum harmonic oscillator algebra is the only non-trivial additive deformation of the polynomial algebra in two commuting adjoint variables.

In the next three examples we take as Hopf algebra the group algebra $\mathbb{C}G$ over a group G with comultiplication given by $\Delta g = g \otimes g$ for elements of G. We identify linear functionals on $(\mathbb{C}G)^{\otimes k}$ with functions on G^k in the obvious way. If $(\mu_t)_{t \in \mathbb{R}}$ is an additive deformation of $\mathbb{C}G$ with generator $L : G \times G \to \mathbb{C}$, the deformed multiplication on groupelements $g_1, g_2 \in G$ is given by $\mu_t(g_1 \otimes g_2) = e^{tL(g_1, g_2)} g_1 g_2$.

3.3.17 Example. We saw that in the cocommutative case it is possible to split an additive deformation into a trivial part and a part that corresponds to constant antipodes. But it is still possible that the part with constant antipodes is trivial, as this example shows. Consider the function $L : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ defined by

$$L(m, n) = m^2 n + mn^2.$$

The function L is a coboundary, since $L = \partial\psi$ for $\psi(k) = -\frac{1}{3}k^3$. We also see that $L(0, 0) = 0$ and L is commuting, so $L \in B^{(\mathbf{NC})}$. Therefore it generates a trivial deformation. In the following, group elements inside $\mathbb{C}\mathbb{Z}$ are denoted by (k) to avoid confusion with the complex number k. The deformation generated by L is non-constant, since

$$\mu_1\big((1)\otimes(1)\big) = e^{L(1,1)}(2) = e^2(2) \;\neq\; (2) = \mu((1)\otimes(1)).$$

Since $\sigma(k) = L(k,-k) = -k^3 + k^3 = 0$ for all $k \in \mathbb{Z}$, the antipodes are constant. We can calculate the isomorphisms Φ_t for the trivial deformation and get

$$\Phi_t((k)) = e^{-t\psi(k)}(k) = e^{tk^3}(k).$$

The second way for calculating the S_t yields

$$S_t(k) = \Phi_t \circ S \circ \Phi_t(k) = e^{-tk^3}\Phi_t(-k) = e^{-tk^3}e^{tk^3}(-k) = (-k).$$

So this is a situation, where we have $S \circ \Phi_t = \Phi_{-t} \circ S$, compare Theorem 3.3.6.

3.3.18 Example. On $\mathbb{Z}^d$ every $d \times d$-matrix A with complex entries defines a 2-cocycle L via

$$L(\underline{k},\underline{l}) := \underline{k}A\underline{l}^t$$

for $\underline{k},\underline{l} \in \mathbb{Z}^d$, since the functions $((k_1,\ldots,k_d),(l_1,\ldots,l_d)) \mapsto k_i l_j$ define cocycles for $i,j = 1,\ldots,d$, as is easily checked. These cocycles are of course normalized and commuting, so they are generators of additive deformations on a cocommutative Hopf algebra. L is hermitian if and only if A is hermitian. We want to apply Theorem 3.3.12, so we calculate

$$\sigma(\underline{k}) = L(\underline{k},-\underline{k}) = -\underline{k}A\underline{k}^t$$

and

$$\partial\frac{\sigma}{2}(\underline{k},\underline{l}) = \frac{1}{2}\big(-\underline{k}A\underline{k}^t + (\underline{k}+\underline{l})A(\underline{k}+\underline{l})^t - \underline{l}A\underline{l}^t\big) = \frac{1}{2}\big(\underline{k}A\underline{l}^t + \underline{l}A\underline{k}^t\big) = \underline{k}\frac{A+A^t}{2}\underline{l}^t$$

which gives

$$\widetilde{L}(\underline{k},\underline{l}) = \left(L - \frac{1}{2}\partial\sigma\right)(\underline{k},\underline{l}) = \underline{k}\frac{A - A^t}{2}\underline{l}^t.$$

So every such cocycle is equivalent to one which comes from an antisymmetric matrix.

3.3.19 Example. Let G be a group. then $\mathbb{C}G$ can be turned into a Hopf $*$-algebra in a natural way by extending the map $* : g \mapsto g^{-1}$ antilinearly to the whole of $\mathbb{C}G$. On the group elements the involution $*$ coincides with the antipode S. Now let L be a generator of an additive $*$-deformation, that is L is a normalized hermitian 2-cocycle. Then

$$\begin{aligned}\partial\frac{\sigma}{2}(g,h) &= \big(L + L\circ(S\otimes S)\circ\tau\big)(g,h)\\ &= \frac{1}{2}\big(L(g,h) + L(h^*,g^*)\big) = \frac{1}{2}\big(L(g,h) + \overline{L(g,h)}\big) = \operatorname{Re} L(g,h)\end{aligned}$$

and, consequently,

$$\widetilde{L}(g,h) = L - \frac{1}{2}\partial\sigma(g,h) = \operatorname{Im} L(g,h).$$

So one has to consider only the case where L is purely imaginary on the group elements.

3.4 Additive Deformations of Braided Hopf Algebras

In [Ger09] quantum Lévy processes on additive deformations of $*$-bialgebras were constructed and for the additive deformation on $\mathbb{C}[x, x^*]$ (discussed in the beginning of this chapter) this resulted in a pair of operator processes fulfilling canonical commutation relations.

If one wants to mimic the constructions on $\mathbb{C}[x, x^*]$ for the algebra with two adjoint anti-commuting generators, there is the problem that this is not a $*$-bialgebra, but a graded $*$-bialgebra (see for example [Sch93]). In this section we generalize the definition of an additive deformation even to the case of a braided bialgebra in the sense of [Maj95, Definition 9.4.5] (respectively [FS99] for $*$-bialgebras). However, we do not work with braided tensor categories, but it is sufficient in our context to define braided vector spaces as in [Ufe04], because we are concerned only with tensor powers of the same vector space.

For more clarity we use a graphic calculus, in the literature known as “braid diagrams”.

We show how the generator calculus and the Schoenberg correspondence of [Wir02] can be carried over to braided $*$-bialgebras (see Section 3.4.2 respectively 3.4.4). Whereas most results can be proved along the lines of [Wir02] for the bialgebra case, the diagrammatic approach gives more insight into the structure of these proofs and things get more involved in the $*$-bialgebra case, where we use the definition of a braided $*$-bialgebra of [FS99]. Our version of the Schoenberg correspondence (Theorem 3.4.14) generalizes and unifies two older versions:

- In [FSS03] there are no additive deformations allowed.
- In [Wir02] additive deformations are considered, but no braidings.

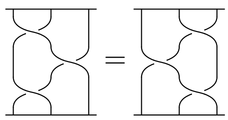
The existence of deformed antipodes, which was established for the non-braided case in Section 3.3.1, also remains true, as we show in Section 3.4.3.

3.4.1 Braided Structures and Braid Diagrams

A *braided vector space* is a pair (V, β) consisting of a vector space V and a *braiding* $\beta \in \operatorname{Aut}(V \otimes V)$, that is a linear automorphism of $V \otimes V$ which satisfies the braid equation

$$(\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \circ (\beta \otimes \mathrm{id}) = (\mathrm{id} \otimes \beta) \circ (\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta).$$

For reasons of clarity and comprehensibility, we use well known braid diagrams (see for example [Maj95, p. 430 f.]) to express coherences with braidings. In this notation the braid equation above can be visualized by



In our case we do not assume that the braiding fulfills the symmetry condition $\beta^2 = \mathrm{id}_{V \otimes V}$. So we distinguish them by under and over crossing like the figure explains:

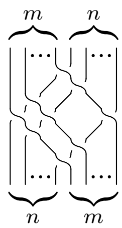
Other morphisms will be represented as nodes with the corresponding number of input and ouput strings. For the multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit $\mathbb{1} : \mathbb{K} \to \mathcal{A}$ on an algebra $\mathcal{A}$ respectively for the comultiplication $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the counit $\delta : \mathcal{C} \to \mathbb{K}$ on a coalgebra $\mathcal{C}$ we use the shorthand notations:

$\mu =$  $\quad \mathbb{1} =$  $\quad \Delta =$  $\quad \delta =$ 

One defines $\beta_{m,n} : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ for every braiding $\beta \in \mathrm{Aut}(V \otimes V)$ inductively by

$$\begin{aligned} \beta_{0,0} &:= \mathrm{id}_{\mathbb{C}}, \quad \beta_{1,m+1} := (\mathrm{id}_V \otimes \beta_{1,m}) \circ (\beta \otimes \mathrm{id}_{V^{\otimes m}}), \\ \beta_{1,0} = \beta_{0,1} &:= \mathrm{id}_V, \quad \beta_{n+1,m} := (\beta_{n,m} \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V^{\otimes n}} \otimes \beta_{1,m}). \end{aligned}$$

The braiding $\beta_{m,n}$ can be illustrated by the figure

$\beta_{m,n} =$ 

Note that $(\beta^{-1})_{n,m}$ is its inverse.

Crucial properties for linear maps on braided vector spaces (V, β) are the following. A linear map $f : V^{\otimes m} \to V^{\otimes n}$ is called *β-invariant*, if

$$(f \otimes \mathrm{id}) \circ \beta_{1,m} = \beta_{1,n} \circ (\mathrm{id} \otimes f)$$

and accordingly *β^{-1}-invariant* if

$$(\mathrm{id} \otimes f) \circ \beta_{m,1} = \beta_{n,1} \circ (f \otimes \mathrm{id}).$$

In case f fulfils both invariance conditions, we refer to f as *β-compatible*.

3.4.1 Remark. One can easily see that the tensor product and the composition of β-

invariant (respectively β^{-1}-invariant and β-compatible) linear maps is again β-invariant (respectively β^{-1}-invariant and β-compatible). For a β-invariant linear map $f : V^{\otimes m} \to V^{\otimes n}$ and a β^{-1}-invariant linear map $g : V^{\otimes k} \to V^{\otimes l}$, we get

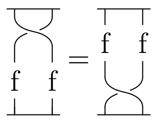
$$(f \otimes g) \circ \beta_{k,m} = \beta_{l,n} \circ (g \otimes f).$$

As an example for a (trivial) braiding we get the flip-operator $\tau(v \otimes w) = w \otimes v$. Obviously, all linear maps are τ-compatible.

To switch between two braided vector spaces (V_1, β_1) and (V_2, β_2), we use the notion of a braided morphism. A linear map $f : V_1 \to V_2$ will be called *braided morphism*, if

$$(f \otimes f) \circ \beta_1 = \beta_2 \circ (f \otimes f).$$

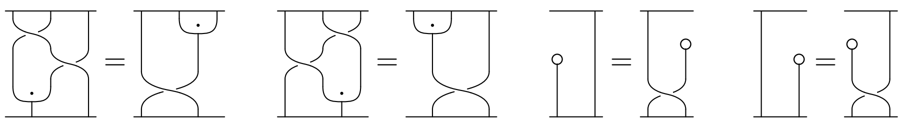
Expressed by braid diagrams, this equation looks like



A *braided algebra* $(\mathcal{A}, \mu, \mathbb{1}, \beta)$ is a unital associative algebra $(\mathcal{A}, \mu, \mathbb{1})$ and a braided vector space $(\mathcal{A}, \beta)$, such that μ and $\mathbb{1}$ are β-compatible, that is

$$(\mu \otimes \mathrm{id}) \circ \beta_{1,2} = \beta \circ (\mathrm{id} \otimes \mu), \qquad (\mathrm{id} \otimes \mu) \circ \beta_{2,1} = \beta \circ (\mu \otimes \mathrm{id}),$$
$$(\mathbb{1} \otimes \mathrm{id}) = \beta \circ (\mathrm{id} \otimes \mathbb{1}), \qquad (\mathrm{id} \otimes \mathbb{1}) = \beta \circ (\mathbb{1} \otimes \mathrm{id}).$$

We can visualize these four conditions by



We define $M_1 := \mu$ and M_n for $n \geq 2$ inductively via

$$M_n := (\mu \otimes M_{n-1}) \circ \left(\mathrm{id} \otimes \beta_{n-1,1} \otimes \mathrm{id}^{\otimes(n-1)}\right).$$

Then $\left(\mathcal{A}^{\otimes n}, M_n, \mathbb{1}^{\otimes n}, \beta_{n,n}\right)$ becomes a braided algebra. In particular,

$$\left(\mathcal{A}\otimes\mathcal{A}, M, \mathbb{1}\otimes\mathbb{1}, \beta_{2,2}\right)$$

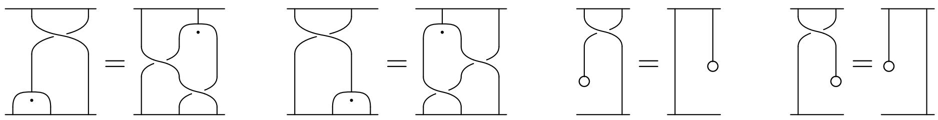
is a braided algebra, whereby $M := M_2 = (\mu\otimes\mu)\circ(\mathrm{id}\otimes\beta\otimes\mathrm{id})$. Note that the usual multiplication map on $\mathcal{A}\otimes\mathcal{A}$ has changed: We get the braiding β instead of the flip operator τ.

Dually, a *braided coalgebra* $(\mathcal{C}, \Delta, \delta, \beta)$ is a coalgebra $(\mathcal{C}, \Delta, \delta)$ and a braided vector space $(\mathcal{C}, \beta)$, such that Δ and δ are β-compatible, that is

$$(\Delta\otimes\mathrm{id})\circ\beta = \beta_{1,2}\circ(\mathrm{id}\otimes\Delta), \qquad (\mathrm{id}\otimes\Delta)\circ\beta = \beta_{2,1}\circ(\Delta\otimes\mathrm{id}),$$

$$(\delta\otimes\mathrm{id})\circ\beta = (\mathrm{id}\otimes\delta), \qquad (\mathrm{id}\otimes\delta)\circ\beta = (\delta\otimes\mathrm{id}).$$

The corresponding diagrams are:



Analogous to the multiplication map in the algebra case, we define $\Lambda_1 := \Delta$ and for $n \geq 2$ inductively by

$$\Lambda_n := \left(\mathrm{id}\otimes\beta_{1,n-1}\otimes\mathrm{id}^{\otimes(n-1)}\right)\circ(\Delta\otimes\Lambda_{n-1}).$$

Then $\left(\mathcal{C}^{\otimes n}, \Lambda_n, \delta^{\otimes n}, \beta_{n,n}\right)$ becomes a braided coalgebra. In particular,

$$\left(\mathcal{C}\otimes\mathcal{C}, \Lambda, \delta\otimes\delta, \beta_{2,2}\right)$$

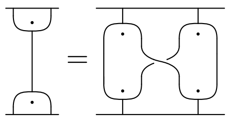
is a braided coalgebra, whereby $\Lambda := \Lambda_2 = (\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\Delta\otimes\Delta)$.

3.4.2 Remark. Given a braided algebra $\mathcal{A}$ and a braided coalgebra $\mathcal{C}$, it can be easily shown that the *opposite algebra* $\mathcal{A}^{\mathrm{op}} := (\mathcal{A}, \mu\circ\beta, \mathbb{1}, \beta)$ is a braided algebra and the *coopposite coalgebra* $\mathcal{C}^{\mathrm{cop}} := (\mathcal{C}, \beta\circ\Delta, \delta, \beta)$ is also a braided coalgebra. The algebra $\mathcal{A}$ is said to be *commutative*, if $\mu = \mu\circ\beta$ and the coalgebra $\mathcal{C}$ is referred to as *cocommutative* in case $\beta\circ\Delta = \Delta$.

A *braided bialgebra* $(\mathcal{B}, \Delta, \delta, \mu, \mathbb{1}, \beta)$ is a braided algebra $(\mathcal{B}, \mu, \mathbb{1}, \beta)$ and a braided coalgebra $(\mathcal{B}, \Delta, \delta, \beta)$, such that δ, Δ are braided algebra homomorphisms, that is

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \tag{3.4.1}$$
$$\delta \circ \mu = \delta \otimes \delta$$

is fulfilled. Equation (3.4.1) is called *braided bialgebra condition* and differs from the usual bialgebra condition. In the used graphical calculus the picture

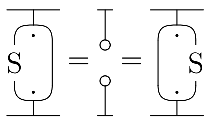


represents this equation.

A homomorphism $f : (\mathcal{B}_1, \beta_1) \to (\mathcal{B}_2, \beta_2)$ of braided bialgebras is defined as a homomorphism of algebras and coalgebras such that $(f \otimes f) \circ \beta_1 = \beta_2 \circ (f \otimes f)$.

If $\mathcal{B}$ is a braided bialgebra, $\mathcal{B}^{\otimes n}$ is a coalgebra and $\mathcal{B}^{\otimes m}$ is an algebra for all $n, m \in \mathbb{N}$. It follows directly from Remark 3.4.1 that the convolution $R \star T$ of compatible linear maps $R, T : \mathcal{B}^{\otimes n} \to \mathcal{B}^{\otimes m}$ is compatible.

A braided bialgebra $\mathcal{H}$ is called *braided Hopf algebra* if it is also a Hopf algebra. Graphically, the antipode condition is expressed by



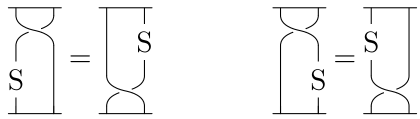
3.4.3 Remark. For a braided Hopf algebra $(\mathcal{H}, \Delta, \delta, \mu, \mathbb{1}, S, \beta)$, similar to the nonbraided case, the following properties are fulfilled.

- The antipode S is uniquely determined if it exists.
- S is a *braided algebra* and *coalgebra anti-homomorphism*, that is

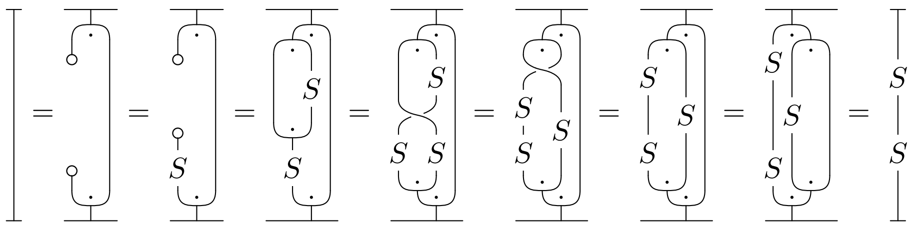
$$S \circ \mu = \mu \circ \beta \circ (S \otimes S), \quad S \circ \mathbb{1} = \mathbb{1}, \qquad \Delta \circ S = \beta \circ (S \otimes S) \circ \Delta, \quad \delta \circ S = \delta$$

are satisfied. The first equation follows from the fact that both sides of this equation are convolution inverses of μ. The third equation follows analogously to the first. The second and fourth equation can be shown as in the non-braided case.

- S is β-compatible, which can be visualized by



- If in addition $\mathcal{H}$ is commutative as an algebra or cocommutative as a coalgebra, then $S^2 = \mathrm{id}$ holds. If $\mathcal{H}$ is cocommutative, that is $\beta \circ \Delta = \Delta$, the diagrams



show $S^2 = \mathrm{id}$. The proof in the commutative situation is analogous.

- It can be easily seen that the braiding β is determined by the formula

$$\beta = (\mu \otimes \mu) \circ \big(S \otimes (\Delta \circ \mu) \otimes S\big) \circ (\Delta \otimes \Delta).$$

Now we want to define involutions on braided algebraic structures. We follow the definition of an involutive braided bialgebra or braided $*$-bialgebra given by Franz, Schott, and Schürmann (see [FSS03] or [Fra06], Section 3.8), which differs from that given by Majid in [Maj95, Proposition 10.3.2].

A *braided $*$-bialgebra* $(\mathcal{B}, \Delta, \delta, \mu, \mathbb{1}, \beta, *)$ is a braided bialgebra $(\mathcal{B}, \Delta, \delta, \mu, \mathbb{1}, \beta)$ with an anti-linear map $* : \mathcal{B} \to \mathcal{B}$ such that $(\mathcal{B}, \mu, \mathbb{1}, *)$ is a $*$-algebra and there exists an involution $*_{\mathcal{B}\otimes\mathcal{B}}$ on $\mathcal{B} \otimes \mathcal{B}$ for which the canonical embeddings $\mathcal{B} \to \mathcal{B} \otimes \mathcal{B} \leftarrow \mathcal{B}$ and Δ are $*$-algebra homomorphisms.

3.4.4 Remark. From the definition above we get the following properties concerning braided $*$-bialgebras:

- If the canonical embeddings $a \mapsto a \otimes \mathbb{1}$ and $a \mapsto \mathbb{1} \otimes a$ are $*$-algebra homomorphisms, we have $(a \otimes \mathbb{1})^* = a^* \otimes \mathbb{1}$ and $(\mathbb{1} \otimes a)^* = \mathbb{1} \otimes a^*$. Since the involution on $\mathcal{B} \otimes \mathcal{B}$ shall be an anti-algebra homomorphism, we get

$$\begin{aligned}(a \otimes b)^* = \big((a \otimes \mathbb{1})(\mathbb{1} \otimes b)\big)^* &= (\mathbb{1} \otimes b)^*(a \otimes \mathbb{1})^* = (\mathbb{1} \otimes b^*)(a^* \otimes \mathbb{1}) \\ &= (\mu \otimes \mu) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id})(\mathbb{1} \otimes b^* \otimes a^* \otimes \mathbb{1}) \\ &= \beta(b^* \otimes a^*) = \beta \circ (* \otimes *) \circ \tau(a \otimes b).\end{aligned}$$

  (Note that we used the β-compatible multiplication $M = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id})$ on $\mathcal{B} \otimes \mathcal{B}$ instead of the usual one.) It follows that the involution on $\mathcal{B} \otimes \mathcal{B}$ is given by

$$*_{\mathcal{B} \otimes \mathcal{B}} = \beta \circ (* \otimes *) \circ \tau,$$

  where τ is the usual flip operator $\tau(a \otimes b) = b \otimes a$.

- Summarizing, we get an equivalent definition for braided $*$-bialgebras: A braided $*$-bialgebra $\mathcal{B}$ is a braided bialgebra $\mathcal{B}$ with an involution $*$, such that

$$(*_{\mathcal{B} \otimes \mathcal{B}})^2 = \big(\beta \circ (* \otimes *) \circ \tau\big)^2 = \mathrm{id}_{\mathcal{B} \otimes \mathcal{B}}.$$

  The involution $*$ is, in general, not β-compatible, but fulfills

$$\beta \circ (* \otimes *) \circ \tau = (* \otimes *) \circ \tau \circ \beta^{-1}.$$

  This condition contains the flip operator and the braiding. To avoid confusion, we did not use braid diagrams in calculation with $*$.

- Note that the multiplication μ and the comultiplication Δ fulfill

$$* \circ \mu = \mu \circ (* \otimes *) \circ \tau, \quad \Delta \circ * = *_{\mathcal{B} \otimes \mathcal{B}} \circ \Delta.$$

Now we want to show a central but not obvious property of hermitian bilinear functionals on a braided coalgebra. We call a bilinear functional $K : \mathcal{C} \otimes \mathcal{C} \to \mathbb{C}$ on a braided coalgebra $\mathcal{C}$ *hermitian* if $K(a^* \otimes b^*) = \overline{K(b \otimes a)}$ for all $a, b \in \mathcal{C}$. Note that this condition differs from $K\big((a \otimes b)^{*_{\mathcal{B}\otimes\mathcal{B}}}\big) = \overline{K(a \otimes b)}$.

3.4.5 Proposition. *Suppose we have two hermitian, bilinear functionals K, L on a braided $*$-coalgebra $(\mathcal{C}, \Delta, \delta, \beta)$. If K is β-invariant or L is β^{-1}-invariant, the convolution $K \star L$ is hermitian, too.*

Proof. That K and L are hermitian means $K = \overline{K} \circ (* \otimes *) \circ \tau$ and $L = \overline{L} \circ (* \otimes *) \circ \tau$. We calculate

$$
\begin{aligned}
&(\overline{K \star L}) \circ (* \otimes *) \circ \tau \\
&\quad = (\overline{K} \otimes \overline{L}) \circ \Lambda \circ (* \otimes *) \circ \tau \\
&\quad = (K \otimes L) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (* \otimes *) \circ \tau \\
&\quad = (K \otimes L) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \\
&\qquad\quad \circ (\beta \otimes \beta) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\Delta \otimes \Delta) \circ \tau \\
&\quad = (K \otimes L) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \\
&\qquad\quad \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\beta^{-1} \otimes \beta^{-1}) \circ \tau_{2,2} \circ (\Delta \otimes \Delta) \\
&\quad = (K \otimes L) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \\
&\qquad\quad \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ \tau_{2,2} \circ (\beta^{-1} \otimes \beta^{-1}) \circ (\Delta \otimes \Delta) \\
&\quad = (K \otimes L) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \\
&\qquad\quad \circ (* \otimes * \otimes * \otimes *) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ \tau_{(14)} \circ (\beta^{-1} \otimes \beta^{-1}) \circ (\Delta \otimes \Delta) \\
&\quad = (K \otimes L) \circ (\tau \otimes \tau) \circ (* \otimes * \otimes * \otimes *) \circ (* \otimes * \otimes * \otimes *) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \\
&\qquad\quad \circ (\mathrm{id} \otimes \beta^{-1} \otimes \mathrm{id}) \circ \tau_{(14)} \circ (\beta^{-1} \otimes \beta^{-1}) \circ (\Delta \otimes \Delta) \\
&\quad = (K \otimes L) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ \tau_{(14)} \\
&\qquad\quad \circ (\mathrm{id} \otimes \beta^{-1} \otimes \mathrm{id}) \circ (\beta^{-1} \otimes \beta^{-1}) \circ (\Delta \otimes \Delta) \\
&\quad = (K \otimes L) \circ \tau_{2,2} \circ \beta_{2,2}^{-1} \circ \Lambda \\
&\quad = (L \otimes K) \circ \beta_{2,2}^{-1} \circ \Lambda = (K \otimes L) \circ \Lambda = K \star L,
\end{aligned}
$$

wherein $\tau_{(14)}$ is defined by $a \otimes b \otimes c \otimes d \mapsto d \otimes b \otimes c \otimes a$. We used the β-invariance of K

(respectively β^{-1}-invariance of L) twice at the last step. □

It follows directly from Proposition 3.4.5 that for a hermitian and β-compatible bilinear functional $L : \mathcal{C} \otimes \mathcal{C} \to \mathbb{C}$ the convolution exponential $\mathrm{e}_\star^{tL}$ is also hermitian for every $t \in \mathbb{R}$. We will need this in the following section.

3.4.2 Generator Calculus

Let $(\mathcal{B}, \Delta, \delta, \mu, \mathbb{1}, \beta)$ be a braided bialgebra. Then we call a family $(\mu_t)_{t\in\mathbb{R}}$ of β-compatible maps $\mu_t : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ an *additive deformation* if

- $\mu_0 = \mu$,
- $\mathcal{B}_t = (\mathcal{B}, \mu_t, \mathbb{1}, \beta)$ is a braided unital algebra for all $t \in \mathbb{R}$,
- $\Delta \circ \mu_{t+s} = (\mu_t \otimes \mu_s) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$ for all $t, s \in \mathbb{R}$,
- $\delta \circ \mu_t \xrightarrow[t\to 0]{} \delta \circ \mu_0 = \delta \otimes \delta$ pointwise.

Assume $\mathcal{B}$ is a braided $*$-bialgebra. Then we call $(\mu_t)_{t\in\mathbb{R}}$ an *additive $*$-deformation* if in addition

- $\mu_t(a^* \otimes b^*) = \mu_t(b \otimes a)^*$ for all $t \in \mathbb{R}$,

that is $* \circ \mu_t = \mu_t \circ (* \otimes *) \circ \tau$.

3.4.6 Remark. The third condition states that the comultiplication Δ is a $*$-algebra homomorphism from $\mathcal{B}_{s+t}$ into $\mathcal{B}_s \otimes \mathcal{B}_t$, as the comultiplication on the bialgebra $\mathcal{B} \otimes \mathcal{B}$ is defined by $\Lambda = (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$.

3.4.7 Theorem. *Suppose that $\mathcal{B}$ is a braided bialgebra and $(\mu_t)_{t\in\mathbb{R}}$ an additive deformation. Then*

$$L = \frac{\mathrm{d}}{\mathrm{d}t} \delta \circ \mu_t \Big|_{t=0} \equiv \lim_{t\to 0^+} \frac{1}{t} \left(\delta \circ \mu_t - \delta \otimes \delta \right)$$

exists pointwise and defines a β-compatible, commuting, normalized 2-cocycle which fulfills

$$\mu_t = \mu \star \mathrm{e}_\star^{tL}. \tag{3.4.2}$$

If $\mathcal{B}$ is even a braided $*$-bialgebra and $(\mu_t)_{t\in\mathbb{R}}$ an additive $*$-deformation, then we have additionally

$$\overline{L(a \otimes b)} = L(b^* \otimes a^*). \tag{3.4.3}$$

Conversely, for every β-compatible, commuting, normalized 2-cocycle $L : \mathcal{B}\otimes\mathcal{B} \to \mathbb{C}$ on a braided bialgebra $\mathcal{B}$, equation (3.4.2) *defines an additive deformation. If $\mathcal{B}$ is a braided $*$-bialgebra and L satisfies additionally* (3.4.3)*, then* (3.4.2) *defines an additive $*$-deformation.*

Proof. (in the non-braided case due to Wirth, see [Wir02]) Let $(\mu_t)_{t\in\mathbb{R}}$ be an additive deformation. It follows that

$$(\delta \circ \mu_s) \star (\delta \circ \mu_t) = (\delta \otimes \delta) \circ (\mu_s \otimes \mu_t) \circ \Lambda = (\delta \otimes \delta) \circ \Delta \circ \mu_{s+t} = \delta \circ \mu_{s+t}.$$

Thus $(\delta \circ \mu_t)_{t\in\mathbb{R}}$ is a continuous convolution semigroup, which implies that there exists a generator $L = \lim_{t\to 0} \frac{1}{t}\,(\mu_t - \delta \otimes \delta)$ with $\delta \circ \mu_t = \mathrm{e}_\star^{tL}$. Moreover,

$$\mu \star (\delta \circ \mu_t) = \big(\mu \otimes (\delta \circ \mu_t)\big) \circ \Lambda = (\mathrm{id} \otimes \delta) \circ (\mu \otimes \mu_t) \circ \Lambda = (\mathrm{id} \otimes \delta) \circ \Delta \circ \mu_t = \mu_t.$$

Analogously, $(\delta \circ \mu_t) \star \mu = \mu_t$ holds. The differentiation of

$$\mathrm{e}_\star^{tL} \star \mu = \mu \star \mathrm{e}_\star^{tL}, \quad \mu_t(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1}, \quad \mu_t \circ (\mu_t \otimes \mathrm{id}) = \mu_t \circ (\mathrm{id} \otimes \mu_t),$$

the β-compatibility of μ_t and $* \circ \mu_t = \mu_t \circ (* \otimes *) \circ \tau$ (in the $*$-case) at $t = 0$ gives the claimed properties of L.

Conversely, assume $L : \mathcal{B} \otimes \mathcal{B} \to \mathbb{C}$ is a β-compatible, commuting, normalized 2-cocycle. We want to show the associativity of $\mu_t := \mu \star \mathrm{e}_\star^{tL}$. Obviously, the multiplication μ_t is β-compatible due to Remark 3.4.1. First we show that

$$\big(\mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes \mu)\big) \star (\delta \otimes \mathrm{e}_\star^{tL}) = \big(\mathrm{e}_\star^{tL} \circ (\mu \otimes \mathrm{id})\big) \star (\mathrm{e}_\star^{tL} \otimes \delta). \tag{3.4.4}$$

Since $(\mathrm{id} \otimes \mu)$ is a coalgebra homomorphism, $\mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes \mu) = \mathrm{e}_\star^{tL \circ (\mathrm{id} \otimes \mu)}$. From $(\delta \otimes K_1) \star (\delta \otimes K_2) = \delta \otimes (K_1 \star K_2)$ we conclude that $\delta \otimes \mathrm{e}_\star^{tL} = \mathrm{e}_\star^{t\delta \otimes L}$. It is easy to see that

$L \circ (\mathrm{id} \otimes \mu)$ and $\delta \otimes L$ commute under convolution, so we have

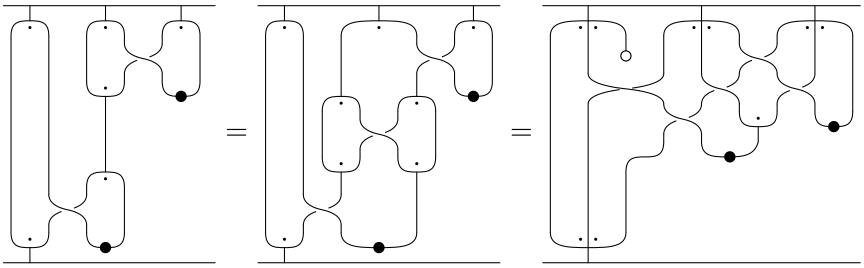
$$\left(\mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes \mu)\right) \star \left(\delta \otimes \mathrm{e}_\star^{tL}\right) = \left(\mathrm{e}_\star^{tL\circ(\mathrm{id}\otimes\mu)}\right) \star \left(\mathrm{e}_\star^{t\delta\otimes L}\right) = \mathrm{e}_\star^{t(L\circ(\mathrm{id}\otimes\mu)+\delta\otimes L)}.$$

Analogously, it holds that

$$\left(\mathrm{e}_\star^{tL} \circ (\mu \otimes \mathrm{id})\right) \star \left(\mathrm{e}_\star^{tL} \otimes \delta\right) = \mathrm{e}_\star^{t(L\circ(\mu\otimes\mathrm{id})+L\otimes\delta)}$$

and (3.4.4) follows from the cocycle property $\delta \otimes L - L \circ (\mu \otimes \mathrm{id}) + L \circ (\mathrm{id} \otimes \mu) - L \otimes \delta = 0$.

Now we calculate $\mu_t \circ (\mathrm{id} \otimes \mu_t)$ with braid diagrams, where $\bullet$ means $\mathrm{e}_\star^{tL}$. We get



The last diagram is equal to $\mu^{(3)} \star \left(\mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes \mu)\right) \star \left(\delta \otimes \mathrm{e}_\star^{tL}\right)$. In the same manner, we get $\mu_t \circ (\mu_t \otimes \mathrm{id}) = \mu^{(3)} \star \left(\mathrm{e}_\star^{tL} \circ (\mu \otimes \mathrm{id})\right) \star \left(\mathrm{e}_\star^{tL} \otimes \delta\right)$. So associativity of μ_t follows from (3.4.4). We also have

$$\mu_t(\mathbb{1} \otimes \mathbb{1}) = \mu \star \mathrm{e}_\star^{tL}(\mathbb{1} \otimes \mathbb{1}) = \mathrm{e}_\star^{tL(\mathbb{1}\otimes\mathbb{1})}\mu(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1}$$

for all $t \in \mathbb{R}$ and, obviously, $\mu_0 = \mu$ is fulfilled. Now we prove that $\Delta : \mathcal{B}_{t+s} \to \mathcal{B}_t \otimes \mathcal{B}_s$ is an algebra homomorphism

$$\begin{aligned}(\mu_t \otimes \mu_s) \circ \Lambda &= (\mathrm{e}_\star^{tL} \otimes \mu \otimes \mu \otimes \mathrm{e}_\star^{sL}) \circ \Lambda^{(4)} \\ &= (\mathrm{e}_\star^{tL} \otimes \mu \otimes \mu \otimes \mathrm{e}_\star^{sL}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \Lambda \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Lambda^{(3)} \\ &= \Delta \circ (\mathrm{e}_\star^{tL} \otimes \mu \otimes \mathrm{e}_\star^{sL}) \circ \Lambda^{(3)} \\ &= \Delta \circ (\mu \otimes \mathrm{e}_\star^{tL} \otimes \mathrm{e}_\star^{sL}) \circ \Lambda^{(3)} \\ &= \Delta \circ (\mu \otimes \mathrm{e}_\star^{(t+s)L}) \circ \Lambda = \Delta \circ \mu_{(t+s)}.\end{aligned}$$

At last we need the implication: If L is hermitian, then $\mathcal{B}_t$ becomes a $*$-algebra, that is $\mu_t(a \otimes b) = \mu_t(b^* \otimes a^*)^*$:

$$\begin{aligned} &* \circ \mu_t \circ (* \otimes *) \circ \tau \\ &\quad = * \circ (\mu \star \mathrm{e}_\star^{tL}) \circ (* \otimes *) \circ \tau \\ &\quad = * \circ (\mu \otimes \mathrm{e}_\star^{tL}) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (* \otimes *) \circ \tau \\ &\quad = (\mu \otimes \mathrm{e}_\star^{tL}) \circ (* \otimes * \otimes * \otimes *) \circ (\tau \otimes \tau) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (* \otimes *) \circ \tau \end{aligned}$$

This last expression is the same as line three in the proof of Proposition 3.4.5 with K replaced by μ and L replaced by $e_\star^{tL}$. The same manipulations can be performed and we arrive at

$$\left(\mu \otimes \mathrm{e}_\star^{tL}\right) \circ \tau_{2,2} \circ \beta_{2,2}^{-1} \circ \Lambda = \left(\mathrm{e}_\star^{tL} \otimes \mu\right) \circ \beta_{2,2}^{-1} \circ \Lambda = \left(\mu \otimes \mathrm{e}_\star^{tL}\right) \circ \Lambda = \mu_t,$$

using the β^{-1}-invariance of L. □

3.4.8 Remark. A cohomological description in the sense of Section 3.2.1 is possible. Put

$$C_n^{(\beta)}\{f : \mathcal{B}^{\otimes n} \to \mathbb{C} \mid f \text{ is } \beta\text{-compatible}\}.$$

Then $\partial C_n^{(\beta)} \subset C_{n+1}^{(\beta)}$ by Remark 3.4.1, so the $C_n^{(\beta)}$ form a subcomplex of the Hochschild complex. We define $C_n^{\mathbf{CN}\beta} := C_n^{\mathbf{CN}} \cap C_n^{\beta}$ and $C_n^{\mathbf{CNH}\beta} := C_n^{\mathbf{CNH}} \cap C_n^{\beta}$ to obtain cochain complexes such that the generators of additive deformations of braided ($*$)-bialgebras are exactly the 2-cocycles. The characterization of trivial deformations works analogously to Section 3.2.2. The only change is that the appearing linear maps are required to be β-compatible.

3.4.3 Hopf-Deformations

In this section, we want to show the existence of deformed antipodes on braided Hopf ($*$-)algebras and explore their properties. The use of Sweedler notation is heavily reduced due to the braiding. Thus, the proofs are sometimes a bit more complicated than the proofs of the corresponding statements for the non-braided case in Section 3.3.

Let $\big(\mathcal{H}, \Delta, \delta, \mu, \mathbb{1}, S, \beta, (*)\big)$ be a braided Hopf ($*$-)algebra. If we have an additive deformation $(\mu_t)_{t\in\mathbb{R}}$ with generator L, the equation

$$L \circ (\mathrm{id} \otimes S) \circ \Delta = L \circ (S \otimes \mathrm{id}) \circ \Delta \tag{3.4.5}$$

holds because of

$$\begin{aligned} 0 &= \partial L \left(a_{(1)} \otimes S(a_{(2)}) \otimes a_{(3)}\right) \\ &= L\left(S(a_{(1)}) \otimes a_{(2)}\right) - \underbrace{L(\mathbb{1} \otimes a)}_{=0} + \underbrace{L(a \otimes \mathbb{1})}_{=0} - L\left(a_{(1)} \otimes S(a_{(2)})\right) \end{aligned}$$

just as in the non-braided case, see Lemma 3.3.7.

3.4.9 Lemma. *Let K be a β-invariant linear functional on $\mathcal{H} \otimes \mathcal{H}$ and define $\widetilde{K} := K \circ (S \otimes \mathrm{id}) \circ \Delta$. Then $K \star \mu = \mu \star K$ implies $\widetilde{K} \star \mathrm{id} = \mathrm{id} \star \widetilde{K}$.*

Proof. First we use $\Delta \circ S = \beta \circ (S \otimes S) \circ \Delta$ in order to get

$$\begin{aligned} \Lambda \circ (S \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (S \otimes \mathrm{id}) \circ \Delta \\ &= (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\beta \otimes \mathrm{id} \otimes \mathrm{id}) \circ (S \otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(4)} \\ &= (\beta_{1,2} \otimes \mathrm{id}) \circ (S \otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(4)}. \end{aligned}$$

This allows us to calculate

$$\begin{aligned} (K \star \mu) \circ (S \otimes \mathrm{id}) \circ \Delta &= (K \otimes \mu) \circ (\beta_{1,2} \otimes \mathrm{id}) \circ (S \otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(4)} \\ &= \mu \circ (S \otimes \widetilde{K} \otimes \mathrm{id}) \circ \Delta^{(3)} \end{aligned}$$

using β-invariance of K. Next we get

$$\begin{aligned} (\mu \star K) \circ (S \otimes \mathrm{id}) \circ \Delta &= (\mu \otimes K) \circ (\beta_{1,2} \otimes \mathrm{id}) \circ (S \otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(4)} \\ &= (\mathrm{id} \otimes K) \circ (\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu \otimes \mathrm{id}) \circ (S \otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(4)} \\ &= (\mathrm{id} \otimes K) \circ (\beta \otimes \mathrm{id}) \circ (S \otimes \mathbb{1} \otimes \mathrm{id}) \circ \Delta = \mathbb{1} \circ \widetilde{K} \end{aligned}$$

using β-invariance of μ and $\mathbb{1}$ as well as the antipode equation. Combining these two equations, it follows from $K \star \mu = \mu \star K$ that

$$\mathbb{1} \circ \widetilde{K} = (\mu \star K) \circ (S \otimes \mathrm{id}) \circ \Delta = (K \star \mu) \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ (S \otimes \widetilde{K} \otimes \mathrm{id}) \circ \Delta^{(3)},$$

or in Sweedler notation (suppressing the unit)

$$\widetilde{K}(a) = S(a_{(1)})\widetilde{K}(a_{(2)})a_{(3)}.$$

We conclude that

$$(\mathrm{id} \star \widetilde{K})(a) = a_{(1)}\widetilde{K}(a_{(2)}) = a_{(1)}\big(S(a_{(2)})\widetilde{K}(a_{(3)})a_{(4)}\big) = \widetilde{K}(a_{(1)})a_{(2)} = (\widetilde{K} \star \mathrm{id})(a)$$

for all $a \in \mathcal{H}$. □

3.4.10 Corollary. *The family $F_t := \mathrm{e}_\star^{tL} \circ (S \otimes \mathrm{id}) \circ \Delta$ is a continuous convolution semigroup and*

$$\mathrm{e}_\star^{tL} \circ (S \otimes \mathrm{id}) \circ \Delta = \mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta = \mathrm{e}_\star^{t\sigma}$$

with $\sigma := L \circ (S \otimes \mathrm{id}) \circ \Delta$.

Proof. The continuous convolution semigroup $e_\star^{tL}$ fulfills $\mathrm{e}_\star^{tL} \star \mu = \mu \star \mathrm{e}_\star^{tL}$. Because of Lemma 3.4.9, we have $F_t \star \mathrm{id} = \mathrm{id} \star F_t$. So we get

$$\begin{aligned}
F_t \star F_s &= (F_t \otimes F_s) \circ \Delta = \mathrm{e}_\star^{tL} \circ (S \otimes \mathrm{id} \otimes F_s) \circ \Delta^{(3)} = \mathrm{e}_\star^{tL} \circ (S \otimes F_s \otimes \mathrm{id}) \circ \Delta^{(3)} \\
&= \mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes \mathrm{e}_\star^{sL} \otimes \mathrm{id}) \circ (S \otimes S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(4)} \\
&= \mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes \mathrm{e}_\star^{sL} \otimes \mathrm{id}) \circ (\beta^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (S \otimes \mathrm{id}) \circ \Delta \\
&= \mathrm{e}_\star^{tL} \circ (\mathrm{e}_\star^{sL} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\beta_{1,2} \otimes \mathrm{id}) \circ (\beta^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (S \otimes \mathrm{id}) \circ \Delta \\
&= \mathrm{e}_\star^{tL} \circ (\mathrm{e}_\star^{sL} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (S \otimes \mathrm{id}) \circ \Delta \\
&= (\mathrm{e}_\star^{sL} \otimes \mathrm{e}_\star^{tL}) \circ \Lambda \circ (S \otimes \mathrm{id}) \circ \Delta \\
&= \mathrm{e}_\star^{(t+s)L} \circ (S \otimes \mathrm{id}) \circ \Delta = F_{t+s}.
\end{aligned}$$

The continuity of F_t follows from the continuity of $\mathrm{e}_\star^{tL}$ and differentiating gives us the generator $\sigma = L \circ (S \otimes \mathrm{id}) \circ \Delta$.

Analogously, one concludes that the linear functionals $\mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta$ constitute a continuous convolution semigroup with generator $L \circ (\mathrm{id} \otimes S) \circ \Delta$. But this equals σ due to (3.4.5). □

3.4.11 Theorem. *Every additive deformation on a braided Hopf algebra provides a family of deformed antipodes $(S_t)_{t\in\mathbb{R}}$ with*

$$S_t = S \star \mathrm{e}_\star^{-t\sigma},$$

where $\sigma = L \circ (\mathrm{id} \otimes S) \circ \Delta$.

Proof. For $F_t = \mathrm{e}_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta$, we get

$$\begin{aligned}
&\mu_t \circ \big(\mathrm{id} \otimes (S \star F_{-t})\big) \circ \Delta \\
&\quad = (\mathrm{e}_\star^{tL} \otimes \mu) \circ \Lambda \circ (\mathrm{id} \otimes S \otimes F_{-t}) \circ \Delta^{(3)} \\
&\quad = (\mathrm{e}_\star^{tL} \otimes \mu) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (\mathrm{id} \otimes S \otimes F_{-t}) \circ \Delta^{(3)} \\
&\quad = (\mathrm{e}_\star^{tL} \otimes \mu) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \beta) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S \otimes S \otimes F_{-t}) \circ \Delta^{(5)} \\
&\quad = (\mathrm{e}_\star^{tL} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \circ (\mathrm{id} \otimes \mathbb{1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes F_{-t}) \circ \Delta^{(3)} \\
&\quad = (\mathrm{e}_\star^{tL} \otimes \mathbb{1}) \circ (\mathrm{id} \otimes S \otimes F_{-t}) \circ \Delta^{(3)} \\
&\quad = \mathbb{1} \circ (F_t \otimes F_{-t}) \circ \Delta = \mathbb{1}\delta.
\end{aligned}$$

□

3.4.12 Theorem. *Let $(\mu_t)_{t\in\mathbb{R}}$ be an additive deformation of a braided Hopf algebra with generator L. Then the deformed antipodes S_t have the properties*

(*i*) $S_t(\mathbb{1}) = \mathbb{1}$,

(*ii*) $S_t \circ \mu_{-t} = \mu_t \circ (S_t \otimes S_t) \circ \beta$,

(*iii*) $\Delta \circ S_{t+r} = (S_t \otimes S_r) \circ \beta \circ \Delta$,

(*iv*) *if $\mathcal{B}$ is commutative or cocommutative, we get* $S_t \circ S_{-t} = \mathrm{id}$ *and* (*v*) *if we have a braided Hopf* $*$*-algebra,* $S_{-t} \circ * \circ S_t \circ * = \mathrm{id}$ *is fulfilled.*

Proof. The proof of (*i*) to (*iv*) is quite similar to the proof of Theorem 3.3.4 for the non-braided case. For convenience of the reader we rewrite the main calculation for the proof of (*ii*) avoiding Sweedler notation. Following the proof of Theorem 3.3.4, at one point we have to show that $\mu_t \star_t (\mu_t \circ (S_t \otimes S_t) \circ \beta) = \mathbb{1}\delta \otimes \delta$. Using $(\mathrm{id} \otimes \beta) \circ (\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \beta_{1,2} \circ (\mathrm{id} \otimes \Delta) = (\Delta \otimes \mathrm{id}) \circ \beta$, we calculate

$$\begin{aligned}
&\mu_t \star_t (\mu_t \circ (S_t \otimes S_t) \circ \beta) \\
&\quad = \mu_t^{(4)} \circ \Big(\mathrm{id}_2 \otimes \big((S_t \otimes S_t) \circ \beta\big)\Big) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \\
&\quad = \mu_t^{(3)} \circ (\mathrm{id} \otimes \mu_t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S_t \otimes S_t) \circ (\mathrm{id} \otimes \Delta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \circ (\Delta \otimes \mathrm{id}) \\
&\quad = \mu_t^{(3)} \circ (\mathrm{id} \otimes \mathbb{1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S_t) \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \circ (\Delta \otimes \mathrm{id}) \\
&\quad = \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta \circ (\mathrm{id} \otimes \delta) \\
&\quad = \mathbb{1}\delta \otimes \delta.
\end{aligned}$$

The corresponding calculation for (*iii*) works similar, making use of $(\mathrm{id} \otimes \mu_r) \circ (\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) = (\mathrm{id} \otimes \mu_r) \circ \beta_{2,1} = \beta \circ (\mu_r \otimes \mathrm{id})$.

For (*v*) we calculate

$$\begin{aligned}
&\mu_{-t} \circ \Big(S_{-t} \otimes (S_{-t} \circ * \circ S_t \circ *)\Big) \circ \Delta \\
&\quad = \mu_t \circ (S_{-t} \otimes S_{-t}) \circ (* \otimes *) \circ (\mathrm{id} \otimes S_t) \circ (* \otimes *) \circ \Delta \\
&\quad = S_{-t} \circ \mu_t \circ \beta^{-1} \circ (* \otimes *) \circ (\mathrm{id} \otimes S_t) \circ (* \otimes *) \circ \Delta \\
&\quad = S_{-t} \circ \mu_t \circ (* \otimes *) \circ \tau \circ \beta \circ \tau \circ (\mathrm{id} \otimes S_t) \circ (* \otimes *) \circ \Delta \\
&\quad = S_{-t} \circ * \circ \mu_t \circ (\mathrm{id} \otimes S_t) \circ \beta \circ \tau \circ (* \otimes *) \circ \Delta \\
&\quad = S_{-t} \circ * \circ \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta \circ * \\
&\quad = S_{-t}(\mathbb{1})(* \circ \delta \circ *) = \mathbb{1}\delta,
\end{aligned}$$

which shows that $S_{-t} \circ * \circ S_t \circ *$ is inverse to S_{-t} with respect to the convolution $\star_{-t}$. Hence, $S_{-t} \circ * \circ S_t \circ * = \mathrm{id}$. □

3.4.4 Schoenberg Correspondence on Braided $*$-Bialgebras

A Schoenberg correspondence is a 1-1 correspondence between positive semigroups and conditionally positive generators. The original version is for matrices and Schürmann transferred this to a result for coalgebras.

3.4.13 Theorem (Special case of [Sch85, Theorem 4.2])**.** *Let K be a hermitian bilinear form on a $*$-coalgebra $\mathcal{C}$. Then the following two statements are equivalent:*

(i) $\mathrm{e}_\star^{tK}(c^* \otimes c) \geq 0$ *for all* $c \in \mathcal{C}, t \in \mathbb{R}_+$,

(ii) $K(c^* \otimes c) \geq 0$ *for all* $c \in \ker \delta$ *(K is* conditionally positive*).*

In this section we prove the following theorem, which generalizes the version of Wirth for additive deformations [Wir02, Theorem 2.1.11] as well as the version of Franz, Schott and Schürmann for braided $*$-bialgebras [FSS03, Theorem 2.1].

A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ defined on a unital $*$-algebra $\mathcal{A}$ is called a *state* if $\varphi(\mathbb{1}) = 1$ and $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. A linear functional $\psi : \mathcal{B} \to \mathbb{C}$ defined on a $*$-bialgebra $\mathcal{B}$ is called *L-conditionally positive* for a 2-cocycle L if $(\psi \circ \mu + L)(b^* \otimes b) \geq 0$ for all $b \in \ker \delta$.

3.4.14 Theorem (Schoenberg correspondence for braided additive deformations)**.** *Let $\mathcal{B}$ be a braided $*$-bialgebra with an additive deformation $(\mu_t)_{t\in\mathbb{R}}$ and let $\psi : \mathcal{B} \to \mathbb{C}$ be a hermitian, β-invariant linear functional with $\psi(\mathbb{1}) = 0$. Then the following two statements are equivalent:*

(i) $\varphi_t := \mathrm{e}_\star^{t\psi}$ *is a state on* $\mathcal{B}_t$ *for all* $t \geq 0$,

(ii) ψ *is L-conditionally positive.*

Proof of (i) $\Rightarrow$ (ii). The function $t \mapsto \varphi_t \circ \mu_t\,(c^* \otimes c)$ is positive for $t \geq 0$. For $c \in \ker \delta$ this function vanishes at 0, since

$$\varphi_0 \circ \mu_0\,(c^* \otimes c) = (\delta \otimes \delta)(c^* \otimes c) = |\delta(c)|^2 = 0.$$

So the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t \circ \mu_t(b^* \otimes b)\big)\Big|_{t=0} = (\psi \circ \mu + L)(b^* \otimes b)$ must be positive in this case. □

The aim of the remainder of this short section is to prove the converse implication. For a vector space V turn the set $\overline{V} := \{\overline{v} : v \in V\}$ into a vector space by defining $\overline{v} + \lambda\overline{w} := \overline{v + \overline{\lambda}w}$.

Now let $(\mathcal{C}, \Delta, \delta)$ be a β-braided $*$-coalgebra. Then $*$ can be interpreted as a linear map from $\mathcal{C}$ to $\overline{\mathcal{C}}$ and from $\overline{\mathcal{C}}$ to $\mathcal{C}$. Recall that $*_{\mathcal{B}\otimes\mathcal{B}} = (* \otimes *) \circ \tau \circ \beta^{-1}$. We define $\overline{a \otimes b} := \overline{a} \otimes \overline{b}$ and set

$$\begin{aligned} \overline{\Delta} &:= (* \otimes *) \circ \tau \circ \beta^{-1} \circ \Delta \circ *, && \text{that is} && \overline{\Delta}(\overline{c}) = \overline{\Delta(c)}, \\ \overline{\delta} &:= \delta \circ *, && \text{that is} && \overline{\delta}(\overline{c}) = \overline{\delta(c)}, \\ \overline{\beta} &:= (* \otimes *) \circ \tau \circ \beta^{-1} \circ (* \otimes *) \circ \tau, && \text{that is} && \overline{\beta}(\overline{a} \otimes \overline{b}) = \overline{\beta(a \otimes b)}. \end{aligned}$$

Then $\left(\overline{\mathcal{C}}, \overline{\Delta}, \delta\right)$ is a $\overline{\beta}$-braided $*$-coalgebra and $\left(\overline{\mathcal{C}} \otimes \mathcal{C}, (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\overline{\Delta} \otimes \Delta), \overline{\delta} \otimes \delta\right)$ is a usual $*$-coalgebra, that is a $\tau_{2,2}$-braided $*$-coalgebra.

We call a linear map $\mathcal{C} \otimes \mathcal{C} \to \mathbb{C}$ *bilinear form* on $\mathcal{C}$ and a linear map $\overline{\mathcal{C}} \otimes \mathcal{C} \to \mathbb{C}$ *sesquilinear form* on $\mathcal{C}$. For a bilinear form K we define the corresponding sesquilinear form $\widetilde{K} := K \circ (* \otimes \mathrm{id})$. This is a bijection of bilinear forms and sesquilinear forms on $\mathcal{C}$.

3.4.15 Lemma. *Let $\star$ be the convolution of bilinear forms with respect to the comultiplication $\Lambda = (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$ on $\mathcal{C} \otimes \mathcal{C}$ and let $\circledast$ be the convolution of sesquilinear forms with respect to the comultiplication $(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\overline{\Delta} \otimes \Delta)$ on $\overline{\mathcal{C}} \otimes \mathcal{C}$. For two bilinear forms M and K on the β-braided $*$-coalgebra $\mathcal{C}$ the following is fulfilled. If M is β-invariant, we have*

$$\widetilde{M \star K} = \widetilde{M} \circledast \widetilde{K}.$$

Proof.

$$\begin{aligned} \widetilde{M \star K} &= (M \star K) \circ (* \otimes \mathrm{id}) \\ &= (M \otimes K) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta) \circ (* \otimes \mathrm{id}) \\ &= (M \otimes K) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\beta \otimes \mathrm{id} \otimes \mathrm{id}) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\overline{\Delta} \otimes \Delta) \\ &= K \circ (\mathrm{id} \otimes M \otimes \mathrm{id}) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\overline{\Delta} \otimes \Delta) \\ &= (M \otimes K) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (* \otimes * \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\overline{\Delta} \otimes \Delta) \end{aligned}$$

$$= (M \otimes K) \circ (* \otimes \mathrm{id} \otimes * \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\overline{\Delta} \otimes \Delta)$$
$$= \widetilde{M} \circledast \widetilde{K}.$$

□

With this lemma we get for a β-invariant bilinear form K on $\mathcal{C}$

$$\mathrm{e}_\star^{tK}(c^* \otimes c) = \mathrm{e}_\circledast^{t\widetilde{K}}(\overline{c} \otimes c)$$

so the following is now a direct consequence of Theorem 3.4.13.

3.4.16 Lemma. *Let K be a β-invariant, hermitian bilinear form on a the β braided $*$-coalgebra $\mathcal{C}$. Then the following two statements are equivalent:*

(i) $\mathrm{e}_\star^{tK}(c^* \otimes c) \geq 0$ *for all* $c \in \mathcal{C}, t \in \mathbb{R}_+$,

(ii) $K(c^* \otimes c) \geq 0$ *for all* $c \in \ker \delta$.

With this we are able to prove the Schoenberg correspondence.

Proof of Theorem 3.4.14, (ii) ⇒ (i). Let L be the generator of the additive deformation $(\mu_t)_{t\in\mathbb{R}}$ and define $K := \psi \circ \mu + L$, which is a hermitian, conditionally positive bilinear form on the β-braided $*$-bialgebra $\mathcal{B}$. With the previous lemma we conclude

$$0 \leq \mathrm{e}_\star^{tK}(c^* \otimes c) = \mathrm{e}_\star^{t\psi\circ\mu+tL}(c^* \otimes c) = \mathrm{e}_\star^{t\psi\circ\mu} \star \mathrm{e}_\star^{tL}(c^* \otimes c) = \mathrm{e}_\star^{t\psi} \circ (\mu \star \mathrm{e}_\star^{tL})(c^* \otimes c)$$
$$= \varphi_t \circ \mu_t(c^* \otimes c),$$

since $(\psi\circ\mu)\star L = \psi\circ(L\star\mu) = \psi\circ(\mu\star L) = L\star(\psi\circ\mu)$ and μ is a coalgebra homomorphism.

From $\psi(\mathbb{1}) = 0$ it follows directly that $\mathrm{e}_\star^{t\psi}(\mathbb{1}) = \mathrm{e}^{t\psi(\mathbb{1})} = \mathrm{e}^0 = 1$, since $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$. □

3.4.5 The Fermi Harmonic Oscillator

We will need the corresponding result to 3.3.14, which determines additive deformations on primiteve elements.

3.4.17 Proposition. *Let $\mathcal{B}$ be a β-braided bialgebra with additive deformation $\mu_t = \mu \star e_\star^{tL}$ and $a, b \in \mathcal{B}$. If a and b are primitive, we have*

$$\mu_t(a \otimes b) = ab + tL(a \otimes b)\mathbb{1}.$$

Proof. The same as the proof of 3.3.14 for the unbraided case with τ replaced by β. This works, because $\mathbb{1}$ is β-compatible. □

Consider the polynomial algebra $\widetilde{\mathcal{B}} := \mathbb{C}\langle x, x^* \rangle$ in two non-commuting adjoint indeterminates. For a monomial M we define the grade $g(M)$ as the degree of the monomial M. Then

$$\beta(M \otimes N) := (-1)^{g(M)g(N)} N \otimes M$$

for monomials M, N defines a braiding on $\widetilde{\mathcal{B}}$, which is a symmetry, that is $\beta^2 = \mathrm{id} \otimes \mathrm{id}$. So β-invariance of a map is equivalent to β^{-1}-invariance. It is easily checked that the multiplication is β-invariant. $\widetilde{\mathcal{B}}$ is turned into a β-braided Hopf $*$-bialgebra by defining comultiplication, counit and antipode on the generators as

$$\Delta\big(x^{(*)}\big) = x^{(*)} \otimes \mathbb{1} + \mathbb{1} \otimes x^{(*)}, \quad \delta\big(x^{(*)}\big) = 0, \quad S(x^{(*)}) = -x^{(*)}$$

and extending them as algebra homomorphisms, respectively anti-homomorphism in the case of S. The ideal I generated by elements of the form $xx^* + x^*x$ is a coideal. One has to show $\delta(I) = 0$, which is obvious, and $\Delta(I) \subset I \otimes \widetilde{\mathcal{B}} + \widetilde{\mathcal{B}} \otimes I$. Therefore we calculate

$$\begin{aligned} \Delta(xx^*) = \Delta(x)\Delta(x^*) &= xx^* \otimes \mathbb{1} + x \otimes x^* + \beta(x \otimes x^*) + \mathbb{1} \otimes xx^* \\ &= xx^* \otimes \mathbb{1} + x \otimes x^* - x^* \otimes x + \mathbb{1} \otimes xx^* \end{aligned}$$

and analogously $\Delta(x^*x) = x^*x \otimes \mathbb{1} + x^* \otimes x - x \otimes x^* + \mathbb{1} \otimes x^*x$. Combining these two equations, we get

$$\Delta(xx^* + x^*x) = (xx^* + x^*x) \otimes \mathbb{1} + \mathbb{1} \otimes (xx^* + x^*x) \in I \otimes \widetilde{\mathcal{B}} + \widetilde{\mathcal{B}} \otimes I.$$

Furthermore, we have $\beta(I\otimes\widetilde{\mathcal{B}}+\widetilde{\mathcal{B}}\otimes I)\subset I\otimes\widetilde{\mathcal{B}}+\widetilde{\mathcal{B}}\otimes I$, so $\mathcal{B}:=\widetilde{\mathcal{B}}/I$ is also a braided Hopf $*$-algebra. A hermitian 2-cocycle on $\mathcal{B}$ is given by $L(x^*\otimes x)=1$ and $L(M\otimes N)=0$ for all other monomials. We want to show that it is commuting and β-compatible. We use the following general proposition.

3.4.18 Proposition. *Let $\mathcal{B}$ be a braided bialgebra and β be a symmetry, that is $\beta\circ\beta=\mathrm{id}\otimes\mathrm{id}$. Then $\beta\circ\Delta(a)=\Delta(a)$ and $\beta\circ\Delta(b)=\Delta(b)$ implies that $\beta\circ\Delta(ab)=\Delta(ab)$. In particular, $\mathcal{B}$ is cocommutative if $\mathcal{B}$ is generated by primitive elements.*

Proof. Assume $\beta\circ\Delta(a)=\Delta(a)$ and $\beta\circ\Delta(b)=\Delta(b)$. Then we calculate

$$\begin{aligned}
&\beta\circ\Delta\circ\mu(a\otimes b)\\
&\quad=\beta\circ(\mu\otimes\mu)\circ(\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\Delta\otimes\Delta)(a\otimes b)\\
&\quad=(\mu\otimes\mu)\circ\underbrace{(\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\beta\otimes\beta)\circ(\mathrm{id}\otimes\beta\otimes\mathrm{id})}_{\beta_{2,2}}\circ(\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\Delta\otimes\Delta)(a\otimes b)\\
&\quad=(\mu\otimes\mu)\circ(\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\beta\otimes\beta)(\Delta(a)\otimes\Delta(b))\\
&\quad=(\mu\otimes\mu)\circ(\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\Delta\otimes\Delta)(a\otimes b)\\
&\quad=\Delta\circ\mu(a\otimes b).
\end{aligned}$$

The second statement is a direct consequence of the first, since for a primitive element $a\in\mathcal{B}$

$$\beta\circ\Delta(a)=\beta(a\otimes\mathbb{1}+\mathbb{1}\otimes a)=\mathbb{1}\otimes a+a\otimes\mathbb{1}=\Delta(a)$$

and finite products of generators span $\mathcal{B}$. $\square$

In our example β is a symmetry and $\mathcal{B}$ is generated by primitive elements, so $\mathcal{B}$ is cocommutative. Then also $\mathcal{B}\otimes\mathcal{B}$ is cocommutative, as

$$\beta_{2,2}\circ\Lambda=(\mathrm{id}\otimes\beta\otimes\mathrm{id})\circ(\beta\otimes\beta)\circ(\mathrm{id}\otimes\beta^2\otimes\mathrm{id})\circ(\Delta\otimes\Delta)=\Lambda.$$

Hence, $L \star \mu = \mu \star L$ is fulfilled. To see that L is β-invariant, we only need to calculate

$$\begin{aligned}(L \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \circ (\beta \otimes \mathrm{id})(M \otimes x^* \otimes x) &= (-1)^{2g(M)} M \\ &= M = (\mathrm{id} \otimes L)(M \otimes x^* \otimes x)\end{aligned}$$

for all monomials M, as L vanishes for other terms. L is obviously hermitian. We have now completed showing that L is the generator of an additive $*$-deformation.

We calculate $\mu_t = (\mu \otimes e_\star^{tL}) \circ \Lambda$. First, we know from Proposition 3.4.17 that

$$\mu_t(x^* \otimes x) = \mu(x^* \otimes x) + tL(x^* \otimes x)\mathbb{1} = -xx^* + t\mathbb{1} = -\mu_t(x \otimes x^*) + t\mathbb{1}, \qquad (3.4.6)$$

since x and x^* are primitive. Next, notice that $\mu_t(M \otimes N)$ and $\mu(M \otimes N) = MN$ can only differ, when M contains a factor x^* and N contains a factor x. With these two facts we know μ_t, because of associativity. Let $M = x^{m_1}(x^*)^{m_2}$ and $N = x^{n_1}(x^*)^{n_2}$ with $m_1, m_2, n_1, n_2 \in \mathbb{N}$ be two monomials. Write $m = m_1 + m_2$ and $n = n_1 + n_2$. Then

$$\begin{aligned}\mu_t(M \otimes N) &= \mu_t \circ \left(\mu^{(m)} \otimes \mu^{(n)}\right) \left((x^{\otimes m_1} \otimes (x^*)^{\otimes m_2}) \otimes (x^{\otimes n_1} \otimes (x^*)^{\otimes n_2})\right) \\ &= \mu_t \circ \left(\mu_t^{(m)} \otimes \mu_t^{(n)}\right) \left((x^{\otimes m_1} \otimes (x^*)^{\otimes m_2}) \otimes (x^{\otimes n_1} \otimes (x^*)^{\otimes n_2})\right) \\ &= \mu_t^{(m+n)} \left(x^{\otimes m_1} \otimes (x^*)^{\otimes m_2} \otimes x^{\otimes n_1} \otimes (x^*)^{\otimes n_2}\right).\end{aligned}$$

Now one can use (3.4.6) to calculate this. The $*$-algebra $\mathcal{B}_t = (\mathcal{B}, \mu_t)$ is isomorphic to the $*$-algebra $\mathcal{A}_t$ generated by a, a^* and $\mathbb{1}$ with the relation $aa^* + a^*a = t\mathbb{1}$. The map $a \mapsto x$, $a^* \mapsto x^*$ can be extended as an algebra homomorphism $\widetilde{\Phi}_t : \mathbb{C}\langle a, a^* \rangle \to \mathcal{B}_t$. Since the relation is respected, that is

$$\widetilde{\Phi}_t(aa^* + a^*a) = \mu_t(x \otimes x^* + x^* \otimes x) = t = \widetilde{\Phi}_t(t\mathbb{1}),$$

we get an algebra homomorphism $\Phi_t : \mathcal{A}_t \to \mathcal{B}_t$. It is clear from our considerations on μ_t that this is an isomorphism as it maps the vector space basis $\left\{a^k(a^*)^l \mid k, l \in \mathbb{N}\right\}$ of $\mathcal{A}_t$ to the vector space basis $\left\{x^k(x^*)^l \mid k, l \in \mathbb{N}\right\}$ of $\mathcal{B}_t$.

Since 0 is a hermitian, L-conditionally positive linear functional vanishing at $\mathbb{1}$, the exponential $e_\star^{t0} = \delta$ is a state on every $\mathcal{B}_t$. Note that this is less trivial than it seems at a first glance because for example $\delta\big(\mu_t(x^* \otimes x)\big) = \delta(-xx^* + t\mathbb{1}) = t$.

For every $q \neq 0$ there is a unique braiding β_q on the algebra $\mathbb{C}\langle x, x^*\rangle$ of two non-commuting, adjoint indeterminates such that

- $\mathbb{C}\langle x, x^*\rangle$ is a β_q-braided $*$-algebra
- β_q is defined on the generators in the following way:

$$\begin{aligned} \beta_q(x \otimes x) &= q\ x \otimes x & \qquad \beta_q(x \otimes x^*) &= q\ x^* \otimes x \\ \beta_q(x^* \otimes x) &= q^{-1}\ x \otimes x^* & \qquad \beta_q(x^* \otimes x^*) &= q^{-1}\ x^* \otimes x^* \end{aligned}$$

These equations determine β_q on all pairs of monomials due to the compatibility of the unit and the multiplication. There exists a compatible Hopf $*$-algebra structure such that $\Delta(x^{(*)}) = x^{(*)} \otimes \mathbb{1} + \mathbb{1} \otimes x^{(*)}$ and the ideal I_q generated by elements of the form $xx^* - qx^*x$ is a coideal with $\beta_q(I \otimes \widetilde{\mathcal{B}} + \widetilde{\mathcal{B}} \otimes I) \subset I \otimes \widetilde{\mathcal{B}} + \widetilde{\mathcal{B}} \otimes I$. Dividing by this biideal yields a β_q-braided Hopf $*$-algebra $\mathcal{B}_q$ with two q-commuting, primitive, adjoint generators. Note that for $q = -1$ the previous example is obtained. But for $q \neq \pm 1$ a multiplication μ_t on $\mathcal{B}_q$ such that

$$\mu_t(x \otimes x^* - qx^* \otimes x) = t\mathbb{1}$$

cannot be β_q-compatible, as it would follow that

$$(\mu_t \otimes \mathrm{id}) \circ \underbrace{(\mathrm{id} \otimes \beta_q) \circ (\beta_q \otimes \mathrm{id})}_{(\beta_q)_{1,2}} \big(x \otimes (x \otimes x^* - q\ x^* \otimes x)\big) = q^2\ t\mathbb{1} \otimes x,$$

but

$$\beta_q \circ (\mathrm{id} \otimes \mu_t)\big(x \otimes (x \otimes x^* - q\ x^* \otimes x)\big) = t\mathbb{1} \otimes x.$$

So this can only work for the considered cases $q = \pm 1$. Still, our version of the Schoenberg correspondence applies to the braided Hopf $*$-algebras $(\widetilde{\mathcal{B}}, \beta_q)$ and $(\mathcal{B}_q, \beta_q)$.

4 Dimension of Subproduct Systems

Most of the material in this chapter is taken from [GS14b] which is joint work with Michael Skeide. The results in Section 4.2.2 for rational-time systems and Section 4.2.3 for continuous-time systems are not yet published.

We can think of a subproduct system as a family of Hilbert spaces $(H_t)_{t \in \mathbb{S}}$ indexed by some (abelian) monoid $\mathbb{S}$ with $H_{s+t} \subset H_s \otimes H_t$ for all $s, t \in \mathbb{S}$. The precise definition coincides with that of a comonoidal system over $\mathbb{S}$ in the tensor category of Hilbert spaces with isometries as morphisms; see Section 2.3.1, Hilbert Spaces, or Definition 4.2.1. A subproduct system is a product system if H_{s+t} fills the whole of $H_s \otimes H_t$ or more precisely if it is full as a comonoidal system. Shalit and Solel study subproduct systems over $\mathbb{N}_0^n$ in order to deal with the dilation problem for commuting CP_0-semigroups. Bhat and Mukherjee study subproduct systems over $\mathbb{R}_+$ (under the name of inclusion systems) as a tool for the construction and analysis of amalgamated products of product systems.

Since the classification of product and subproduct systems is extremely difficult, we adress a somewhat simpler question. Which possibilities are there for the *dimension function* $t \mapsto \dim H_t$ of a product or subproduct system $(H_t)_{t \in \mathbb{S}}$. We will treat the monoids $\mathbb{S} = \mathbb{N}_0$ (discrete), $\mathbb{S} = \mathbb{Q}_+$ (rational time), and $\mathbb{S} = \mathbb{R}_+$ (continuous time). An obvious necessary condition is *submultiplicativity*, that is $\dim H_{s+t} \leq \dim H_s \dim H_t$ for all $s, t \in \mathbb{S}$. Shalit and Solel explicitly raised the question whether for every submultiplicative sequence there exists a discrete subproduct system with dimensions given by the sequence in [SS09].

Before we come to subproduct systems themselves, we treat the somewhat simpler case of *Cartesian systems*, which are comonoidal systems in the tensor category $(\mathbf{Set}^{inj}, \times)$ of sets with injections as morphisms and the Cartesian product as tensor product. We exhibit a special kind of discrete Cartesian systems called *word systems*, and show that each Cartesian system is isomorphic to a word system. For word systems the question of all possible cardinality sequences has quite some history, which we lay out in 4.1.2. In 4.1.3 we prove some results which might be new, or at least we could not find them in the literature. The problem is that the literature is vast and the terminology far from unique. It seems many results in this area have been reproved several times. Still we expect that especially the results which explicitly use the notion of Cartesian systems are either new or get a much simpler proof by using our approach. The detailed study of Cartesian systems proves useful, as we show in Section 4.2.1 that the dimension sequences of discrete subproduct systems coincide with the cardinality sequences of Cartesian systems (and give some other characterizations). The difficult step of the proof turned out to be a classical result on graded algebras, see Remark 4.2.6. A simple conclusion is that not every submultiplicative sequence appears as dimension sequence of a subproduct system, but of course the result tells us much more than this. We apply the results for discrete systems in the study of rational-time and continuous-time systems. For rational-time subproduct (and Cartesian) systems we present a simple characterization of the dimension functions via the inequality (4.2.1), see Corollary 4.2.9. The same inequality yields a characterization of the cardinality functions of continuous-time Cartesian systems (Theorem 4.2.18), but we do not know this for continuous-time subproduct systems yet. At least, we get a necessary condition for the dimension function if the subproduct system fulfills a mild continuity condition, see Theorem 4.2.19.

4.1 Cartesian Systems

We already gave a definition of Cartesian systems as comonoidal systems in $(\mathbf{Set}^{inj}, \times)$, see Section 2.3.1. But for convenience of the reader who is unfamiliar with category theory, as well as to fix our standard notation for Cartesian systems in this chapter, we repeat the definition in a more direct language.

4.1.1 Definition. A *Cartesian system* (over $\mathbb{S}$) is a family $X^{>} = (X_t)_{t\in\mathbb{S}}$ of sets X_t with $X_0 = \{\Lambda\}$ a one point set and with injections

$$i_{s,t} \colon X_{s+t} \longrightarrow X_s \times X_t$$

such that the diagrams

$$
\begin{array}{ccc}
X_{r+s+t} & \xrightarrow{i_{r+s,t}} & X_{r+s} \times X_t \\
\downarrow{\scriptstyle i_{r,s+t}} & & \downarrow{\scriptstyle i_{r,s}\times \mathrm{id}_t} \\
X_r \times X_{s+t} & \xrightarrow{\mathrm{id}_r \times i_{s,t}} & X_r \times X_s \times X_t
\end{array}
\qquad
\begin{array}{ccccc}
 & & X_s & & \\
 & \swarrow{\scriptstyle i_{0,s}} & \downarrow{\scriptstyle \mathrm{id}_s} & \searrow{\scriptstyle i_{s,0}} & \\
X_0 \times X_s & \xrightarrow{\cong} & X_s & \xleftarrow{\cong} & X_s \times X_0
\end{array}
$$

commute. A family $Y^{>} = (Y_t)_{t\in\mathbb{S}}$ of subsets $Y_t \subset X_t$ is a *Cartesian subsystem* of $X^{>}$ if $i_{s,t}(Y_{s+t}) \subset Y_s \times Y_t$, where, as customary, we identify $Y_s \times Y_t \subset X_s \times X_t$.

The aim of this section is to characterize sequences $(d_n)_{n\in\mathbb{N}_0}$ for which there exists a Cartesian system $X^{>}$ with $\#X_n = d_n$ for all $n \in \mathbb{N}_0$. Therefore we will present a certain standard form of a Cartesian system named word systems. Those are Cartesian systems for which there is a set A such that $X_n \subset A^n$ and the injection $i_{m,n}$ is simply the restriction of the canonical identification $A^{m+n} \cong A^m \times A^n$ to X_{m+n}. In this case the set $L := \bigcup_{n\in\mathbb{N}_0} X_n$ is a *factorial language*, that is a subset of A^* which is closed under taking *subwords* or *factors*. Of course, one can recover the word system from the language L as $X_n = L \cap A^n$, so factorial languages and word systems are 1-1. The sequence $(\#X_n)_{n\in\mathbb{N}_0}$ is known as *complexity* of the factorial language. We show that every Cartesian system is isomorphic to a word system. Hence all results about the complexities of factorial languages translate to cardinality sequences of Cartesian systems. We list some known results and prove some new results. The most relevant for the other parts of this chapter is Theorem 4.1.9, [BB05, Theorem 8] of Balogh and Bollobás.

4.1.1 Discrete Cartesian Systems, Word Systems and Factorial Languages

Let us fix a set A, called *alphabet*. We refer to the elements of A as *letters*, the tuples in $A^* = \bigcup_{n\in\mathbb{N}_0} A^n$ are called *words* and the subsets of A^* are called *languages*. The concatenation product of words turns A^* into a monoid with the *empty word* $\Lambda = ()$ as neutral element. In this section, we simply write

$$(a_1, \dots, a_n)(b_1, \dots, b_m) := (a_1, \dots, a_n, b_1, \dots, b_m),$$

for the concatenation, omitting the symbol $\smile$ which was used in Chapter 2. The *length* of a word is $|(a_1, \ldots, a_n)| := n$. The *complexity* of a language $L \subset A^*$ is the sequence $(\#(L \cap A^n))_{n\in\mathbb{N}_0}$.

4.1.2 Proposition. *Let A be an alphabet. Then the following statements hold:*

(*i*) *Every word w in $A^{\ell_1+\cdots+\ell_k}$ factors uniquely as $w = w_1 \cdots w_k$ with $w_i \in A^{\ell_i}$.*

(*ii*) *Suppose $X_i \subset A^{\ell_i}$. Then $w \in X_1 \cdots X_k \subset A^{\ell_1+\cdots+\ell_k}$ if and only if $w_i \in X_i$ for all $i = 1, \ldots, k$.*

Proof. This proposition is a simple consequence of the fact that $A^{\ell_1} \times \cdots \times A^{\ell_k}$ may be identified with A^n by sending $(w_1, \ldots, w_k)$ to $w_1 \cdots w_k$, and of the fact that an element s in a product $S_1 \times \cdots \times S_k$ is a unique tuple $(s_1, \ldots, s_k)$. □

4.1.3 Example. Choose an alphabet A. By Proposition 4.1.2(i), the restriction of the concatenation product to $A^m \times A^n$ is an invertible map onto A^{m+n}. Define $i_{m,n}$ to be the inverse of this map. Then $A^\times := (A^n)_{n\in\mathbb{N}_0}$ with the maps $i_{m,n}$ is a Cartesian system, the *full word system over A*.

4.1.4 Definition. Let A be an alphabet. A *word system over A* is a Cartesian subsystem of the full word system over A.

4.1.5 Theorem. *Every discrete Cartesian system $X^> = (X_n)_{n\in\mathbb{N}_0}$ is isomorphic to a word system over X_1.*

Proof. The fastest way is to use Theorem 2.3.15. The full Cartesian system generated by $X^>$ is clearly $(X_1^n)_{n\in\mathbb{N}_0}$. The images of the canonical maps $D_n : X_n \to X_1^n$ form the desired word system. □

We say a word y is a *subword* of w if there are words $x, z \in A^*$ with $w = xyz$. One may check that the relation defined by y being a subword of x, is a partial order. Note that some authors call this a *factor*, while using the term "subword" for a more general concept.

4.1.6 Definition. A language $L \subset A^*$ is called *factorial* if it is closed with respect to taking subwords, that is if

$$xyz \in L \Rightarrow y \in L$$

for all $x, y, z \in A^*$.

4.1.7 Proposition. *Put $X_n := L \cap A^n$. L is factorial if $X_{n+1} \subset AX_n \cap X_n A$ for all $n \in \mathbb{N}_0$.*

Proof. By repeated application of the inclusion, we obtain $X_{m+n+k} \subset A^m X_n A^k$. So, by Proposition 4.1.2(ii), if $xyz \in X_{m+n+k} \subset A^m X_n A^k$ with $x \in A^m, y \in A^n, z \in A^k$, then $y \in X_n$. □

Obviously, the converse is true, too.

4.1.8 Theorem. *Let A be an alphabet. For a family $(X_n)_{n\in\mathbb{N}_0}$ of subsets $X_n \subset A^n$ the following conditions are equivalent.*

(i) *$(X_n)_{n\in\mathbb{N}_0}$ is a word system over A.*

(ii) *$L := \bigcup_{n\in\mathbb{N}_0} X_n$ is a factorial language over A.*

Proof. If $w \in X^>$ and $w = xy$, then x and y are subwords. So, (ii)⇒(i) is immediate.

Conversely, suppose $X_{m+n} \subset X_m X_n$ for all $m, n \in \mathbb{N}_0$. This means, in particular, that $X_{n+1} \subset X_1 X_n \cap X_n X_1 \subset AX_n \cap X_n A$. By Proposition 4.1.7, L is factorial. This shows (i)⇒(ii). □

Frequently, we will identify a word system $X^>$ with the corresponding factorial language $L = \bigcup_{n\in\mathbb{N}_0} X_n$. In particular, we will write $A^* \setminus X^>$ instead of $A^* \setminus L$.

4.1.2 Complexity of Factorial Languages: Known Results

We started with the question, what are the possible cardinality sequences of discrete Cartesian systems. As we have seen, all Cartesian systems are isomorphic to word systems and word systems are in 1-1 correspondence with factorial languages. Complexity of factorial languages is a well studied area, correspondingly there is a long list of known results. Additionally, there are different names and equivalent descriptions, still multiplying the number of applicable results. Unfortunately, the publications dealing with this structure frequently seem not to interact. (We hope it may be forgiven that we add a further name. The term "word system" is inspired from the analogy with subproduct systems.) Thus, it is quite difficult to get a an idea about the real status of the theory. In this section we intend to give an overview over important known results. It should be clear that this cannot be exhaustive. But we hope we can at least provide a small guide pointing into interesting directions. We cite sources where the interested reader can find more information.

It seems that it is easier to get estimates for the complexity when restricting to the subclass of factorial languages L^w consisting of all (finite) subwords of a single (usually) infinite word w. (These languages are particularly relevant for comupter science.) The complexity of L^w is usually referred to as *subword complexity* of w. As early as 1938, Morse and Hedlund [MH38] provided a necessary condition for that a sequence occurs as subword complexity: Let $X_n^w := L^w \cap A^n$. Then, either $\#X_{n+1}^w > \#X_n^w$ for all $n \in \mathbb{N}$ or $\#X_n^w$ is eventually constant. Ferenczi [Fer99, Section 3] provides several results on subword complexities. For instance, if for some $\alpha > 0$ it holds that $\#X_n^w \leq \alpha n$ for all $n \in \mathbb{N}$, then there exists a $C > 0$ such that $\#X_{n+1}^w - \#X_n^w \leq C\alpha^3$ for all n.

The following result of Balogh and Bollobás on general factorial languages, which is related to Morse and Hedlund's theorem, will prove very useful later on. We prefer to formulate the results in this section for the corresponding word systems. For $a \in \mathbb{R}$, we write $\lceil a \rceil := \min\{n \in \mathbb{Z} \mid n \geq a\}$ and $\lfloor a \rfloor := \max\{n \in \mathbb{Z} \mid n \leq a\}$.

4.1.9 Theorem ([BB05, Theorem 8])**.** *Let $X^>$ be a word system. Suppose $d := \#X_k \leq k$ for some $k \in \mathbb{N}$. Then $(\#X_n)_{n\in\mathbb{N}}$ is bounded. Furthermore,*

$$\#X_m \leq \left\lceil \frac{d+1}{2} \right\rceil \left\lfloor \frac{d+1}{2} \right\rfloor \tag{4.1.1}$$

for all $m \geq k + d$.

The bound in the above theorem is best possible, even if one restricts to word systems over a two-letter alphabet, see [BB05, Theorem 7]. We present Cartesian systems for which equality holds in (4.1.1). These Cartesian systems have the advantage that the construction can be generalized to rational-time and continuous-time Cartesian systems without any difficulties.

4.1.10 Lemma. *Let* $X := \mathbb{N} \times \mathbb{N}$ *and* $\iota : X \to X \times X$ *defined by* $\iota(i,j) := \big((i,1),(1,j)\big)$. *Then* ι *is injective and satisfies*

$$(\iota \times \mathrm{id}) \circ \iota = (\mathrm{id} \times \iota) \circ \iota. \tag{4.1.2}$$

Proof. Injectivity is obvious. The direct calculation

$$(\iota \times \mathrm{id})\big((i,1)(1,j)\big) = \big((i,1),(1,1),(1,j)\big) = (\mathrm{id} \times \iota)\big((i,1)(1,j)\big)$$

proves (4.1.2). □

As an obivious conclusion we have:

4.1.11 Corollary. *Let* $X_n := X$ *and* $i_{m,n} := \iota$ *for all* $m,n \neq 0$. *Furthermore put* $X_0 := \{\Lambda\}$ *and* $i_{m,n}$ *canonical for* $m = 0$ *or* $n = 0$. *Then this defines a discrete Cartesian system* $\mathcal{X}^>$.

4.1.12 Example. For a natural number $k \in \mathbb{N}$, we denote by $[k]$ the set $\{1,\dots,k\}$. Let $m,n \in \mathbb{N}$ and $I \subset \mathbb{N}$. Put

$$X_k := \begin{cases} \big([m] \times \{1\}\big) \cup \big(\{1\} \times [n]\big) & \text{for } k \in I \\ [m] \times [n] & \text{for } k \in \mathbb{N} \setminus I \\ \{\Lambda\} & \text{for } k = 0. \end{cases}$$

Then for every $(i,j) \in [m]\times[n]$ one has $\iota(i,j) = \big((i,1),(1,j)\big) \in \big([m]\times\{1\}\big)\times\big(\{1\}\times[n]\big)$, hence the X_k form a Cartesian subsystem $X^> \subset \mathcal{X}^>$. The dimension sequence is given by

$$\#X_k = \begin{cases} m+n-1 & \text{for } k \in I \\ mn & \text{for } k \in \mathbb{N} \setminus I \\ 1 & \text{for } k = 0. \end{cases}$$

If we let $d \in \mathbb{N}_0$ and put $m := \left\lceil \frac{d+1}{2} \right\rceil, n := \left\lfloor \frac{d+1}{2} \right\rfloor$, it follows that $m + n = d + 1$ and

$$\#X_k = \begin{cases} d & \text{for } k \in I \\ \left\lceil \frac{d+1}{2} \right\rceil \left\lfloor \frac{d+1}{2} \right\rfloor & \text{for } k \in \mathbb{N} \setminus I \\ 1 & \text{for } k = 0. \end{cases}$$

This shows that (4.1.1) in Theorem 4.1.9 gives indeed the best possible bound.

Reduced sets of excluded words. A word system can be described by indicating which words do not occur as subwords. The following results are well known (see for example [CMR98]), but we prefer to give independent proofs, first, to illustrate how arguments work and, second, to be self-contained in the following section. They also promise to be relevant in analyzing the structure of associated graph C^*–algebras.

4.1.13 Observation. Let $E \subset A^*$ be any set of words. Then the sets

$$X_n(E) := \{w \in A^n \mid w \text{ has no subword from } E\}$$

form a word system. Indeed, if w does not contain a subword from E and y is a subword of w, then, by transitivity, also y cannot contain a subword from E.

Reflexivity means that every word is a subword of itself. From this it immediately follows that E and $X^>(E)$ are disjoint.

4.1.14 Observation. Every word system $X^>$ can be obtained as $X^> = X^>(E)$. Indeed, take $E := A^* \setminus X^>$, the set of all words in A^* that do not belong to $X^>$. A word belongs to the word system $X^>$ if and only if all its subwords belong to $X^>$. Equivalently, $x \in X^>$ if and only if none of its subwords is in $A^* \setminus X^>$, that is, $X^> = X^>(A^* \setminus X^>)$.

$E = A^* \setminus X^>$ is, clearly, the maximal choice. We now show that there is a unique minimal choice.

4.1.15 Definition. A subset $E \subset A^* \setminus \{\Lambda\}$ is called *reduced* (or *antifactorial*) if no word of E is a proper subword of another word of E.

Note that $E = A^* \setminus X^>$ is reduced if and only if it is empty, that is, if $X^>$ is the full word system over A. (Indeed, suppose that E is reduced. If $A = \emptyset$, so that $A^* = \{\Lambda\}$, then a reduced subset E of A, by definition, is empty. And if A is nonempty, then every word x is a proper subword of another word y. If $y \in X^>$, then $x \in X^>$, because $X^>$ is a word system. If $y \notin X^>$, that is, if $y \in E$, then $x \notin E$, that is, $x \in X^>$, because E is reduced. So, every word x is in $X^>$, that is, $X^> = A^*$. The other direction is obvious.)

4.1.16 Proposition. *Let E be reduced and $X^>(E) = X^>(E')$. Then $E \subset E'$.*

Proof. We conclude indirectly. Suppose $w \in E \setminus E'$. Since $w \in E$, w does not belong to $X^>(E) = X^>(E')$. Therefore, w contains a subword $y \in E'$. Since $w \notin E'$, y is a proper subword of w. Since E is reduced, y and all subwords of y are not in E. Therefore, $y \in X^>(E)$. But, $y \in E'$, so $y \notin X^>(E') = X^>(E)$. Contradiction! □

This proposition shows that if there is a reduced set R such that $X^>(R) = X^>$, then $R = \bigcap_{X^>(E)=X^>} E$. In particular, R is unique. The following theorem settles existence by giving an explicit formula for R. The unique reduced set R generating $X^>$ as $X^>(R)$ is also called the *antidictionary* of $X^>$.

4.1.17 Theorem. *For every word system $X^>$ over A,*

$$R := \bigcup_{n \geq 1} R_n, \quad R_n := (X_{n-1}A \cap AX_{n-1}) \setminus X_n$$

is the unique reduced set of words such that $X^> = X^>(R)$.

Proof. A word $w = (a_1, \ldots, a_n)$ is in $X_{n-1}A \cap AX_{n-1}$ if and only if the two subwords $w_{\widehat{n}} = (a_1, \ldots, a_{n-1})$ and $w_{\widehat{1}} = (a_2, \ldots, a_n)$ are in X_{n-1}. Now, each proper subword y of w is a subword of $w_{\widehat{n}}$ or a subword of $w_{\widehat{1}}$. Since $X^>$ is a word system, $y \in X^>$. In other words, $w = (a_1, \ldots, a_n)$ is in $X_{n-1}A \cap AX_{n-1}$ if and only if each of its proper subwords is in $X^>$.

In order to illustrate some different techniques, we continue in two versions.

Version 1: Since all proper subwords of $w \in R_n$ are in $X^>$, these subwords are not in R. Therefore, R is reduced.

To show $X^> \subset X^>(R)$ take any $w \notin X^>(R)$. Then w has a subword $r \in R$. Since R and $X^>$ are disjoint, r is not in $X^>$. Hence, the word w containing r is not in the word system $X^>$.

For the other inclusion, we show $X_n(R) \subset X_n$ by induction on n. Since a reduced set may not contain the empty word, $X_0 = \{\Lambda\} = X_0(R)$. Now let $n \geq 1$ and suppose $X_{n-1}(R) \subset X_{n-1}$. Let $w = (a_1, \ldots, a_n) \in X_n(R)$. As $X^>(R)$ is a word system, the two subwords $w_{\widehat{n}}$ and $w_{\widehat{1}}$ of w belong to $X_{n-1}(R)$. By assumption, $X_{n-1}(R)$ is a subset of X_{n-1}. In other words, $w \in (X_{n-1}A) \cap (AX_{n-1})$. Since R and $X^>(R)$ are disjoint, w is not an element of R_n. Since $R_n = ((X_{n-1}A) \cap (AX_{n-1})) \setminus X_n$, this implies $w \in X_n$, so $X_n(R) \subset X_n$ for all n. In conclusion, $X^>(R) \subset X^>$.

Version 2: Given any $E \subset A^*$ we may obtain the unique reduced E^{red} such that $X^>(E^{red}) = X^>(E)$ by replacing $E_n := E \cap A^n$ with

$$E_n^{red} := \Big\{w \in E_n \mid w \text{ has no subword from } E_k,\ k = 1, \ldots, n-1\Big\}.$$

(We omit the proof.) So, for our word system $X^>$, appealing to Observation 4.1.14, put $E := A^* \setminus X^>$. We find

$$\begin{aligned} E_n^{red} &= \Big\{w \in A^n \setminus X_n \mid w \text{ has no subword from } A^k \setminus X_k,\ k = 1, \ldots, n-1\Big\} \\ &= \Big\{w \in A^n \setminus X_n \mid \text{all proper subwords of } w \text{ are in } X^>\Big\} \\ &= \Big\{w \in A^n \setminus X_n \mid w \in X_{n-1}A \cap AX_{n-1}\Big\}. \end{aligned}$$

So, $E_n^{red} = R_n$.

□

The second proof also illustrates the feature of exclusion sets with only one word as *atoms*, and further exploitation of the problem's *inductive structure* will be demonstrated in Theorem 4.1.18. For simplicity assume $X_1 = A$. To understand the reduced set R of a given word system $X^>$ over X_1, for R_2 simply take all words of length 2 that do not occur in $X^>$. Then to get R_3 from $X^>(R_2)$ take all words of length 3 that do not occur in $X^>$. Then proceed with $X^>(R_2 \cup R_3)$ and words of length 4 to get R_4, and so forth.

In general, we have

$$X^{>}(E \cup E') = X^{>}(E) \cap X^{>}(E').$$

So, not only do we get $X^{>}(R_2 \cup \cdots \cup R_n) = X^{>}(R_2) \cap \cdots \cap X^{>}(R_n)$, but

$$X^{>} = \bigcap_{r \in R} X^{>}\big(\{r\}\big).$$

Being the smallest building blocks (the maximal proper word subsystems) it is important to understand the case $R = \{r\}$, that is the word systems $X^{>}(\{r\})$. Also the case $R = R_2$ is important; in Section 4.1.4 it will lead to word systems of graphs.

Generating functions. Guibas and Odlyzko [GO81, Theorem 1.1] find the generating function $\sum_{n=0}^{\infty} \#X_n(R) z^{-n}$ for a word system with a finite reduced set R of excluded words as the solution of a system of linear equations only depending on the so-called *correlation* of the words in R. As a special case, they give an explicit formula for the generating function in the case $R = \{r\}$, which depends only on the *autocorrelation* of the only one excluded word r. In [GO81, Section 7], they decide which word r gives the "biggest" word system: $\#X_n(\{r\}) \geq \#X_n(\{s\})$ if and only if the autocorrelation of r is less or equal to the autocorrelation of s. There is a nice survey in Odlyzko [Odl85]. Some more methods to determine the generating function can be found in Goulden and Jackson [GJ79].

Growth rates. One may analyze the asymptotic behaviour of the cardinality sequence. It is clear, that the sequence may break down simply by setting $X_n = \emptyset$ for all $n \geq N$, or that d_n is limited by d^n for the full word system A^* with $\#A = d$. But there are more interesting results. For instance, Shur shows in [Shu06] that for all $s \in \mathbb{R}_+$ there are word systems with asymptotic growth rate n^s. In [Shu09], it is shown that there are word systems with asymptotic growth rate larger than every polynomial and smaller than every exponential function.

4.1.3 Complexity of Factorial Languages: New Results

In this section we present some results which, we believe, may be new. The results are formulated for cardinality sequences of word systems. Of course, from Theorem 4.1.5 it follows that all these results remain true for Cartesian systems. It also should be noted that some results (Theorem 4.1.19 and its consequences) are much easier to prove for Cartesian systems than for word systems.

Local to global

Let $X^>$ be a word system over A and let R be the unique reduced set of excluded words such that $X^> = X^>(R)$. Put $d_i := \#X_i$. It is noteworthy that in order to determine $X_1, \ldots, X_k$, and therefore $d_1, \ldots, d_k$, up to some finite k, we only need to know the R_i up to the same k. In order to realize the partial sequence $d_1, \ldots, d_k$ as a cardinality sequence of a word system, it does not matter what the word system $X^>$ does for $i > k$, nor, equivalently, what the R_i are for $i > k$. We may cut down $X^>$ by assuming $X_i = \emptyset$ for $i > k$; this is an easy choice but, possibly, not the most clever, because it makes the corresponding R_i rather big. We also may cut down R by assuming $R_i = \emptyset$ for $i > k$; this gives the biggest word system with the partial sequence X_i for $i = 1, \ldots, k$ with the corresponding R_i for $i = 1, \ldots, k$. This choice has the advantage that now the resulting truncated set of excluded words is finite, so, all results for generating functions for finite sets of excluded words (for instance, those in [GO81]) are applicable for checking if the partial sequence $d_1, \ldots, d_k$ can be realized for suitable choices of $R_1, \ldots, R_k$.

If for a sequence $d_1, d_2, \ldots$ we can realize $d_1, \ldots, d_k$ for each finite k by choosing $R_1, \ldots, R_k$ in such a way that each R_i does not depend on $k \geq i$, then, of course, the whole sequence of R_k determines a word system $X^>$ with $\#X_k = d_k$ for all k. But what, if we can realize each finite subsequence $d_1, \ldots, d_k$, but without being able to fix the R_i? The following theorem shows that this local realizability of the sequence $d_1, d_2, \ldots$ is sufficient.

4.1.18 Theorem. *Let $(d_n)_{n \geq 1}$ be a sequence of nonnegative integers. Suppose for every $k \in \mathbb{N}$ there exists a word system $Y^>$ with $\#Y_i = d_i$ for $i = 1, \ldots, k$. Then there exists a word system $X^>$ with $\#X_i = d_i$ for all $i \geq 1$.*

Proof. Every word system $X^>$ may be considered as a word system over X_1 as alphabet. And if two word systems $X^>$ and $X'^>$ (over potentially different alphabets) fulfill $\#X_1 = \#X'_1$, then there is a bijection from X'_1 to X_1, which corresponds to an isomorphism from $X'^>$ to another word system $X''^>$, also over X_1. So, we may assume that the $Y^{(k)>}$ realizing $d_1, \dots, d_k$ are over a fixed finite alphabet A.

Let us consider word systems as elements of the product

$$\mathcal{W}(A) := \bigtimes_{n \in \mathbb{N}_0} \mathcal{P}(A^n)$$

of the power sets $\mathcal{P}(A^n)$ of A^n. By $\mathcal{WS}(A) \subset \mathcal{W}(A)$ we denote the set of all word systems over A. Let $(Y^{(k)>})_{k\in\mathbb{N}}$ be a sequence in $\mathcal{WS}(A)$ fulfilling $\#Y_i^{(k)} = d_i$ for $i = 1, \dots, k$. We will show:

1. There is a subsequence $(Y^{(k_n)>})_{n\in\mathbb{N}}$ of $(Y^{(k)>})_{k\in\mathbb{N}}$ such that for each $i \in \mathbb{N}$ the sequence $(Y_i^{(k_n)>})_{n\in\mathbb{N}}$ is eventually constant, say, $Y_i^{(k_n)>} =: X_i$ for sufficiently big n.

2. The X_i form a word system $X^>$ with $\#X_i = d_i$ for all i.

Such a sequence can be constructed explicitly by hand. But one has to introduce *ad hoc* total orders on $\mathcal{P}(A^n)$, and writing it down requires lots of more indices. We prefer to introduce a topology on $\mathcal{W}(A)$ that allows to apply Tychonoff's theorem.

We equip $\mathcal{P}(A^n)$ with the discrete topology and $\mathcal{W}(A)$ with the product topology. So, convergence in $\mathcal{P}(A^n)$ means eventually constant, and convergence in $\mathcal{W}(A)$ means eventually constant entrywise. Since A is assumed finite, $\mathcal{P}(A^n)$ is finite, hence, compact. By Tychonoff's theorem, $\mathcal{W}(A)$ is compact. Since $\mathcal{W}(A)$ is first countable, it is even sequentially compact. This proves (1) and, of course, it proves that the limit $X^>$ of the subsequence of $(Y^{(k)>})_{k\in\mathbb{N}}$ fufills $\#X_i = d_i$ for all i.

To show that $X^>$ is a word system, we show that $\mathcal{WS}(A)$ is closed in $\mathcal{W}(A)$. Suppose $Z \in \mathcal{W}(A)$ is not a word system. That is, there exists a word $w \in Z_k$ with a subword y of w with $y \in A^m \setminus Z_m$. Then the set $U := \{Z_0\} \times \{Z_1\} \times \cdots \times \{Z_k\} \times \bigtimes_{n>k} \mathcal{P}(A^n)$ is an open neighbourhood of Z and no element of U is a word system. This shows that $\mathcal{W}(A) \setminus \mathcal{WS}(A)$ is open, hence, $\mathcal{WS}(A)$ is closed. □

'Thinning out' Cartesian systems

We present some results how to select from a Cartesian system a subsequence and turn that subsequence again into a Cartesian system. We know from Theorem 4.1.5 that every Cartesian system is isomorphic to a word system, and all results about cardinality sequences also apply to word systems. But it would be very cumbersome, indeed, if we had to turn these fresh Cartesian systems into word systems, explicitly. These results are, therefore, instances that illustrate how powerful the considerably more flexible notion of Cartesian system can be as compared with the more restrictive notion of word system.

Let us start with the following triviality - and imagine how notationally complicated it would be to prove it, using only word systems.

4.1.19 Theorem. *Let $X^>$ be a Cartesian system with injections $i_{m,n}$ and fix $k \in \mathbb{N}$. Then the family $Y^> = (Y_n)_{n \in \mathbb{N}_0}$ with $Y_n := X_{nk}$ and with the injections $j_{m,n} := i_{mk,nk}$ is a Cartesian system.*

The next result relies on the important property that, unlike for tensor products, in a Cartesian product of sets there are canonical projections onto the factors; see Proposition 4.1.2. For all sets S_1 and S_2, define $P_i \colon S_1 \times S_2 \to S_i$ by $P_i(s_1, s_2) = s_i$.

4.1.20 Theorem. *Let $X^>$ be a Cartesian system with injections $i_{m,n}$ and fix $k \in \mathbb{N}$. Then the family $Y^> = (Y_n)_{n \in \mathbb{N}_0}$ with*

$$Y_n = \begin{cases} X_{n+k}, & n > 0 \\ \{\Lambda\}, & n = 0 \end{cases}$$

together with the injections

$$j_{m,n} := (P_1 \circ i_{m+k,n}, P_2 \circ i_{m,k+n}) \tag{4.1.3}$$

for $m, n \geq 1$ and $j_{m,0}$ and $j_{0,n}$ being (necessarily) the canonical injections is a Cartesian system.

Proof. Note that the construction 'commutes' with isomorphisms $\alpha^> \colon X^> \to X'^>$. Indeed, since $i'_{m,n} \circ \alpha_{m+n} = (\alpha_m \times \alpha_n) \circ i_{m,n}$, we find

$$\begin{aligned}
&(P_1 \circ i'_{m+k,n}, P_2 \circ i'_{m,k+n}) \circ \alpha_{m+k+n} \\
&\quad = (P_1 \circ (\alpha_{m+k} \times \alpha_n) \circ i_{m+k,n}, P_2 \circ (\alpha_m \times \alpha_{k+n}) \circ i_{m,k+n}) \\
&\quad = (\alpha_{m+k} \circ P_1 \circ i_{m+k,n}, \alpha_{k+n} \circ P_2 \circ i_{m,k+n}) \\
&\quad = (\alpha_{m+k} \times \alpha_{k+n}) \circ (P_1 \circ i_{m+k,n}, P_2 \circ i_{m,k+n}),
\end{aligned}$$

so that $j'_{m,n} \circ \alpha_{m+k+n} = (\alpha_{m+k} \times \alpha_{k+n}) \circ j_{m,n}$. By Theorem 4.1.5, every Cartesian system is isomorphic to a word system. Therefore, we may assume that $X^>$ is a word system.

For a word system $X^>$, the definition in (4.1.3) leads to

$$j_{m,n}(a_1, \dots, a_{m+n+k}) := \big((a_1, \dots, a_{m+k}), (a_{m+1}, \dots, a_{m+n+k})\big) \tag{4.1.4}$$

for all $m, n \geq 1$. In other words, the k letters 'in the middle' $a_{m+1}, \dots, a_{m+k}$ are 'replicated' once to the right part of the left factor and once to the left part of the right factor. The maps $j_{m,n} : X_{m+n+k} \to X_{m+k} \times X_{n+k}$ defined in (4.1.4) are clearly injective. The computation

$$\begin{aligned}
&\big((j_{m,n} \times \mathrm{id}_{Y_\ell}) \circ j_{m+n,\ell}\big)(a_1, \dots, a_{m+n+\ell+k}) \\
&\quad = (j_{m,n} \times \mathrm{id}_{Y_\ell})\big((a_1, \dots, a_{m+n+k}), (a_{m+n+1}, \dots, a_{m+n+\ell+k})\big) \\
&\quad = \big((a_1, \dots, a_{m+k}), (a_{m+1}, \dots, a_{m+n+k}), (a_{m+n+1}, \dots, a_{m+n+\ell+k})\big) \\
&\quad = (\mathrm{id}_{Y_m} \times j_{n,\ell})\big((a_1, \dots, a_{m+k}), (a_{m+1}, \dots, a_{m+n+\ell+k})\big) \\
&\quad = \big((\mathrm{id}_{Y_m} \times j_{n,\ell}) \circ j_{m,n+\ell}\big)(a_1, \dots, a_{m+n+\ell+k})
\end{aligned}$$

proves associativity for $m, n, \ell \geq 1$. For the cases involving $m = 0$ or $n = 0$ or $\ell = 0$ there is nothing to prove. So the $Y_n = X_{n+k}$ together with the maps $j_{m,n}$ form a Cartesian system. □

4.1.21 Corollary. *Suppose for $n \in \mathbb{N}$ there is a function $f \colon \mathbb{N}_0^{n-1} \to \mathbb{N}_0$ such that for every word system $X^>$ we have*

$$\#X_n \leq f(\#X_1, \#X_2, \dots, \#X_{n-1}).$$

Then for every Cartesian system $X^>$ we have

$$\#X_{na+b} \leq f(\#X_{a+b}, \#X_{2a+b}, \dots, \#X_{(n-1)a+b}).$$

Proof. By the preceding two theorems the $Y_n = X_{na+b}$ form a Cartesian system. □

4.1.22 Corollary. *For every cardinality sequence $d_n = \#X_n$ of a word system $X^>$, we have*

$$d_{m+n+k} \leq d_{m+k} d_{n+k} \tag{4.1.5}$$

for all $m, n, k \in \mathbb{N}_0$. In particular, $d_{k+1} \leq d_k^2$ for every $k \geq 1$.

Proof. Equation (4.1.5) follows from Corollary 4.1.21, because every cardinality sequence $\#X_n$ is submultiplicative ($\#X_{m+n} \leq \#X_m \#X_n$). The formula $d_{k+1} \leq d_k^2$ follows from (4.1.5) with $m = n = 1$. □

4.1.23 Corollary. *Not every submultiplicative sequence d_n is the cardinality sequence of a word system.*

Proof. A cardinality sequence fulfills $d_3 \leq d_2^2$. However, the sequence $d_1 = 2$, $d_2 = 1$, $d_3 = 2$, and $d_k = 0$ for $k > 3$ is submultiplicative, but $d_3 \not\leq d_2^2$. □

A sufficient criterion motivated by submultiplicativity

It is well known that for every submultiplicative sequence $(d_n)_{n \in \mathbb{N}}$ of nonnegative integers we have $\lim_{n\to\infty} \sqrt[n]{d_n} = \inf_n \sqrt[n]{d_n}$. On the other hand, if we assume the limit is approached monotonously, that is, if we assume $\sqrt[m+1]{d_{m+1}} \leq \sqrt[m]{d_m}$ for all $m \in \mathbb{N}$, from

$$d_{m+n} = \sqrt[m+n]{d_{m+n}}^m \sqrt[m+n]{d_{m+n}}^n \leq \sqrt[m]{d_m}^m \sqrt[n]{d_n}^n = d_m d_n$$

we get that $(d_n)_{n \in \mathbb{N}}$ is submultiplicative. We may ask, if this condition is sufficient to be the cardinality sequence of a word system. It turns out that this condition is neither sufficient (Example 4.1.29) nor necessary (Example 4.1.26). However, we may modify the condition to make it at least sufficient.

Recall our notation $\lceil a \rceil := \min\{n \in \mathbb{Z} \mid n \geq a\}$ and $\lfloor a \rfloor := \max\{n \in \mathbb{Z} \mid n \leq a\}$ for $a \in \mathbb{R}$.

4.1.24 Theorem. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative integers such that*

$$\left\lceil \sqrt[n+1]{d_{n+1}} \right\rceil \leq \left\lfloor \sqrt[n]{d_n} \right\rfloor \tag{4.1.6}$$

for all $n \in \mathbb{N}$. Then there exists a word system $X^{>}$ with $\#X_n = d_n$ for all $n \in \mathbb{N}$.

Proof. Set $X_0 = \{\Lambda\}$, $X_1 = \{1, \ldots, d_1\}$, and choose arbitrary $X_n \subset X_1^n$ such that

$$\left\{1, \ldots, \left\lfloor \sqrt[n]{d_n} \right\rfloor\right\}^n \subset X_n \subset \left\{1, \ldots, \left\lceil \sqrt[n]{d_n} \right\rceil\right\}^n.$$

Since $\left\lfloor \sqrt[n]{d_n} \right\rfloor^n \leq d_n \leq \left\lceil \sqrt[n]{d_n} \right\rceil^n$, this is always possible. We find

$$\begin{aligned}
X_{m+n} &\subset \left\{1, \ldots, \left\lceil \sqrt[m+n]{d_{m+n}} \right\rceil\right\}^{m+n} \\
&= \left\{1, \ldots, \left\lceil \sqrt[m+n]{d_{m+n}} \right\rceil\right\}^m \times \left\{1, \ldots, \left\lceil \sqrt[m+n]{d_{m+n}} \right\rceil\right\}^n \\
&\subset \left\{1, \ldots, \left\lfloor \sqrt[m]{d_m} \right\rfloor\right\}^m \times \left\{1, \ldots, \left\lfloor \sqrt[n]{d_n} \right\rfloor\right\}^n \\
&\subset X_m \times X_n
\end{aligned}$$

for all $m, n \in \mathbb{N}_0$. So, the X_n form a word system over X_1. □

Example 4.1.26 will also show that the sufficient condition (4.1.6) is not necessary.

4.1.4 (Counter)examples with word systems of graphs

We establish a connection between word systems and directed graphs. In fact, the paths of a directed graph without multiple edges form a word system over its vertex set. Moreover, every word system is a subsystem of such a *graph system*. As application, we provide examples that show that $\sqrt[n+1]{d_{n+1}} \leq \sqrt[n]{d_n}$ for all $n \in \mathbb{N}_0$ is neither necessary nor sufficient for the existence of a word system with cardinality sequence d_n. Of course, this implies that the sufficient condition in Theorem 4.1.24, which is even stronger, is not necessary.

By a *graph*, we will always mean a directed graph, possibly with loops, but without multiple edges. That is, a graph is a pair (V, E), where V is a set, whose elements are called *vertices*, and a subset E of $V \times V$, whose elements are called *edges*.

4.1.25 Theorem. *The following statements hold:*

1. *Let $\Gamma = (V, E)$ be a graph and set*

$$E_0 := \{\Lambda\}, \qquad E_1 := V, \qquad E_2 := E,$$
$$E_n := \Big\{(v_1, \dots, v_n) \in V^n \;\Big|\; (v_i, v_{i+1}) \in E \text{ for all } i = 1, \dots, n-1\Big\}.$$

   (That is, E_n consists of all paths *of length $n-1$.)*

   Then $E_n = X_n\big((V \times V) \setminus E\big)$. In particular, the E_n form a word system over V, the graph system *$X_\Gamma^>$.*

2. *Every word system $X^>$ is a subsystem of the graph system $X_{(X_1,X_2)}^>$ associated with the graph (X_1, X_2).*

Proof.

1. By definition,

$$X_n\big((V \times V) \setminus E\big)$$
$$:= \Big\{(v_1, \dots, v_n) \in V^n \;\Big|\; (v_i, v_{i+1}) \notin (V \times V) \setminus E \;\forall\, i = 1, \dots, n-1\Big\} = E_n.$$

2. For each word $(v_1, \dots, v_n) \in X_n$, the (v_i, v_{i+1}) are subwords, hence belong to X_2. So X_n is a subset of E_n.

□

Of course, $(V \times V) \setminus E$ is reduced. So graph systems are precisely those word systems which have a reduced set of excluded words consisting only of words of length 2.

Let $\Gamma = (V, E)$ be a graph with $V = \{1, \dots, d\}$. Its *adjacency matrix* is the $d \times d$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{for } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then, obviously, the number of paths of length n from i to j is given by the i-j-entry of A^n. For any $d \times d$ matrix B, denote by

$$\Sigma(B) := \sum_{i,j=1,\dots,d} B_{i,j}$$

the sum of all its entries. So $\Sigma(A^{n-1}) = \#E_n$. Denote by $\mathbb{1}_{m\times n}$ the $m \times n$ matrix with all entries equal to 1 and put $\mathbb{1}_d := \mathbb{1}_{d\times 1}$ and $\mathbb{1}_d^t := \mathbb{1}_{1\times d}$. Note that in this notation $\mathbb{1}_{d\times d}\mathbb{1}_d = d\mathbb{1}_d$.

4.1.26 Example. Let (V, E) be the graph with $d+1$ vertices and adjacency matrix $A = \begin{pmatrix} 0 & \mathbb{1}_d^t \\ \mathbb{1}_d & 0 \end{pmatrix}$. Then $\#E_2 = \Sigma(A) = 2d$. Since $A^2 = \begin{pmatrix} d & 0 \\ 0 & \mathbb{1}_{d\times d} \end{pmatrix}$ we have $\#E_3 = \Sigma(A^2) = d^2 + d$.

4.1.27 Example. Let (V, E) be a graph with $\#E = 1$, so $E = \{(v, w)\}$. If $v \neq w$, then for $n > 2$ there is no path of length $n-1$, so $E_n = \emptyset$. If $v = w$, then $E_n = \{(v, v, \dots, v)\}$, so, $\#E_n = 1$.

Let (V, E) be a graph with $\#E = 2$, that is, its adjacency matrix A is the sum of two distinct matrix units E_{ij} and E_{kl}. We find

$$\#E_3 = \Sigma(A^2) = \Sigma\big((E_{ij} + E_{kl})^2\big) = \delta_{ij} + \delta_{il} + \delta_{kj} + \delta_{kl}.$$

Since three of the equalities $i = j, i = l, k = j$ and $k = l$ necessarily lead to $i = j = k = l$, it follows that $\#E_3 \leq 2$. Since every word system is a subsystem of its graph system, the implication $\#X_2 \leq 2 \Rightarrow \#X_3 \leq 2$ holds for all word systems. In other words, if we define

$$f(d_1, d_2) := \begin{cases} 2 & \text{for } d_2 \leq 2, \\ d_1^3 & \text{otherwise,} \end{cases}$$

then $d_3 \leq f(d_1, d_2)$ for all cardinality sequences of word systems. By Corollary 4.1.21, $d_4 = d_{3+1} \leq f(d_{1+1}, d_{2+1}) = f(d_2, d_3) = 2$, because $d_3 \leq 2$, and so forth. Hence, $\#E_2 \leq 2$ implies $\#E_{2+k} \leq 2$ for all $k \in \mathbb{N}_0$.

4.1.28 Observation. A straightforward calculation gives

$$\Sigma(\mathbb{1}_{d\times d}A) = \Sigma(A\mathbb{1}_{d\times d}) = d\Sigma(A)$$

for all $A \in M_d$. Put $\overline{A} := \mathbb{1}_{d\times d} - A$. Combining the two equations

$$\Sigma(\overline{A}A) = \Sigma\big((\mathbb{1}_{d\times d} - A)A\big) = \Sigma(\mathbb{1}_{d\times d}A - A^2) = d\Sigma(A) - \Sigma(A^2)$$

and

$$\Sigma(\overline{A}A) = \Sigma\big(\overline{A}(\mathbb{1}_{d\times d} - \overline{A})\big) = \Sigma(\overline{A}\mathbb{1}_{d\times d} - \overline{A}^2) = d\Sigma(\overline{A}) - \Sigma(\overline{A}^2),$$

we get

$$\Sigma(A^2) = \Sigma(\overline{A}^2) + d\big(\Sigma(A) - \Sigma(\overline{A})\big). \tag{4.1.7}$$

4.1.29 Example. Let $\Gamma = (V, E)$ be a graph with 3 vertices and 7 edges. For its adjacency matrix A we, thus, have $\Sigma(A) = 7$ and $\Sigma(\overline{A}) = 2$. Note that $\overline{A}$ is the adjacency matrix of the *complementary graph* $\overline{\Gamma} := (V, (V \times V) \setminus E)$. Therefore, by Example 4.1.27, we obtain $\Sigma(\overline{A}^2) \leq 2$. So, (4.1.7) yields

$$\#E_3 = \Sigma(A^2) = \Sigma(\overline{A}^2) + d\big(\Sigma(A) - \Sigma(\overline{A})\big) \leq 2 + 3(7 - 2) = 17.$$

This shows that a graph with 3 vertices and 7 edges has at most 17 paths of length two.

4.1.30 Example. In a graph with d vertices and $d^2 - 1$ edges we have $\Sigma(A) = d^2 - 1$, $\Sigma(\overline{A}) = 1$ and $\Sigma(\overline{A}^2)$ is either 1 or 0, depending on whether the missing edge is a loop or not. Using again (4.1.7), we find

$$\#E_3 = \Sigma(\overline{A}^2) + d(d^2-2) = \begin{cases} d^3-2d & \text{if the missing edge is a loop,} \\ d^3-2d+1 & \text{if the missing edge is not a loop.} \end{cases}$$

We learn from these examples that the condition $\sqrt[m+1]{d_{m+1}} \leq \sqrt[m]{d_m}$ is neither sufficient nor necessary for the existence of a Cartesian system $X^>$ with $\#X_n = d_n$. By Example 4.1.29 there is no system with $\#X_1 = 3, \#X_2 = 7, \#X_3 = 18$. But $18^2 = 324 < 343 = 7^3$. So, the sequence $d_1 = 3, d_2 = 7, d_3 = 18, d_n = 0$ for $n > 3$ fulfills the condition. So the condition is not sufficient. In Example 4.1.26, putting $d = 10$, we get a system with $\#X_1 = 11, \#X_2 = 20, \#X_3 = 110$. But $110^2 > 10000 > 8000 = 20^3$. So the condition is not necessary. As mentioned in the beginning of this section, this implies that the stronger condition (4.1.6) is not necessary either.

Especially in view of Theorem 4.1.18, the following class of questions is interesting: Fixing (some of) the cardinalities $\#X_1, \ldots, \#X_n$, what is the maximal possibility for $\#X_{n+1}$ in a word system $X^>$? The question, which graph with d_1 vertices and d_2 edges has the maximal number of paths of length 2, is clearly of the above type with $n = 2$. It was first investigated by Katz in [Kat71], who gave an answer only for special values of d_1 and d_2. A complete answer was given by Aharoni in [Aha80] by exhibiting four special types of graphs, (two of them are close to being complete graphs, two of them are close to being complements of complete graphs) one of which is maximal for any choice of d_1 and d_2. This allows one to determine the maximal d_3 such that there is a word system $X^>$ with $\#X_1 = d_1, \#X_2 = d_2$ and $\#X_3 = d_3$. Similar results for undirected graphs can be found in [AK78], [PPS99] and [ÁFMNW09]. It seems the questions for higher n are still open problems.

4.2 Subproduct Systems

For convenience of the reader, we repeat the definition of a subproduct system in a plain way. See also Section 2.3.1, Hilbert Spaces.

4.2.1 Definition. A *subproduct system* (over $\mathbb{S}$) is a family $H^{\otimes} = (H_t)_{t\in\mathbb{S}}$ of Hilbert spaces H_t with $H_0 = \mathbb{C}$ and with coisometries

$$w_{s,t} \colon H_s \otimes H_t \longrightarrow H_{s+t}$$

such that the product defined by $x_s y_t := w_{s,t}(x_s \otimes y_t)$ is associative and such that $w_{0,t}$ and $w_{t,0}$ are the canonical identifications.

It is more common to write subproduct systems with the adjoint maps $v_{s,t} := w_{s,t}^* \colon H_{s+t} \to H_s \otimes H_t$, which have to fulfill the coassociativity and marginal conditions expressed in the following two diagrams.

$$\begin{array}{ccc} H_{r+s+t} & \xrightarrow{v_{r+s,t}} & H_{r+s} \otimes H_t \\ \downarrow{\scriptstyle v_{r,s+t}} & & \downarrow{\scriptstyle v_{r,s} \otimes \mathrm{id}_t} \\ H_r \otimes H_{s+t} & \xrightarrow{\mathrm{id}_r \otimes v_{s,t}} & H_r \otimes H_s \otimes H_t \end{array} \qquad \begin{array}{ccccc} & \swarrow{\scriptstyle v_{0,s}} & H_s & \searrow{\scriptstyle v_{s,0}} & \\ & & \downarrow{\scriptstyle \mathrm{id}_s} & & \\ H_0 \otimes H_s & \xrightarrow{\cong} & H_s & \xleftarrow{\cong} & H_s \otimes H_0 \end{array}$$

The isometries $v_{s,t}$ emphasize the idea, of considering H_{s+t} as a subspace of $H_s \otimes H_t$. It also makes clear that subproduct systems are the same as comonoidal systems in the tensor category $(\mathbf{Hilb}^{isom}, \otimes)$ with $H_0 = \mathbb{C}$.

The aim of this section is to characterize the dimension functions $s \mapsto \dim H_s$. In the discrete case we show that the dimension sequences of subproduct systems are the same as the dimension sequences of certain $\mathbb{N}_0$-graded algebras and the same as the cardinality sequences of word systems and Cartesian systems. Thus, the results of the previous section apply also to discrete subproduct systems. We can use this fact and Theorem 4.1.9 to give a simple characterization of all dimension functions of rational-time subproduct systems and cardinality functions of rational-time cartesian systems. Finally, we discuss the continuous-time situation. For Cartesian systems the simple characterization remains true, but for continuous-time subproduct systems we need additional continuity assumptions to extend the rational-time results.

4.2.1 Dimension of Discrete Subproduct Systems

Recall that a *product system* is simply a full subproduct system, that is all the isometries $v_{m,n}$ are unitaries.

4.2.2 Example. Discrete product systems $H^{\otimes} = (H_n)_{n\in\mathbb{N}_0}$ are easy to understand. If we identify H_n with $H_1^{\otimes n}$ $(n \geq 1)$ via the inverse of the unitary determined by $x_n\otimes\cdots\otimes x_1 \mapsto x_n\cdots x_1$, it is clear that the product of $H^{\otimes} = (H_1^{\otimes n})_{n\in\mathbb{N}_0}$ is nothing but the tensor product $(x_m \otimes\cdots\otimes x_1)(y_n \otimes\cdots\otimes y_1) = x_m \otimes\cdots\otimes x_1 \otimes y_n \otimes\cdots\otimes y_1$. By mentioning that we are working in a tensor category, this is nothing but the identification map $H_1^{\otimes m} \otimes H_1^{\otimes n} \equiv H_1^{\otimes(m+n)}$. We say a (discrete) product system $H^{\otimes} = (H^{\otimes n})_{n\in\mathbb{N}_0}$ with the identity as product is in *standard form*.

As far as discrete product systems are concerned, there is not more to be said than what is said in the preceding example. The situation gets more interesting for subproduct systems. We start with an obvious relation between subproduct systems and Cartesian systems.

4.2.3 Example. Let $X^{>} = (X_n)_{n\in\mathbb{N}_0}$ be a Cartesian system. Denote by H_n the canonical Hilbert space with orthonormal basis X_n. Then, clearly, the embeddings $i_{m,n}$ of X_{m+n} into $X_m \times X_n$ extend as isometries $v_{m,n}\colon H_{m+n} \to H_m \otimes H_n$ and the $v_{m,n}$ define a subproduct system structure, the subproduct system *associated* with $X^{>}$. Moreover, if $X^{>}$ is a word system over A, so that $X_n \subset A^n$ and $H_n \subset H_1^{\otimes n}$, this subproduct system is a subproduct subsystem of a product system in standard form. We say the subproduct system is in *standard form*.

Obviously, $\dim H_n = \#X_n$. We see, for every Cartesian (word) system there is a subproduct system (in standard form) such that the dimension sequence of the latter coincides with the cardinality sequence of the former. Before we show the converse statement in Proposition 4.2.5, let us mention that not every subproduct system is isomorphic to one that is associated with a word system.

4.2.4 Observation. If at least one X_n in a word system contains a word with at least two different letters, then the associated subproduct system in standard form is not commutative. But there are commutative subproduct systems. See, for instance, the *symmetric subproduct system* introduced by Shalit and Solel [SS09], which is obtained by considering the symmetric tensor power $H^{\otimes_s n}$ as subspace of $H^{\otimes n}$.

By an $\mathbb{N}_0$-*graded algebra* we mean a unital algebra $\mathcal{A}$ with a vector space direct sum decomposition $\mathcal{A} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{A}_n$ such that $\mathcal{A}_m\mathcal{A}_n \subset \mathcal{A}_{m+n}$ for all $m, n \in \mathbb{N}_0$. The vector spaces $\mathcal{A}_m$ are called *homogeneous components* of $\mathcal{A}$. An $\mathbb{N}_0$-graded algebra is called *connected* if $\mathcal{A}_0 = \mathbb{C}\mathbb{1}$ and *standard graded* if it is connected and $\mathcal{A}_n = \mathcal{A}_1^n = \operatorname{span}\{a_1 \cdots a_n \mid a_i \in \mathcal{A}_1, i = 1, \ldots, n\}$. Equivalently, one can define a graded algebra as a monoidal system $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ over $\mathbb{N}_0$ in $(\mathbf{Vect}, \otimes)$ and a standard graded algebra as a monoidal system over $\mathbb{N}_0$ in $(\mathbf{Vect}^{surj}, \otimes)$; compare Section 2.3.1, Vector Spaces. An $\mathbb{N}_0$-graded algebra is called *locally finite-dimensional* if all $\mathcal{A}_n$ are finite-dimensional.

Every finite-dimensional discrete subproduct system can be considered as a locally finite-dimensional standard graded algebra with respect to the multiplication induced by the structure maps $w_{m,n} = v_{m,n}^*$; compare also Remark 2.3.3. In particular, for every finite-dimensional subproduct system there is a standard graded algebra such that their dimension sequences coincide. The next proposition closes the circle:

4.2.5 Proposition. *Let $\mathcal{A} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{A}_n$ be a locally finite-dimensional standard graded algebra with $\operatorname{dim} \mathcal{A}_n < \infty$ for all $n \in \mathbb{N}_0$. Then there exists a word system $(X_n)_{n\in\mathbb{N}_0}$ with $\#X_n = \operatorname{dim} \mathcal{A}_n$ for all $n \in \mathbb{N}_0$.*

4.2.6 Remark. It turned out that this has been proven already in [Ani82]. Anick's proof is almost the same as that of [Sta78], who showed a corresponding result for commutative graded algebras, and who tributes this result to MacAulay [Mac27]. The proof presented here is basically Anick's proof, but it was found independently and first written down directly for subproduct systems. It is left in the thesis on the one hand to be self contained and on the other hand to present it in a modern terminology and notation.

We give a short preparation for the proof.

4.2.7 Definition. Let A be a partially ordered alphabet. Then the *lexicographical order* $\leq_{lex}$ on A^n is given by $(a_1, \ldots, a_n) \leq_{lex} (b_1, \ldots, b_n)$ if $a_k = b_k$ for all k or if $a_k < b_k$ for k being the smallest index i with $a_i \neq b_i$.

It is easy to show that the lexicographical order is a total order on A^n whenever $\leq$ is a total order on A. Without the obvious proof, we state:

4.2.8 Lemma. *Let $y, y' \in A^n$. Then*

$$y \leq_{lex} y' \iff \exists\, x, z \in A^*\colon xyz \leq_{lex} xy'z \iff \forall\, x, z \in A^*\colon xyz \leq_{lex} xy'z.$$

Proof of Poposition 4.2.5. Let $(a_1, \ldots, a_d)$ be a basis of $\mathcal{A}_1$. Set $A = \{1, \ldots, d\}$. For a word $w = (i_1, \ldots, i_n)$ in A^n we define the element $a_w \in \mathcal{A}_n, a_w := a_{i_1} \cdots a_{i_n}$. The multiplication $\mathcal{A}_m \times \mathcal{A}_n \to \mathcal{A}_{m+n}$ is surjective for all $m, n \in \mathbb{N}_0$. Therefore, a simple induction yields $\mathcal{A}_n = \operatorname{span}\{a_w \mid w \in A^n\}$. Set

$$X_n := \Big\{ w \in A^n \;\Big|\; a_w \notin \operatorname{span}\{a_v \mid v <_{lex} w\} \Big\}.$$

Since $\{a_w \mid w \in X_n\}$ is linearly independent and still spans $\mathcal{A}_n$, it is a basis of $\mathcal{A}_n$. Thus, $\# X_n = \operatorname{dim} \mathcal{A}_n$ for all $n \in \mathbb{N}_0$.

We claim $X^> = (X_n)_{n \in \mathbb{N}_0}$ is a word system over A. Let $y \in A^k$ be a subword of $w \in A^n$ $(k \leq n)$, that is $w = xyz$ for some $x, z \in A^*$. We are done if we show $y \notin X_k \Rightarrow xyz \notin X_n$. Suppose $y \notin X_k$, that is

$$a_y = \sum_{y' < y} \alpha_{y'} a_{y'}.$$

Then, we have

$$a_w = a_x a_y a_z = \sum_{y' < y} \alpha_{y'} a_x a_{y'} a_z = \sum_{y' < y} \alpha_{y'} a_{xy'z}.$$

Since, by Lemma 4.2.8, $y' <_{lex} y$ implies $xy'z <_{lex} xyz$, we obtain $a_w \notin X_n$. □

The content of this section is summarized in:

4.2.9 Corollary. *Let $(d_n)_{n \in \mathbb{N}_0}$ be a sequence of nonnegative integers. Then the following are equivalent*

(i) There exists a discrete subproduct system $H^{\otimes}$ with $\operatorname{dim} H_n = d_n$ for all $n \in \mathbb{N}_0$.

(ii) There exists a standard graded algebra $\mathcal{A}$ with $\operatorname{dim} \mathcal{A}_n = d_n$ for all $n \in \mathbb{N}_0$.

(iii) There exists a factorial language L with $\#(L \cap A^n) = d_n$ for all $n \in \mathbb{N}_0$.

(iv) There exists a word system $X^>$ with $\# X_n = d_n$ for all $n \in \mathbb{N}_0$.

(v) There exists a discrete Cartesian system $X^>$ with $\# X_n = d_n$ for all $n \in \mathbb{N}_0$.

Thus, results on the complexity of factorial languages hold for the dimension sequence of a subproduct system. In particular, Theorem 4.1.9 translates to:

4.2.10 Corollary. *Suppose* $d := \operatorname{dim} H_k \leq k$ *for some* $k \in \mathbb{N}$*. Then*

$$\operatorname{dim} H_m \leq \left\lceil \frac{d+1}{2} \right\rceil \left\lfloor \frac{d+1}{2} \right\rfloor$$

for all $m \geq k + d$*.*

4.2.2 Rational Time

For rational-time subproduct systems we can give a simple characterization of the possible dimension functions.

4.2.11 Proposition. *Let* $H^{\circleddash} = (H_t)_{t \in \mathbb{Q}_+}$ *be a rational-time subproduct system with* $\operatorname{dim} H_t = d(t)$*. Then*

$$d(s) \leq \left\lceil \frac{d(t)+1}{2} \right\rceil \left\lfloor \frac{d(t)+1}{2} \right\rfloor \tag{4.2.1}$$

holds for all $s > t$*. The analogous statement holds for a rational-time Cartesian system.*

Proof. Denote the isometries of $H^{\circleddash}$ by $v_{m,n}$. For every $t \in \mathbb{Q}_+$ and every $N \in \mathbb{N}$, the family $(H_n^{t,N})_{n \in \mathbb{N}_0}$ with $H_n^{t,N} := H_{t\frac{n}{N}}$ is a discrete subproduct system with respect to the isometries $(v_{m,n}^{t,N}) := v_{t\frac{m}{N}, t\frac{n}{N}}$; compare Theorem 2.3.2. Suppose $d(t) < \infty$. Applying Corollary 4.2.10 to the subproduct system $(H_n^{t,N})_{n \in \mathbb{N}_0}$ yields

$$\operatorname{dim} H_{t\frac{m}{N}} \leq \left\lceil \frac{d(t)+1}{2} \right\rceil \left\lfloor \frac{d(t)+1}{2} \right\rfloor$$

for all $m \geq N + d$. Every rational $s > t$ can be written in the form $s = t\frac{m}{N}$ with $m \geq N + 1$. Expanding the fraction by d yields $s = t\frac{dm}{dN}$ with $dm \geq dN + d$. So we may conclude that (4.2.1) holds for all $s > t$.

The same proof works for a Cartesian system when we use Theorem 4.1.9 instead of Corollary 4.2.10. □

We will show that the inverse also holds, that is if $d : \mathbb{Q}_+ \to \mathbb{N}_0$ is a function with $d(0) = 1$ and which satisfies (4.2.1) for all $s > t$, then there exists a rational-time

Cartesian system $X^{>}$ with $\#X_t = d(t)$ for all $t \in \mathbb{Q}_+$. Then, analogous to Example 4.2.3 for the discrete case, there is also a rational-time subproduct system $H^{\otimes}$ with $\mathsf{dim}\, H_t = d(t)$ for every $t \in \mathbb{Q}_+$. We start with the following obvious corollary to Lemma 4.1.10.

4.2.12 Corollary. *There exists a rational-time Cartesian system $\mathcal{X}^{>}$ with $\mathcal{X}_t := \mathbb{N} \times \mathbb{N}$ and $i_{s,t}(i,j) := \big((i,1),(1,j)\big)$ for all $s,t \neq 0$.*

4.2.13 Lemma. *Suppose $m,n : \mathbb{Q}_+ \setminus \{0\} \to \mathbb{N}_0$ are decreasing and $X_t \subset \mathbb{N} \times \mathbb{N}$ such that*

$$\big([m(t)] \times \{1\}\big) \cup \big(\{1\} \times [n(t)]\big) \subset X_t \subset [m(t)] \times [n(t)].$$

Then the X_t together with $X_0 = \{\Lambda\}$ form a Cartesian subsystem $X^{>} \subset \mathcal{X}^{>}$.

Proof. Since m and n are decreasing, we have $[m(s+t)] \subset [m(s)]$ and $[n(s+t)] \subset [n(t)]$ for all s,t. Thus,

$$\begin{aligned} i_{s,t}(X_{s+t}) \subset i_{s,t}\big([m(s+t)] \times [n(s+t)]\big) \subset i_{s,t}\big([m(s)] \times [n(t)]\big) \\ \subset \big([m(s)] \times \{1\}\big) \times \big(\{1\} \times [n(t)]\big) \subset X_s \times X_t. \end{aligned}$$

□

4.2.14 Proposition. *Let $d : \mathbb{Q}_+ \to \mathbb{N}_0$ be a function with $d(0) = 1$ and which fulfills* (4.2.1)*. Then there exists a Cartesian system with $\#X_t = d(t)$ for all $t \in \mathbb{Q}_+$.*

Proof. Let $d : \mathbb{Q}_+ \to \mathbb{N}_0$ be a function and assume (4.2.1) holds for all $s > t$. Put $\hat{d}(t) := \min\{d(s) \mid 0 < s \leq t\}, m(t) := \left\lceil \frac{\hat{d}(t)+1}{2} \right\rceil$ and $n(t) := \left\lfloor \frac{\hat{d}(t)+1}{2} \right\rfloor$. We have $m(t) + n(t) = \hat{d}(t) + 1$, which implies $m(t) + n(t) - 1 = \hat{d}(t)$ and, therefore,

$$m(t) + n(t) - 1 \leq d(t) \leq m(t)n(t)$$

for every $t \in Q_+$. Thus, it is possible to choose $X_t \subset \mathbb{N} \times \mathbb{N}$ with $\#X_t = d(t)$ and

$$\big([m(t)] \times \{1\}\big) \cup \big(\{1\} \times [n(t)]\big) \subset X_t \subset [m(t)] \times [n(t)]$$

Since m and n are decreasing, we can apply Lemma 4.2.13 to conclude that the X_t form a Cartesian subsystem of $\mathcal{X}^{>}$. □

Since the subproduct system $H^{\otimes}$ associated with $X^{>}$ fulfills $\mathsf{dim}\, H_t = \#X_t$, the same holds for subproduct systems. We summarize the results of this section in:

4.2.15 Theorem. *Let $d : \mathbb{Q}_+ \to \mathbb{N}_0$ be a function. Then the following are equivalent:*

(i) There is a rational-time subproduct system $H^{\otimes}$ with $\mathsf{dim}\, H_t = d(t)$ for all $t \in \mathbb{Q}_+$.

(ii) There is a rational-time Cartesian system $X^{>}$ with $\#X_t = d(t)$ for all $t \in \mathbb{Q}_+$.

(iii) The function d fulfills $d(0) = 1$ and (4.2.1) for all $s > t > 0$.

4.2.3 Continuous Time

For Cartesian systems the situation is as nice as in the rational-time case. The proof of Proposition 4.2.14 remains valid when we replace $\mathbb{Q}_+$ by $\mathbb{R}_+$, so we have:

4.2.16 Proposition. *Let $d : \mathbb{R}_+ \to \mathbb{N}_0$ be a function with $d(0) = 1$ and which fulfills (4.2.1) for all $s > t > 0$. Then there exists a continuous-time Cartesian system with $\#X_t = d(t)$ for all $t \in \mathbb{R}_+$.*

For the converse we need the continuous-time analogue of Theorem 4.1.20.

4.2.17 Theorem. *Let $X^{>}$ be a continuous-times Cartesian system with injections $i_{s,t}$ and fix $r \in \mathbb{R}_+$. Then the family $Y^{>} = (Y_t)_{t \in \mathbb{N}}$ with*

$$Y_t = \begin{cases} X_{t+r}, & t > 0 \\ \{\Lambda\}, & t = 0 \end{cases}$$

together with the injections

$$j_{s,t} := (P_1 \circ i_{s+r,t}, P_2 \circ i_{s,r+t})$$

for $s, t \geq 1$ and $j_{s,0}$ and $j_{0,t}$ being (necessarily) the canonical injections is a continuous-time Cartesian system.

Proof. This is just a tedious calculation to check the associativity of the $j_{s,t}$. □

4.2.18 Theorem. *Let $d : \mathbb{R}_+ \to \mathbb{N}_0$ be a function. Then the following are equivalent:*

(*i*) *There is a continuous-time Cartesian system* $X^{>}$ *with* $\#X_t = d(t)$ *for all* $t \in \mathbb{R}_+$.

(*ii*) *The function* d *fulfills* $d(0) = 1$ *and* (4.2.1) *for all* $s > t > 0$.

Proof. Let $X^{>}$ be a continuous-time Cartesian system and $S > T > 0$. Then, there exist $R \in \mathbb{R}_+$ with $R < T$ and $Q \in \mathbb{Q}_+$ with $Q > 1$ such that $S - R = Q(T - R)$. Now, by Theorem 4.2.17, the $Y_t := X_{t+R}$ form a continuous-time Cartesian system. Put $S' := S - R$ and $T' := T - R$, whence $S' = QT'$. The $Z_q := Y_{qT'}$ form a rational-time Cartesian system $Z^{>}$ by Theorem 2.3.2. Since $Z_1 = Y_{T'} = X_T$ and $Z_Q = Y_{QT'} = Y_{S'} = X_S$, applying Proposition 4.2.11 to $Z^{>}$ shows that (4.2.1) holds for S and T. The other direction is Proposition 4.2.16. □

Proposition 4.2.16 holds likewise for subproduct systems, but the proof of the converse statement breaks down. It might well be that the converse is even false in general for subproduct systems. But the main interest in subproduct systems comes from their relation to product systems and for most applications technical assumptions like measurability or contiuity are needed anyway. So the following weaker version of the converse is still useful.

4.2.19 Theorem. *Let* $H^{\circledcirc}$ *be a continuous-time subproduct system and assume that* $t \mapsto \dim H_t$ *is lower semicontinuous. Then* (4.2.1) *holds for all* $s > t > 0$.

Proof. Fix $t \in \mathbb{R}_+$. Lower-semicontinuity implies that $\{s \mid \dim H_s \leq N\}$ is closed for every N. In particular, $\{s \mid (4.2.1) \text{ holds}\}$ is closed. The H_{qt} for $q \in \mathbb{Q}_+$ form a rational-time subproduct system, so (4.2.1) holds for all $s > t > 0$ with $s = qt$ for some rational q. Thus, $\{s \mid (4.2.1) \text{ holds}\}$ is closed and dense in the interval (t, ∞), so it contains (t, ∞). □

5 Universal Products

We have seen that independence can be defined in any tensor category with inclusions; confer Section 2.2.1. Whereas there is essentially only one notion of independence for classical probability spaces, it turned out that there are several such notions for quantum probability spaces, each with a rich theory and connections to other areas of mathematics. The most prominent example is Voiculescu's *freeness*, which has many interrelations with the theory of random matrices and the theory of operator algebras. Soon, the question arose what are "all" notions of independence of quantum probability spaces. One way to make this question precise is to ask for all ways to turn **AlgQ** into a tensor category with inclusions. This approach naturally leads to universal products. Suppose $(\mathbf{AlgQ}, \boxtimes)$ is a tensor category with inclusions ι^1, ι^2. For quantum probability spaces $(\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)$ let $(\mathcal{A}_1, \varphi_1) \boxtimes (\mathcal{A}_2, \varphi_2) = (\mathcal{A}, \varphi)$. Then $\varphi_1 \boxdot \varphi_2 := \varphi \circ (\iota^1 \sqcup \iota^2)$ defines a linear functional on $\mathcal{A}_1 \sqcup \mathcal{A}_2$. Using this definition one gets a new bifunctor

$$\big((\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)\big) \mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \boxdot \varphi_2) : \mathbf{AlgQ} \times \mathbf{AlgQ} \to \mathbf{AlgQ}$$

which turns **AlgQ** into a tensor category with the canonical embeddings into the free product as inclusions and $\iota^1 \sqcup \iota^2$ defines a natural transformation between the two. The axioms of tensor categories with inclusions translate to axioms for $\boxdot$, which are the definition of universal products.

Recently, Lachs discovered a new family of universal products, the (r, s)-products. As a first step towards a detailed study of (r, s)-independence and (r, s)-Lévy-processes, we present a construction, which allows to calculate the GNS-constructions of (r, s)-product functionals.

Section 5.2 is based on [GL14].

5.1 Universal Products

Let I be an arbitrary index set. We put

$$\mathbb{A}_I := \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \mid m \in \mathbb{N}, \varepsilon_k \in I, \varepsilon_k \neq \varepsilon_{k+1}, k = 1, \ldots, m-1\}$$

and define the *length of* ε as $|\varepsilon| := m$ if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$. We simply denote $\mathbb{A}_{\{1,\ldots,k\}}$ by $\mathbb{A}_k$. For $\varepsilon \in \mathbb{A}_I$, $|\varepsilon| = m$ and vector spaces $\mathcal{V}_i, i \in I$, we set

$$\mathcal{V}_\varepsilon := \mathcal{V}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{V}_{\varepsilon_m}.$$

The *free product* of algebras $\mathcal{A}_i, i \in I$, is defined as the vector space

$$\bigsqcup_{i \in I} \mathcal{A}_i := \bigoplus_{\varepsilon \in \mathbb{A}_I} \mathcal{A}_\varepsilon$$

with the multiplication given by

$$(a_1 \otimes \cdots \otimes a_m)(b_1 \otimes \cdots \otimes b_n) := \begin{cases} a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n, & \epsilon_m \neq \delta_1 \\ a_1 \otimes \cdots \otimes a_m b_1 \otimes \cdots \otimes b_n, & \epsilon_m = \delta_1 \end{cases}$$

for all $a_1 \otimes \cdots \otimes a_m \in \mathcal{A}_\varepsilon, b_1 \otimes \cdots \otimes b_n \in \mathcal{A}_\delta$, where $\varepsilon, \delta \in \mathbb{A}_I$, $|\varepsilon| = m, |\delta| = n$. By slight abuse of notation, expressions of the form $a_1 \cdots a_n \in \mathcal{A}_\varepsilon$ are always supposed to signify $|\varepsilon| = n$ and $a_i \in \mathcal{A}_{\varepsilon_i}$. Similarly $a_1 \cdots a_n \in \bigsqcup_{i \in I} \mathcal{A}_i$ shall mean there is an $\varepsilon \in \mathbb{A}_I$ such that $a_1 \cdots a_n \in \mathcal{A}_\varepsilon$, that is $a_1, \ldots, a_n$ are always assumed to belong to alternating algebras. The free product of two algebras is denoted by $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and has the following universal property: For two algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{A}$, $i \in \{1, 2\}$, one gets a unique algebra homomorphism $j_1 \sqcup j_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{A}$ such that $j_1 \sqcup j_2\,(a) = j_i(a)$ for all $a \in \mathcal{A}_i$. In particular, this implies that for two algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{B}_i$ we have a unique algebra homomorphism $j_1 \amalg j_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{B}_1 \sqcup \mathcal{B}_2$ fulfilling $j_1 \amalg j_2\,(a) = j_i(a)$ for all $a \in \mathcal{A}_i$.

5.1.1 Definition. A *universal product* is a prescription $\boxdot$ that assigns to each pair of linear functionals $\varphi_i : \mathcal{A}_i \to \mathbb{C}$ on algebras $\mathcal{A}_i$ a linear functional $\varphi_1 \boxdot \varphi_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathbb{C}$

such that the following axioms hold:

(UP1) $(\varphi_1 \boxdot \varphi_2) \circ (j_1 \amalg j_2) = (\varphi_1 \circ j_1) \boxdot (\varphi_2 \circ j_2)$ for all algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{B}_i$, where $i \in \{1, 2\}$.

(UP2) $(\varphi_1 \boxdot \varphi_2) \boxdot \varphi_3 = \varphi_1 \boxdot (\varphi_2 \boxdot \varphi_3)$ for all linear functionals $\varphi_i : \mathcal{A}_i \to \mathbb{C}$, where $i \in \{1, 2, 3\}$.

(UP3) $(\varphi_1 \boxdot \varphi_2)(a) = \varphi_i(a)$ for all $a \in \mathcal{A}_i \subset \mathcal{A}_1 \sqcup \mathcal{A}_2$ and $i \in \{1, 2\}$.

5.1.2 Example. For algebraic quantum probability spaces $(\mathcal{A}_i, \varphi_i)$, $i \in \{1, 2\}$, the well-known Boolean product $\diamondsuit$ is given by

$$\varphi_1 \diamondsuit \varphi_2(c_1 \cdots c_n) = \varphi_{\varepsilon_1}(c_1) \cdots \varphi_{\varepsilon_n}(c_n)$$

for $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$. This is a universal product. It made early appearances, although not named this way, in the work of von Waldenfels [vW73] and Bozejko [Boż87]. The theory of Boolean convolution was established in [SW97]. Nowadays it is an important part of non-commutative probability theory, see for example the work of Arizmendi and Hasebe [AH13], [AH14].

We know from Section 2.2.1 that independence can be defined in every tensor category with inclusions. The following two propositions explain the close relationship between universal products and notions of independence in quantum probability.

5.1.3 Proposition. *Let $\boxdot$ be a universal product. Then*

$$\big((\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)\big) \mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \boxdot \varphi_2)$$
$$(j_1, j_2) : (\mathcal{A}_1 \times \mathcal{A}_2) \to (\mathcal{A}_1', \mathcal{A}_2') \mapsto j_1 \amalg j_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{A}_1' \sqcup \mathcal{A}_2'$$

is a bifunctor, which turns **AlgQ** *into a tensor category with the canonical embeddings into the free product as inclusions.*

Proof. (UP1) shows that the prescription is a bifunctor. (UP2) guarantees that the natural isomorphism $\mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3) \cong (\mathcal{A}_1 \sqcup \mathcal{A}_2)\mathcal{A}_3$ induces a natural isomorphism of quantum probability spaces. (UP3) shows that the zero algebra with zero functional is a neutral element and the canonical embeddings induce natural transformations of quantum probability spaces. □

We identify the universal product with the bifunctor from the above proposition and simply write $\mathcal{Q}_1 \boxdot \mathcal{Q}_2$ for $(\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \boxdot \varphi_2)$ when $\mathcal{Q}_i = (\mathcal{A}_i, \varphi_i)$.

Let $\boxtimes$ be a bifunctor, which turns **AlgQ** into a tensor category with inclusion ι^1, ι^2. Note that both, ι^1 and ι^2, are families of algebra homomorphisms indexed by **AlgQ** $\times$ **AlgQ** and that $\iota^1_{Q_1,Q_2}$ and $\iota^2_{Q_1,Q_2}$ have the same target algebra. Thus, we can define a family of algebra homomorphisms $\iota^1 \sqcup \iota^2$ given by $(\iota^1 \sqcup \iota^2)_{Q_1,Q_2} := \iota^1_{Q_1,Q_2} \sqcup \iota^2_{Q_1,Q_2}$. To simplify notation, we will sometimes omit the subscripts and just write ι^i instead of $(\iota^i)_{Q_1,Q_2}$.

5.1.4 Proposition. *Let $\boxtimes$ be a bifunctor, which turns* **AlgQ** *into a tensor category with inclusions ι^1, ι^2. Then there is a unique universal product $\boxdot$ such that the family of algebra homomorphims $\iota^1 \sqcup \iota^2$ is a natural transformation $\iota^1 \sqcup \iota^2 : \cdot \boxdot \cdot \Rightarrow \cdot \boxtimes \cdot$.*

Proof. Morphisms in **AlgQ** are functional preserving, so $\boxdot$ is uniquely determined by

$$\varphi_1 \boxdot \varphi_2 := (\varphi_1 \boxtimes \varphi_2) \circ (\iota^1 \sqcup \iota^2)$$

where $\varphi_1 \boxtimes \varphi_2$ denotes the linear functional of $(\mathcal{A}_1, \varphi_1) \boxtimes (\mathcal{A}_2, \varphi_2)$. It remains to show that the prescription defined this way is indeed a universal product. Since ι^1 is a natural transformation, we have

$$(j_1 \boxtimes j_2) \circ (\iota^1 \sqcup \iota^2) \circ i_1 = (j_1 \boxtimes j_2) \circ \iota^1 = \iota^1 \circ j_1 = (\iota^1 \circ j_1) \sqcup (\iota^2 \circ j_2) \circ i_1$$

where i_1 is the canonical inclusion into the free product. Analogously it holds that

$$(j_1 \boxtimes j_2) \circ (\iota^1 \sqcup \iota^2) \circ i_2 = (\iota^1 \circ j_1) \sqcup (\iota^2 \circ j_2) \circ i_2.$$

By the universal property of the free product we get

$$(j_1 \boxtimes j_2) \circ (\iota^1 \sqcup \iota^2) = (\iota^1 \circ j_1) \sqcup (\iota^2 \circ j_2)$$

and thus

$$\begin{aligned}((\varphi_1 \circ j_1) \boxdot (\varphi_2 \circ j_2)) &= ((\varphi_1 \circ j_1) \boxtimes (\varphi_2 \circ j_2)) \circ (\iota^1 \sqcup \iota^2) \\ &= (\varphi_1 \boxtimes \varphi_2) \circ (j_1 \boxtimes j_2) \circ (\iota^1 \sqcup \iota^2) \\ &= (\varphi_1 \boxtimes \varphi_2) \circ ((\iota^1 \circ j_1) \sqcup (\iota^2 \circ j_2)) \\ &= (\varphi_1 \boxtimes \varphi_2) \circ (\iota^1 \sqcup \iota^2) \circ (j_1 \sqcup j_2) \\ &= (\varphi_1 \boxdot \varphi_2) \circ (j_1 \sqcup j_2)\end{aligned}$$

which proves (UP1). (UP2) and (UP3) are easy to check. □

Let S be a set. Recall that S^* denotes the set of all finite tuples over S and Λ denotes the empty tuple. A *tuple partition* of S is a set of tuples $\Pi = \{V_1, \ldots, V_\ell\}, V_i \in S^* \setminus \{\Lambda\}$ such that every element of S belongs to one and only one of the tuples V_i. The set of all tuple partitions of S is denoted by $\mathrm{TP}(S)$. A tuple partition of $\{1, \ldots, n\}$ is called *compatible* with $\varepsilon \in \mathbb{A}_I$ if $|\varepsilon| = n$ and, for all $i, j \in \{1, \ldots, n\}$ that belong to the same block of Π, one has $\varepsilon_i = \varepsilon_j$. The set of all tuple partitions compatible with ε is denoted by $\mathrm{TP}(\varepsilon)$. We simply write $\mathrm{TP}(n)$ for $\mathrm{TP}(\{1, \ldots, n\})$. Let $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}, \varphi_2 : \mathcal{A}_2 \to \mathbb{C}$ be linear functionals, and $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$. For a tuple $U = (i_1, \ldots, i_m)$ such that all c_{i_k} belong to the same algebra $\mathcal{A}_j$ we use the shorthand notation

$$\varphi_U(c_1 \cdots c_n) := \varphi_j(c_{i_1} \cdots c_{i_m}).$$

Furthermore, for $\varepsilon \in \mathbb{A}_2$, $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$ and a tuple partition $\Pi \in \mathrm{TP}(\varepsilon)$ we write

$$\varphi_\Pi(c_1 \cdots c_n) := \prod_{U \in \Pi} \varphi_U(c_1 \cdots c_n).$$

5.1.5 Theorem. *Let $\boxdot$ be a universal product. Then there exist unique constants $t_\varepsilon(\Pi)$ for every $k \in \mathbb{N}$, $\varepsilon \in \mathbb{A}_k$ and $\Pi \in \mathrm{TP}(\varepsilon)$ such that*

$$\varphi_1 \boxdot \cdots \boxdot \varphi_k(c_1 \cdots c_n) = \sum_{\Pi \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Pi) \varphi_\Pi(c_1 \cdots c_n)$$

for all $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$.

Theorem 5 of [BGS02] deals with the case $k=2$ and commutative universal products, but the proof relies on UP(UP1) only, hence it applies to our more general situation. See also [Mur03, Theorem 3.1].

In the following we call a universal product which fulfills

$$t_{(1,2)}(\{(1),(2)\}) = r \quad \text{and} \quad t_{(2,1)}(\{(1),(2)\}) = s$$

an (r,s)-*universal product*. By Theorem 5.1.5 every universal product is an (r,s)-universal product for unique constants $r,s\in\mathbb{C}$. If $r=s=q$ we speak of a q-*universal product*. A 1-universal product is also called a *normalized* universal product.

5.1.6 Observation. Let $\boxdot$ be an (r,s)-universal product. Then one easily checks that $\varphi_1 \boxdot^{\mathrm{op}} \varphi_2 := \varphi_2 \boxdot \varphi_1$ defines an (s,r)-universal product. For the universal coefficients $t_\varepsilon^{\boxdot}(\Pi)$ and $t_\varepsilon^{\boxdot^{\mathrm{op}}}(\Pi)$ one finds

$$t_\varepsilon^{\boxdot}(\Pi) = t_{\overline{\varepsilon}}^{\boxdot^{\mathrm{op}}}(\Pi)$$

where $\overline{\varepsilon}_k := 1$ if $\varepsilon_k = 2$ and vice versa.

5.2 The (r,s)-Products

For a tuple $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)\in\mathbb{N}^n$ we call a position $k\in\{1,\ldots,n-1\}$ an *up* if $\varepsilon_k<\varepsilon_{k+1}$ and a *down* if $\varepsilon_k>\varepsilon_{k+1}$. Denote the set of all ups by $u(\varepsilon)$ and the set of all downs by $d(\varepsilon)$. The sets $u(\varepsilon)$ and $d(\varepsilon)$ are always disjoint One has $u(\varepsilon)\cup d(\varepsilon)=\{1,\ldots,n-1\}$ if and only if $\varepsilon_k\neq\varepsilon_{k+1}$ for all $k\in\{1,\ldots,n-1\}$, that is if $\varepsilon\in\mathbb{A}_{\mathbb{N}}$.

5.2.1 Definition. Let $r,s\in\mathbb{C}$ be fixed. The (r,s)-*product* of two functionals $\varphi_i:\mathcal{A}_i\to\mathbb{C}$ is defined as $\varphi_1 \curlywedge \varphi_2 : \mathcal{A}_1\sqcup\mathcal{A}_2\to\mathbb{C}$ with

$$\varphi_1 \curlywedge \varphi_2(a_1\ldots a_n) := r^{\#u(\varepsilon)} s^{\#d(\varepsilon)} \varphi_1 \diamond \varphi_2(a_1\ldots a_n)$$

for all $a_1\ldots a_n\in\mathcal{A}_\varepsilon$.

At first, only commutative universal products have been considered. It can be shown that there are exactly three commutative q-universal products for every $q\neq 0$, see Speicher [Spe97] and Ben Ghorbal and Schürmann [BGS02]. The classification of noncommutative 1-universal products was done by Muraki [Mur03] and it can easily be extended to show that there are exactly five q-universal products for every $q \neq 0$. By discovering the (r, s)-products the classification of general universal products was completed (except for the case $q = 0$ which is still open).

5.2.2 Theorem ([GL14, Theorem 3.21], see also [Lac14])**.** *Let* $\boxdot$ *be an* (r, s)*-universal product with* $r \neq s$*. Then* $\boxdot$ *coincides with the* (r, s)*-product* $\curlywedge$*.*

In this section we will perform the GNS-construction for the (r, s)-product of two linear functionals. Since the (r, s)-product is not preserving positivity, we have to generalize the usual GNS-construction to not necessarily positive functionals. When comparing with the case of the Boolean product, that is $r = s = 1$, strange things happen to the dimension of the representation space. Since the (r, s)-product of two homomorphisms is not a homomorphism, the dimension can increase, see Example 5.2.14. It is also possible that the dimensions coincide, as is shown in example 5.2.17, or even that the dimension is smaller than in the Boolean case, see Example 5.2.18.

5.2.1 Dual Pairs

We briefly mention some basic definitions and facts.

5.2.3 Definition. A *semi-dual pair* consists of a pair of vector spaces (E, F) and a bilinear form $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{C}$. A semi-dual pair is called a *dual pair* if its bilinear form is *non-degenerate* in the sense that

- $\langle e, f \rangle = 0$ for all $e \in E$ implies $f = 0$ and
- $\langle e, f \rangle = 0$ for all $f \in F$ implies $e = 0$.

In that case the bilinear form is called *dual pairing* of (E, F).

Given a semi-dual pair (E, F) and a subset $M \subset E$, the *orthogonal space* of M is

$$M^{\perp} := \left\{ f \in F \;\middle|\; \langle e, f \rangle = 0 \; \forall e \in M \right\}$$

and similarly for a subset $N \subset F$

$$N^\perp := \left\{ e \in E \;\middle|\; \langle e, f \rangle = 0 \;\forall f \in N \right\}.$$

The subspaces $F^\perp \subset E$ and $E^\perp \subset F$ are called *degeneracy spaces* of (E, F).

5.2.4 Proposition. *Let (E, F) be a semi-dual pair. Denote by $[e]$ and $[f]$ the equivalence classes of e and f in $E/F^\perp$ and $F/E^\perp$ respectively. Then*

$$\langle [e], [f] \rangle := \langle e, f \rangle$$

gives a well-defined dual pairing on $(E/F^\perp, F/E^\perp)$.

Proof. Let $e \in E, e' \in F^\perp, f \in F, f' \in E^\perp$. Then

$$\langle e + e', f + f' \rangle = \langle e, f \rangle$$

by bilinearity. This shows well-definedness. To show non-degeneracy assume $\langle [e], [f] \rangle = 0$ for all $f \in F$. Since $\langle [e], [f] \rangle = \langle e, f \rangle$ we get $e \in F^\perp$, hence $[e] = 0$. Analogously $\langle [e], [f] \rangle = 0$ for all $e \in E$ implies $[f] = 0$. □

5.2.5 Example. Any complex $m \times n$ matrix B defines a bilinear form

$$\langle x, y \rangle := x^t B y$$

on $\mathbb{C}^m \times \mathbb{C}^n$. We have $F^\perp = \left\{ x \in \mathbb{C}^m \;\middle|\; x^t B y = 0 \;\forall y \in \mathbb{C}^n \right\}$ so $\operatorname{dim} F^\perp = m - \operatorname{rank} B$ and $\operatorname{dim} E/F^\perp = m - \operatorname{dim} F^\perp = \operatorname{rank} B$. Similarly, $\operatorname{dim} F/E^\perp = \operatorname{rank} B$.

5.2.6 Definition. Let $\mathcal{A}$ be an algebra. A *(semi-)dual pair of $\mathcal{A}$-modules* is a (semi-)dual pair (E, F) such that

- E is a right $\mathcal{A}$-module,
- F is a left $\mathcal{A}$-module,
- $\langle ea, f \rangle = \langle e, af \rangle$ for all $e \in E$, $a \in \mathcal{A}$ and $f \in F$.

If $\mathcal{A}$ has a unit $1_{\mathcal{A}}$, (E, F) is called *unital* if $e1_{\mathcal{A}} = e$ and $1_{\mathcal{A}}f = f$ for all $e \in E, f \in F$.

5.2.7 Proposition. *Let $\mathcal{A}$ be an algebra and (E, F) a semi-dual pair of $\mathcal{A}$-modules. If the subspace $U \subset E$ is invariant under the right-action of $\mathcal{A}$ on E, then $U^\perp$ is invariant under the left-action of $\mathcal{A}$ on F.*

Proof. Let $U \subset E$ be invariant, that is $e \in U$ implies $ea \in U$ for all $a \in \mathcal{A}$. For $f \in U^\perp$ and arbitrary $a \in \mathcal{A}$ we get

$$\langle e, af \rangle = \langle ea, f \rangle = 0$$

for all $e \in U$, that is $af \in U^\perp$. □

In particular $F^\perp \subset E$ is invariant. Of course, we can switch the roles of E and F to show $E^\perp \subset F$ is invariant.

5.2.8 Theorem. *Let (E, F) be a semi-dual pair of $\mathcal{A}$ modules. Then $(E/F^\perp, F/E^\perp)$ is a dual pair of $\mathcal{A}$ modules with actions and dual pairing given by*

$$[e]a = [ea], \qquad a[f] = [af] \quad \textit{and} \quad \langle [e], [f] \rangle = \langle e, f \rangle$$

for all $e \in E, f \in F$ and $a \in \mathcal{A}$.

Proof. The pair $(E/F^\perp, F/E^\perp)$ is a dual pair by Proposition 5.2.4. The given actions are well-defined by proposition 5.2.7, . Furthermore $(E/F^\perp, F/E^\perp)$ is a dual pair of $\mathcal{A}$ modules, since

$$\begin{aligned} \langle [e]a, [f] \rangle = \langle [ea], [f] \rangle = \langle ea, f \rangle = \\ \langle e, af \rangle = \langle [e], [af] \rangle = \langle [e], a[f] \rangle \end{aligned}$$

for all $e \in E, f \in F$ and $a \in \mathcal{A}$. □

5.2.2 A Generalized GNS-Construction

Every algebra $\mathcal{A}$ acts on itself from the right and from the left by multiplication. For any linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ the bilinear form $(a, b) \mapsto \varphi(ab) : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ turns

$(\mathcal{A},\mathcal{A})$ into a semi-dual pair of $\mathcal{A}$-modules. We denote by N_R^φ and N_L^φ the degeneracy spaces. Define $E^\varphi := \mathcal{A}/N_R^\varphi$ and $F^\varphi := \mathcal{A}/N_L^\varphi$. By the preceding theorem (E^φ, F^φ) is a dual pair of $\mathcal{A}$-modules. Furthermore, if $\mathcal{A}$ is unital und $\varphi(1) = 1$, then we define $\Omega^\varphi := [1] \in E^\varphi$ and $\Xi^\varphi := [1] \in F^\varphi$. Then it holds that

$$\langle \Omega^\varphi, a\Xi^\varphi \rangle = \varphi(1a1) = \varphi(a) \tag{5.2.1}$$

for all $a \in \mathcal{A}$ and $E^\varphi = \Omega^\varphi \mathcal{A}$, $F^\varphi = \mathcal{A}\Xi^\varphi$.

5.2.9 Definition. Let $\mathcal{A}$ be an algebra and E a right $\mathcal{A}$-module. A vector $\Omega \in E$ is called *cyclic* if $\Omega\mathcal{A} = E$ and *quasi-cyclic* if $\mathbb{C}\Omega + \Omega\mathcal{A} = E$.

In other words, a vector $\Omega \in E$ is quasi-cyclic, if and only if the smallest submodule of E that contains Ω equals E. Cyclicity and quasi-cyclicity for left modules are defined likewise.

For a unital algebra with normalized linear functional (E^φ, F^φ) is a dual pair of $\mathcal{A}$-modules with cyclic vectors $\Omega^\varphi, \Xi^\varphi$ from which we can recover the functional by (5.2.1).

Denote by $\widetilde{\mathcal{A}}$ the unitization of $\mathcal{A}$, that is the unital algebra with underlying vector space $\widetilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$ and product $(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + a\mu + ab)$. Then a right $\mathcal{A}$-module E can be turned into a right $\widetilde{\mathcal{A}}$-module by setting $e(1,a) := e + ea$. Clearly, a vector $\Omega \in E$ is quasi-cyclic for the $\mathcal{A}$-action if and only if it is cyclic for the corresponding $\widetilde{\mathcal{A}}$-action. For a functional $\varphi : \mathcal{A} \to \mathbb{C}$ define $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \mathbb{C}$ with $\widetilde{\varphi}(\lambda, a) := \lambda + \varphi(a)$. Using $\widetilde{\mathcal{A}}$ and $\widetilde{\varphi}$ instead of $\mathcal{A}$ and φ we can always use the construction above to find a dual pair of $\mathcal{A}$ modules which allows us to reconstruct φ by (5.2.1).

5.2.10 Proposition. *Let (E,F) be a dual pair of $\mathcal{A}$-modules, $\Omega \in E$, $\Xi \in F$ quasi-cyclic vectors with*

$$\langle \Omega, \Xi \rangle = 1 \quad \textit{and} \quad \langle \Omega, a\Xi \rangle = \varphi(a) \tag{5.2.2}$$

for all $a \in \mathcal{A}$. Then there is a unique pair of module isomorphisms $U : E^{\widetilde{\varphi}} \to E$, $T : F^{\widetilde{\varphi}} \to F$ with $U(\Omega^{\widetilde{\varphi}}) = \Omega$ and $T(\Xi^{\widetilde{\varphi}}) = \Xi$. It holds that $\langle Ue, Tf \rangle = \langle e, f \rangle$ for all $e \in E^{\widetilde{\varphi}}, f \in F^{\widetilde{\varphi}}$.

Proof. Since $\Omega^{\widetilde{\varphi}}$ and $\Xi^{\widetilde{\varphi}}$ are quasi-cyclic, U and T are uniquily determined, if they exist. We have

$$\left\langle \Omega^{\widetilde{\varphi}} a, b\Xi^{\widetilde{\varphi}} \right\rangle = \varphi(ab) = \langle \Omega a, b\Xi \rangle \quad \text{and} \quad \left\langle \Omega^{\widetilde{\varphi}} a, \Xi^{\widetilde{\varphi}} \right\rangle = \varphi(a) = \langle \Omega a, \Xi \rangle$$

for all $a, b \in \mathcal{A}$. Together this yields $\left\langle \Omega^{\widetilde{\varphi}} a, f \right\rangle = \langle \Omega a, f \rangle$ for all $f \in F$. Provided U and T exist this also settles the last equality in the proposition. Since $(E^{\widetilde{\varphi}}, F^{\widetilde{\varphi}})$ is a dual pair, the pairing is non-degenerate. So

$$\begin{aligned} \Omega^{\widetilde{\varphi}} a = 0 &\Leftrightarrow \left\langle \Omega^{\widetilde{\varphi}} a, f \right\rangle = 0 \text{ for all } f \in F \\ &\Leftrightarrow \langle \Omega a, f \rangle = 0 \text{ for all } f \in F \\ &\Leftrightarrow \Omega a = 0. \end{aligned}$$

This shows that $U : \lambda\Omega^{\widetilde{\varphi}} + \Omega^{\widetilde{\varphi}} a \mapsto \lambda\Omega + \Omega a$ is well defined. The existence of T follows analogously. □

5.2.11 Definition. Let $\mathcal{A}$ be an algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional. A dual pair of $\mathcal{A}$-modules (E, F) with quasi-cyclic vectors Ω, Ξ is called *GNS-pair* of $(\mathcal{A}, \varphi)$ if it fulfills (5.2.2).

We have seen that a GNS-pair always exists and is unique up to *isometric isomorphism* in the sense of Proposition 5.2.10. In particular, for $\mathcal{A}$ a unital algebra and $\varphi(1) = 1$ we have $(E^{\varphi}, F^{\varphi}) \cong (E^{\widetilde{\varphi}}, F^{\widetilde{\varphi}})$, since they are both GNS-pairs of $(\mathcal{A}, \varphi)$.

5.2.12 Example. Let $\mathcal{A}$ be an algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a homomorphism, that is $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. Then $(\mathbb{C}, \mathbb{C})$ becomes a dual pair of $\mathcal{A}$-modules with canonical dual pairing $\langle \lambda, \mu \rangle := \lambda\mu$ and left and right actions on $\mathbb{C}$ by

$$\lambda a := \lambda\varphi(a), \quad a\mu := \varphi(a)\mu.$$

The unit $1 \in \mathbb{C}$ is quasi-cyclic for these actions and $(\mathbb{C}, \mathbb{C})$ with $\Omega := 1, \Xi := 1$ is obviously a GNS-pair of $(\mathcal{A}, \varphi)$. Also note that Ω and Ξ are cyclic if and only if $\varphi \neq 0$.

The converse holds, that is if the GNS-modules are one-dimensional, then φ is a homomorphism. For let (E, F) with quasi-cyclic vektors $\Omega \in E, \Xi \in F$ be a GNS-pair with $\dim E = \dim F = 1$. So $E = \mathbb{C}\Omega, F = \mathbb{C}\Xi$ and $\langle \lambda\Omega, \mu\Xi = \lambda\mu\rangle$. From $\langle \Omega, a\Xi\rangle = \varphi(a)$ we conclude $a\Xi = \varphi(a)\Xi$. Since $\varphi(ab)\Xi = (ab)\Xi = a(b\Xi) = \varphi(a)\varphi(b)\Xi$ for all $a, b \in \mathcal{A}$, φ is a homomorphism.

5.2.3 GNS-Construction for the (r,s)-Products

Fix $r, s \in \mathbb{C}$ and denote by $\curlywedge$ the (r,s)-product. Let (E_i, F_i) with quasi-cyclic vectors Ω_i, Ξ_i be a GNS-pair of $(\mathcal{A}_i, \varphi_i)$ for $i = 1, 2$. In the following we will present a way to express the GNS-pair of $(\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \curlywedge \varphi_2)$ in terms of the respective GNS-pairs (E_i, F_i). Set

$$E := \mathbb{C}\Omega \oplus \Omega_1\mathcal{A}_1 \oplus \Omega_2\mathcal{A}_2 \quad \text{and} \quad F := \mathbb{C}\Xi \oplus \mathcal{A}_1\Xi_1 \oplus \mathcal{A}_2\Xi_2. \tag{5.2.3}$$

and define a semi-dual pairing on (E, F) by

$$\begin{aligned}
&\langle \Omega, \Xi\rangle = 1 && \langle \Omega, b_1\Xi_1\rangle = \langle b_1\rangle && \langle \Omega, b_2\Xi_2\rangle = \langle b_2\rangle \\
&\langle \Omega_1 a_1, \Xi\rangle = \langle a_1\rangle && \langle \Omega_1 a_1, b_1\Xi_1\rangle = \langle a_1 b_1\rangle && \langle \Omega_1 a_1, b_2\Xi_2\rangle = r\langle a_1\rangle\langle b_2\rangle \\
&\langle \Omega_2 a_2, \Xi\rangle = \langle a_2\rangle && \langle \Omega_2 a_2, b_1\Xi_1\rangle = s\langle a_2\rangle\langle b_1\rangle && \langle \Omega_2 a_2, b_2\Xi_2\rangle = \langle a_2 b_2\rangle
\end{aligned} \tag{5.2.4}$$

where $\langle a\rangle := \varphi_i(a)$ for $a \in \mathcal{A}_i$. Furthermore, set

$$(\lambda\Omega + \Omega_1 a_1 + \Omega_2 a_2)b := \begin{cases} \lambda\Omega_1 b + \Omega_1 a_1 b + s\varphi_2(a_2)\Omega_1 b \text{ for } b \in \mathcal{A}_1 \\ \lambda\Omega_2 b + r\varphi_1(a_1)\Omega_2 b + \Omega_2 a_2 b \text{ for } b \in \mathcal{A}_2 \end{cases} \tag{5.2.5}$$

and

$$b(\mu\Xi + c_1\Xi_1 + c_2\Xi_2) := \begin{cases} \mu b\Xi_1 + bc_1\Xi_1 + r\varphi_2(c_2)b\Xi_1 \text{for } b \in \mathcal{A}_1 \\ \mu b\Xi_2 + s\varphi_1(c_1)b\Xi_2 + bc_2\Xi_2 \text{for } b \in \mathcal{A}_2. \end{cases} \tag{5.2.6}$$

5.2.13 Theorem. *The aligns* (5.2.5),(5.2.6) *define actions of* $\mathcal{A}_1 \sqcup \mathcal{A}_2$ *such that the semidual pair* (E, F) *in* (5.2.3) *becomes a semi-dual pair of* $\mathcal{A}_1 \sqcup \mathcal{A}_2$*-modules. The dual pair* $(E/F^\perp, F/E^\perp)$ *with the vectors* $\Omega + F^\perp, \Xi + E^\perp$ *is the GNS-pair of* $\varphi_1 \curlywedge \varphi_2$.

Proof. Straightforward. □

5.2.14 Example. Suppose $0 \neq \varphi_i : \mathcal{A}_i \to \mathbb{C}$ are homomorphisms. Then by Example 5.2.12 $E_i = \mathbb{C}\Omega_i$ and $F_i = \mathbb{C}\Xi_i$ for $i = 1, 2$. So $E, F \cong \mathbb{C}^3$ and the semi-dual pairing (5.2.4) is determined by the matrix

$$B := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & r \\ 1 & s & 1 \end{pmatrix}$$

in the sense of Example 5.2.5. We have

$$\operatorname{rank} B = \begin{cases} 1 & \text{for } r, s = 1 \\ 2 & \text{for } r = 1, s \neq 1 \text{ or } r \neq 1, s = 1 \\ 3 & \text{for } r, s \neq 1 \end{cases}$$

so the dimension of the the GNS-pair of $(\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \curlywedge \varphi_2)$ depends on r and s.

Can it happen that the semi-dual pairing (5.2.4) is degenerate even if $r, s \neq 1$? Before we give two more examples let us do some general considerations.

5.2.15 Lemma. *Let (E, F) be a GNS-pair of $(\mathcal{A}, \varphi)$. Then*

$$\Omega\hat{a} = \Omega \Leftrightarrow \varphi(\hat{a}) = 1 \text{ and } \varphi(\hat{a}b) = \varphi(b) \forall b \in \mathcal{A}.$$

Proof. Straightforward. □

5.2.16 Proposition. *Let $r, s \neq 1$. If Ω_1 and Ω_2 are cyclic, then $E^\perp = \{0\}$.*

Proof. Cyclicity means $\Omega_1\mathcal{A}_1 = E_1$ and $\Omega_2\mathcal{A}_2 = E_2$. Thus we can rewrite E of (5.2.3) as

$$E = \mathbb{C}\Omega \oplus E_1 \oplus E_2.$$

In particular, there exist $\hat{a}_1 \in \mathcal{A}_1, \hat{a}_2 \in \mathcal{A}_2$ with $\Omega_1 = \Omega_1\hat{a}_1, \Omega_2 = \Omega_2\hat{a}_2 \in E$. Let

$f = \mu\Xi + b_1\Xi_1 + b_2\Xi_2 \in E^{\perp}$. Using (5.2.4), we get the system of linear aligns

$$\begin{aligned}
\langle \Omega, f\rangle &= \mu + \varphi_1(b_1) + \varphi_2(b_2) = 0\\
\langle \Omega_1, f\rangle &= \mu + \varphi_1(b_1) + r\varphi_2(b_2) = 0\\
\langle \Omega_2, f\rangle &= \mu + s\varphi_1(b_1) + \varphi_2(b_2) = 0
\end{aligned}$$

which is determinate for $r, s \neq 1$. Hence $\mu = \varphi_1(b_1) = \varphi_2(b_2) = 0$. Furthermore

$$\langle \Omega_1 a_1, f\rangle = \varphi_1(a_1 b_1) = \langle \Omega_1 a_1, b_1\Xi_1\rangle = 0$$

for all $a_1 \in \mathcal{A}_1$. Using, again, cyclicity of Ω_1 and non-degeneracy of (E_1, F_1) we conclude $b_1\Xi_1 = 0$. In the same way we get $b_2\Xi_2 = 0$ and finally $f = 0$. □

So in order to get more interesting examples it is necessary that the quasicyclic vectors for the GNS-pair of at least one of the functionals are not cyclic.

5.2.17 Example. Let $0 \neq \varphi_1 : \mathcal{A}_1 \to \mathbb{C}$ be a homomorphism and $\varphi_2 : \mathbb{C}_0[x] \to \mathbb{C}$ given by

$$\varphi_2(x^m) := \begin{cases} 1 & \text{for } m = 2\\ 0 & \text{for } m \neq 2. \end{cases}$$

We already know the GNS-pair of φ_1 is $E_1 = \mathbb{C}\Omega_1$ and $F_1 = \mathbb{C}\Xi_1$. To determine the GNS-pair of φ_2 we calculate

$$\begin{aligned}
\widetilde{N} &= \left\{ p = \sum_{i=0}^{n} \alpha_i x^i \mid \widetilde{\varphi_2}(pq) = 0 \quad \forall q \in \mathbb{C}[x] \right\}\\
&= \left\{ p \mid \widetilde{\varphi_2}(px^k) = 0 \quad \forall k \in \mathbb{N}_0 \right\}\\
&= \left\{ p = \sum_{i=0}^{n} \alpha_i x^i \mid \alpha_0 = \alpha_1 = \alpha_2 = 0 \right\}
\end{aligned}$$

which yields

$$E_2 = F_2 = {}^{\mathbb{C}[x]}/_{\widetilde{N}} = \operatorname{span}\left\{[1], [x], [x^2]\right\}.$$

Setting $\Omega_2 = \Xi_2 := [1]$ we get

$$\Omega_2\mathbb{C}_0[x] = \operatorname{span}\{\Omega_2 x, \Omega_2 x^2\}, \mathbb{C}_0[x]\Xi_2 = \operatorname{span}\{x\Xi_2, x^2\Xi_2\}.$$

and

$$E = \operatorname{span}\{\Omega, \Omega_1, \Omega_2 x, \Omega_2 x^2\}, F = \operatorname{span}\{\Xi, \Xi_1, x\Xi_2, x^2\Xi_2\}$$

with the semidual pairing determined by

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & r \\ 0 & 0 & 1 & 1 \\ 1 & s & 0 & 0 \end{pmatrix}.$$

We find

$$\operatorname{rank} B = \begin{cases} 3 & \text{for } r = 1 \text{ or } s = 1 \\ 4 & \text{for } r, s \neq 1. \end{cases}$$

So the dimension of the GNS-pair of $\varphi_1 \curlywedge \varphi_2$ can be equal to the one in the Boolean case $r = s = 1$ for other values of r and s.

5.2.18 Example. As a third example consider $\varphi_1 = \varphi_2 : \mathbb{C}_0[x] \to \mathbb{C}$ with

$$\varphi_i(x^m) := \begin{cases} 1 & \text{for } m = 2 \\ 0 & \text{for } m \neq 2. \end{cases}$$

We already know the GNS-pairs of φ_1 and φ_2. From these we construct

$$E = \operatorname{span}\{\Omega, \Omega_1 x, \Omega_1 x^2, \Omega_2 x, \Omega_2 x^2\}, F = \operatorname{span}\{\Xi, x\Xi_1, x^2\Xi_1, x\Xi_2, x^2\Xi_2\}$$

with the semidual pairing determined by

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & s & 0 & 0 \end{pmatrix}.$$

We calculate $\det B = -rs + r + s$ and thus find

$$\operatorname{rank} B = \begin{cases} 4 & \text{for } rs = r + s \\ 3 & \text{otherwise.} \end{cases}$$

This example shows that, surprisingly, the dimension of the GNS-module of $\varphi_1 \curlywedge \varphi_2$ can even be smaller than in the Boolean case $r = s = 1$.

5.2.19 Remark. One may ask if these dimension phenomena can also arise for the normalized universal products, when one allows non-positive linear functionals. This it not the case. The same constructions one uses to build the joint GNS-representations of two states from their respective GNS-representations can be applied to build the joint GNS-modules for general linear functionals. Non-degeneracy of the pairings is automatically preserved.

5.2.4 GNS-Triples

The GNS-construction for nonpositive functionals has also been investigated by Wilhelm [Wil08]. We show that his approach is equivalent to ours. He defines:

5.2.20 Definition. A *GNS-triple* (π, H_0, ω) is a triple, consisting of a vector space H^0, a representation $\pi : A \to L(H)$ with $H = \mathbb{C} \oplus H^0$ and the vector $\omega = (1, 0)$ such that

1. ω is quasi-cyclic for π

2. there is no nonzero invariant subspace of H_0.

Let (π, H_0, ω) be a GNS-triple. Set $F := H = \mathbb{C}\omega \oplus H_0$. The representation $\pi : A \to L(H)$ yields a left action $af := \pi(a)f$ on F and a right action $ea := e \circ \pi(a)$ on the algebraic dual F'. Let $E := \mathbb{C}P_\omega + P_\omega A$ be the submodule generated by P_ω.

5.2.21 Proposition. *(E, F) is a dual pair of A-modules with the canonical dual pairing*

$$\langle e, f \rangle := e(f) \tag{5.2.7}$$

Proof. Clearly, (5.2.7) defines a bilinear form on $E \times F$ and fulfills $\langle ea, f \rangle = \langle e, af \rangle$ for all $e \in E, a \in A, f \in F$. Hence we have a semi-dual pair of A-modules. Obviously $\langle e, f \rangle = 0$ for all $f \in F$ implies $e = 0$, since e is a linear functional on F. Suppose $\langle e, f \rangle = 0$ for all $e \in E$. Since $P_\omega \in E$, we have $f \in H_0$. Furthermore we get $P_\omega af = 0$ for all $a \in A$, hence $\mathbb{C}f + Af \subset H_0$ is an invariant subspace. So $f = 0$. □

5.2.22 Proposition. *(E, F) with the vectors $\Omega := P_\omega, \Xi := \omega$ is a GNS-pair of the functional $\varphi(a) := P_\omega a\omega$.*

Proof. Ξ is quasi-cyclic by (1) in Definition 5.2.20 and Ω is cyclic by definition of E. We have

$$\langle \Omega, a\Xi \rangle = P_\omega \pi(a)\omega = \varphi(a)$$

for all $a \in A$. □

On the other hand

5.2.23 Proposition. *Let (E, F) be a dual pair of A-modules with quasi-cyclic vectors Ω, Ξ. Then $(\pi, \Omega^\perp, \Xi)$ a GNS-triple, where $\pi : A \to L(F)$ is the representation defined by $\pi(a)f := af$.*

Proof. It is immediately clear that π is a representation and Ξ is quasi-cyclic. Let $U \subset \Omega^\perp$ be an invariant subspace and $u \in U$. Then $\langle \Omega, u \rangle = 0$ and $\langle \Omega a, u \rangle = \langle \Omega, au \rangle = 0$ for all $a \in A$. Since Ω is quasi-cyclic, this shows $\langle e, u \rangle = 0$ for all $e \in E$, hence $u = 0$. □

Bibliography

[Abe80] Eiichi Abe, *Hopf algebras*, Cambridge Tracts in Mathematics, vol. 74, Cambridge University Press, Cambridge, 1980, Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka. MR 594432 (83a:16010)

[ÁFMNW09] Bernardo M. Ábrego, Silvia Fernández-Merchant, Michael G. Neubauer, and William Watkins, *Sum of squares of degrees in a graph*, JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 3, Article 64, 34. MR 2551087 (2010k:05060)

[AH13] Octavio Arizmendi and Takahiro Hasebe, *Semigroups related to additive and multiplicative, free and Boolean convolutions*, Studia Math. **215** (2013), no. 2, 157–185. MR 3071490

[AH14] ______, *Classical and free infinite divisibility for Boolean stable laws*, Proc. Amer. Math. Soc. **142** (2014), no. 5, 1621–1632. MR 3168468

[Aha80] Ron Aharoni, *A problem in rearrangements of* (0, 1) *matrices*, Discrete Math. **30** (1980), no. 3, 191–201. MR 573634 (81i:05036)

[AHS04] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories: the joy of cats*, Online Edition, 2004, http://katmat.math.uni-bremen.de/acc.

[AK78] R. Ahlswede and G. O. H. Katona, *Graphs with maximal number of adjacent pairs of edges*, Acta Math. Acad. Sci. Hungar. **32** (1978), no. 1-2, 97–120. MR 505076 (80g:05053)

[Ani82] David J. Anick, *Noncommutative graded algebras and their Hilbert series*, J. Algebra **78** (1982), no. 1, 120–140. MR 677714 (84g:16001)

[Ara70] Huzihiro Araki, *Factorizable representation of current algebra. Non commutative extension of the Lévy-Kinchin formula and cohomology of a solvable group with values in a Hilbert space*, Publ. Res. Inst. Math. Sci. **5** (1969/1970), 361–422. MR 0263326 (41 #7931)

[Arv89] W. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc. **80** (1989), no. 409, iv+66. MR 987590 (90f:47061)

[ASvW88] Luigi Accardi, Michael Schürmann, and Wilhelm von Waldenfels, *Quantum independent increment processes on superalgebras*, Math. Z. **198** (1988), no. 4, 451–477. MR 950578 (89h:46092)

[BB05] József Balogh and Béla Bollobás, *Hereditary properties of words*, Theor. Inform. Appl. **39** (2005), no. 1, 49–65. MR 2132578 (2006c:68148)

[BFS09] *Mini-Workshop: Product Systems and Independence in Quantum Dynamics*, 2009, Abstracts from the mini-workshop held February 15–21, 2009, Organized by B. V. Rajarama Bhat, Uwe Franz and Michael Skeide, Oberwolfach Reports. Vol. 6, no. 1, pp. 493–547. MR 2604064

[BGS02] Anis Ben Ghorbal and Michael Schürmann, *Non-commutative notions of stochastic independence*, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 3, 531–561. MR 1919720 (2003k:46096)

[BGS05] ———, *Quantum Lévy processes on dual groups*, Math. Z. **251** (2005), no. 1, 147–165. MR 2176469 (2006i:81109)

[Bha96] B. V. Rajarama Bhat, *An index theory for quantum dynamical semigroups*, Trans. Amer. Math. Soc. **348** (1996), no. 2, 561–583. MR 1329528 (96g:46059)

[BM10] B. V. Rajarama Bhat and Mithun Mukherjee, *Inclusion systems and amalgamated products of product systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13** (2010), no. 1, 1–26. MR 2646788 (2011f:46087)

[Bou89] Nicolas Bourbaki, *Algebra. I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989, Translated from the French, Reprint of the 1974 edition. MR 979982 (90d:00002)

[Bou04] ———, *Theory of sets*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004, Reprint of the 1968 English translation [Hermann, Paris; MR0237342]. MR 2102219

[Boż87] Marek Bożejko, *Uniformly bounded representations of free groups*, J. Reine Angew. Math. **377** (1987), 170–186. MR 887407 (89a:22009)

[BS00] B. V. Rajarama Bhat and Michael Skeide, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), no. 4, 519–575. MR 1805844 (2001m:46149)

[CMR98] M. Crochemore, F. Mignosi, and A. Restivo, *Automata and forbidden words*, Inform. Process. Lett. **67** (1998), no. 3, 111–117. MR 1638178 (99d:68204)

[DRS11] Kenneth R. Davidson, Christopher Ramsey, and Orr Moshe Shalit, *The isomorphism problem for some universal operator algebras*, Adv. Math. **228** (2011), no. 1, 167–218. MR 2822231 (2012j:46083)

[Fer99] Sébastien Ferenczi, *Complexity of sequences and dynamical systems*, Discrete Math. **206** (1999), no. 1-3, 145–154, Combinatorics and number theory (Tiruchirappalli, 1996). MR 1665394 (2000f:68093)

[FKLS14] Uwe Franz, Anna Kula, J. Martin Lindsay, and Michael Skeide, *Hunt's formula for $SU_q(N)$ and $U_q(N)$*, Preprint, in preparation, 2014.

[Fra06] Uwe Franz, *Lévy processes on quantum groups and dual groups*, Quantum independent increment processes. II, Lecture Notes in Math., vol. 1866, Springer, Berlin, 2006, pp. 161–257. MR 2213449 (2007g:81063)

[FS99] Uwe Franz and René Schott, *Stochastic processes and operator calculus on quantum groups*, Mathematics and its Applications, vol. 490, Kluwer Academic Publishers, Dordrecht, 1999. MR 1746647 (2002e:81103)

[FSS03] Uwe Franz, René Schott, and Michael Schürmann, *Lévy processes and brownian motion on braided spaces*, 2003.

[Ger63] Murray Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288. MR 0161898 (28 #5102)

[Ger64] ———, *On the deformation of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103. MR 0171807 (30 #2034)

[Ger09] Malte Gerhold, *Lévy-Prozesse auf additiven Deformationen von Bialgebren*, Diploma thesis, Universität Greifswald, 2009.

[Ger11] ———, *Additive deformations of Hopf algebras*, J. Algebra **339** (2011), 114–134. MR 2811315 (2012e:16079)

[GJ79] Ian P. Goulden and David M. Jackson, *An inversion theorem for cluster decompositions of sequences with distinguished subsequences*, J. London Math. Soc. (2) **20** (1979), no. 3, 567–576. MR 561149 (83b:05005)

[GKL12] Malte Gerhold, Stefan Kietzmann, and Stephanie Lachs, *Additive deformations of braided Hopf algebras*, Noncommutative harmonic analysis with applications to probability III, Banach Center Publ., vol. 96, Polish Acad. Sci. Inst. Math., Warsaw, 2012, pp. 175–191. MR 2986826

[GL14] Malte Gerhold and Stephanie Lachs, *Classification and GNS-construction for general universal products*, accepted for publication in Infinite Dimensional Analysis, Quantum Probability and Related Topics, 2014.

[GO81] Leonidas J. Guibas and Andrew M. Odlyzko, *String overlaps, pattern matching, and nontransitive games*, J. Combin. Theory Ser. A **30** (1981), no. 2, 183–208. MR 611250 (82g:05007)

[GS14a] M. Gerhold and M. Skeide, *Interacting Fock spaces and subproduct systems*, Preprint, in preparation, 2014.

[GS14b] Malte Gerhold and Michael Skeide, *Discrete subproduct systems and word systems*, 2014.

[Gui72] Alain Guichardet, *Symmetric Hilbert spaces and related topics*, Lecture Notes in Mathematics, Vol. 261, Springer-Verlag, Berlin-New York, 1972, Infinitely divisible positive definite functions. Continuous products and tensor products. Gaussian and Poissonian stochastic processes. MR 0493402 (58 #12416)

[IS08] Masaki Izumi and R. Srinivasan, *Generalized CCR flows*, Comm. Math. Phys. **281** (2008), no. 2, 529–571. MR 2410905 (2009c:46096)

[Joh71] R. E. Johnson, *Unique factorization monoids and domains*, Proc. Amer. Math. Soc. **28** (1971), 397–404. MR 0277453 (43 #3186)

[Kat71] Moshe Katz, *Rearrangements of* $(0, -1)$ *matrices*, Israel J. Math. **9** (1971), 53–72. MR 0271131 (42 #6014)

[KS97] Anatoli Klimyk and Konrad Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997. MR 1492989 (99f:17017)

[Lac14] Stephanie Lachs, *Non-positive universal products*, Ph.D. thesis, EMAU Greifswald, planned 2014.

[Lie09] Volkmar Liebscher, *Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces*, Mem. Amer. Math. Soc. **199** (2009), no. 930, xiv+101. MR 2507929 (2010i:46105)

[LS05] J. Martin Lindsay and Adam G. Skalski, *Quantum stochastic convolution cocycles. I*, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), no. 3, 581–604. MR 2139034 (2006b:81146)

[LS08] ______, *Quantum stochastic convolution cocycles. II*, Comm. Math. Phys. **280** (2008), no. 3, 575–610. MR 2399608 (2009h:81170)

[LS12] ______, *Quantum stochastic convolution cocycles III*, Math. Ann. **352** (2012), no. 4, 779–804. MR 2892452

[Mac27] F. S. MacAulay, *Some Properties of Enumeration in the Theory of Modular Systems*, Proc. London Math. Soc. **S2-26** (1927), no. 1, 531. MR 1576950

[Maj95] Shahn Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995. MR 1381692 (97g:17016)

[McL92] Colin McLarty, *Elementary categories, elementary toposes*, Oxford Logic Guides, vol. 21, The Clarendon Press, Oxford University Press, New York, 1992, Oxford Science Publications. MR 1182992 (94c:18001)

[MH38] Marston Morse and Gustav A. Hedlund, *Symbolic Dynamics*, Amer. J. Math. **60** (1938), no. 4, 815–866. MR 1507944

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872 (2001j:18001)

[MS02] Paul S. Muhly and Baruch Solel, *Quantum Markov processes (correspondences and dilations)*, Internat. J. Math. **13** (2002), no. 8, 863–906. MR 1928802 (2003j:46100)

[Mur03] Naofumi Muraki, *The five independences as natural products*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), no. 3, 337–371. MR 2016316 (2005h:46093)

[Odl85] Andrew M. Odlyzko, *Enumeration of strings*, Combinatorial algorithms on words (Maratea, 1984), NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., vol. 12, Springer, Berlin, 1985, pp. 205–228. MR 815341 (87e:68087)

[Pow04] Robert T. Powers, *Addition of spatial E_0-semigroups*, Operator algebras, quantization, and noncommutative geometry, Contemp. Math., vol.

365, Amer. Math. Soc., Providence, RI, 2004, pp. 281–298. MR 2106824 (2005m:46110)

[PPS99] Uri N. Peled, Rossella Petreschi, and Andrea Sterbini, *(n, e)-graphs with maximum sum of squares of degrees*, J. Graph Theory **31** (1999), no. 4, 283–295. MR 1698746 (2000d:05064)

[PS72] K. R. Parthasarathy and K. Schmidt, *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory*, Lecture Notes in Mathematics, Vol. 272, Springer-Verlag, Berlin-New York, 1972. MR 0622034 (58 #29849)

[RBS05] B. V. Rajarama Bhat and R. Srinivasan, *On product systems arising from sum systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (2005), no. 1, 1–31. MR 2126876 (2006e:46075)

[Sch85] Michael Schürmann, *Positive and conditionally positive linear functionals on coalgebras*, Quantum probability and applications, II (Heidelberg, 1984), Lecture Notes in Math., vol. 1136, Springer, Berlin, 1985, pp. 475–492. MR 819527 (87d:46079)

[Sch93] ———, *White noise on bialgebras*, Lecture Notes in Mathematics, vol. 1544, Springer-Verlag, Berlin, 1993. MR 1238942 (95i:81128)

[Sch05] ———, *Lévy processes on deformations of Hopf algebras*, Infinite dimensional harmonic analysis III, World Sci. Publ., Hackensack, NJ, 2005, pp. 277–287. MR 2230636 (2007e:60034)

[Shu06] Arseny M. Shur, *Factorial languages of low combinatorial complexity*, Developments in language theory, Lecture Notes in Comput. Sci., vol. 4036, Springer, Berlin, 2006, pp. 397–407. MR 2335422

[Shu09] ———, *On intermediate factorial languages*, Discrete Appl. Math. **157** (2009), no. 7, 1669–1675. MR 2510251 (2010h:68095)

[Ske03] Michael Skeide, *Commutants of von Neumann modules, representations of $\mathcal{B}^a(E)$ and other topics related to product systems of Hilbert modules*, Advances in quantum dynamics (South Hadley, MA, 2002), Contemp. Math., vol. 335, Amer. Math. Soc., Providence, RI, 2003, pp. 253–262. MR 2029630 (2004k:46124)

[Ske08] ______, *Isometric dilations of representations of product systems via commutants*, Internat. J. Math. **19** (2008), no. 5, 521–539. MR 2418195 (2009i:46126)

[Ske09] ______, *Unit vectors, Morita equivalence and endomorphisms*, Publ. Res. Inst. Math. Sci. **45** (2009), no. 2, 475–518. MR 2510509 (2010h:46113)

[Spe97] Roland Speicher, *On universal products*, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 257–266. MR 1426844 (98c:46141)

[SS98] Michael Schürmann and Michael Skeide, *Infinitesimal generators on the quantum group* $\mathrm{SU}_q(2)$, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** (1998), no. 4, 573–598. MR 1665276 (2000b:81083)

[SS09] Orr Moshe Shalit and Baruch Solel, *Subproduct systems*, Doc. Math. **14** (2009), 801–868. MR 2608451 (2011k:46095)

[Sta78] Richard P. Stanley, *Hilbert functions of graded algebras*, Advances in Math. **28** (1978), no. 1, 57–83. MR 0485835 (58 #5637)

[Str69] Raymond F. Streater, *Current commutation relations, continuous tensor products and infinitely divisible group representations*, Local quantum theory (R. Jost, ed.), Academic Press, 1969.

[SW97] Roland Speicher and Reza Woroudi, *Boolean convolution*, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 267–279. MR 1426845 (98b:46084)

[Swe69] M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969. MR 0252485 (40 #5705)

[Tsi00a] B. Tsirelson, *From random sets to continuous tensor products: answers to three questions of W. Arveson*, Preprint, arXiv: math.FA/0001070, 2000.

[Tsi00b] ______, *From slightly coloured noises to unitless product systems*, Preprint, arXiv: math.FA/0006165, 2000.

[Tsi09a] Boris Tsirelson, *Graded algebras and subproduct systems: dimension two*, Preprint, arXiv:
math.OA/0906.4255, 2009.

[Tsi09b] ______, *Subproduct systems of hilbert spaces: dimension two*, 2009.

[Ufe04] S. Ufer, *Braided Hopf algebras of triangular type*, Ph.D. thesis, Ludwig-Maximilians-Universität München, 2004.

[Vos13] Stefan Voss, *Realisierung von Quanten-Lévy-Prozessen auf Fockräumen*, Ph.D. thesis, Universität Greif, 2013.

[vW73] Wilhelm von Waldenfels, *An approach to the theory of pressure broadening of spectral lines*, Probability and information theory, II, Springer, Berlin, 1973, pp. 19–69. Lecture Notes in Math., Vol. 296. MR 0421496 (54 #9495)

[vW84] ______, *Itô solution of the linear quantum stochastic differential equation describing light emission and absorption*, Quantum probability and applications to the quantum theory of irreversible processes (Villa Mondragone, 1982), Lecture Notes in Math., vol. 1055, Springer, Berlin, 1984, pp. 384–411. MR 782919 (86e:81165)

[Wil08] Lukas Wilhelm, *Natürliche Produkte von Darstellungen*, Diploma thesis, Universität Greifswald, 2008.

[Wir02] Jerome Wirth, *Formule de Lévy Khintchine et deformations d'algebres*, Ph.D. thesis, Universite Paris VI, 2002.

[Wor87] S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), no. 4, 613–665. MR 901157 (88m:46079)

[Wor98] ———, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884. MR 1616348 (99m:46164)

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Index of Notation

Notation is only listed here if it is used in several places with the same meaning. It can happen that a symbol has a different meaning than listed here in a specific context.

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