The Spinor Bundle on Loop Space and its Fusion Product

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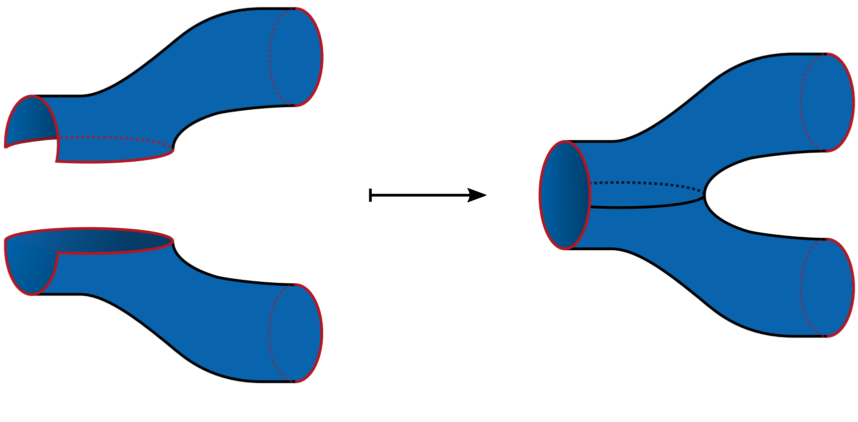


Abstract

Given a manifold with a string structure, we construct a spinor bundle on its loop space. Our construction is in analogy with the usual construction of a spinor bundle on a spin manifold, but necessarily makes use of tools from infinite dimensional geometry. We equip this spinor bundle on loop space with an action of a bundle of Clifford algebras. Given two smooth loops in our string manifold that share a segment, we can construct a third loop by deleting this segment. If this third loop is smooth, then we say that the original pair of loops is a pair of compatible loops. It is well-known that this operation of fusing compatible loops is important if one wants to understand the geometry of a manifold through its loop space. In this work, we explain in detail how the spinor bundle on loop space behaves with respect to fusion of compatible loops. To wit, we construct a family of fusion isomorphisms indexed by pairs of compatible loops in our string manifold. Each of these fusion isomorphisms is an isomorphism from the relative tensor product of the fibres of the spinor bundle over its index pair of compatible loops to the fibre over the loop that is the result of fusing the index pair. The construction of a spinor bundle on loop space equipped with a fusion product as above was proposed by Stolz and Teichner with the goal of studying the "Dirac operator on loop space". Our construction combines facets of the theory of bimodules for von Neumann algebras, infinite dimensional manifolds, and Lie groups and their representations. We moreover place our spinor bundle on loop space in the context of bundle gerbes and bundle gerbe modules.

Contents

1. Introduction

The main contribution of this thesis is to prove a conjecture postulated by Stolz and Teichner in their 2005 survey/preprint [ST, Theorem 1]. The conjecture concerns the existence of a spinor bundle on the free loop space of a string manifold. Moreover, an important part of the conjecture is the existence of a *fusion product* on this spinor bundle. We recall some necessary background and then give an outline of our construction which positively resolves the conjecture.

The Whitehead tower of the orthogonal group in d dimensions, $\mathrm{O}(d)$, starts out

$$\mathrm{O}(d) \leftarrow \mathrm{SO}(d) \leftarrow \mathrm{Spin}(d) \leftarrow \mathrm{String}(d) \leftarrow \dots \tag{1.1}$$

The topological spaces $\mathrm{O}(d)$, $\mathrm{SO}(d)$, and $\mathrm{Spin}(d)$ are equipped with their familiar Lie group structures. The topological space $\mathrm{String}(d)$ can be equipped with a group structure such that it is a topological group, and such that the projection map $\mathrm{String}(d) \to \mathrm{Spin}(d)$ is a group homomorphism, see [Sto96, Theorem 1.2 & Theorem 5.1]. Let M be a smooth spin manifold of dimension d. A *string structure* on M is a lift of the spin frame bundle to a principal $\mathrm{String}(d)$-bundle, or equivalently, it is a lift of the classifying map of the spin frame bundle as in the following diagram

$$\begin{array}{ccc} & & B\,\mathrm{String}(d) \\ & \nearrow & \downarrow \\ M & \longrightarrow & B\,\mathrm{Spin}(d) \end{array}$$

Such a lift exists exactly when the first fractional Pontryagin class, $p_1(M)/2 \in H^4(M, \mathbb{Z})$, vanishes, in which case the string structures on M form a torsor over $H^3(M, \mathbb{Z})$, (see [McL92, ST]).

The study of string structures goes back to Killingback, see [Kil87]. In that article Killingback finds sufficient conditions for the existence of a "spin structure on loop space". We now summarize some of the key findings and definitions of that article. Let M be a spin manifold, and let $\mathrm{Spin}(M)$ be its spin frame bundle. The looped spin frame bundle $L\,\mathrm{Spin}(M) := C^\infty(S^1, \mathrm{Spin}(M)) \to LM$ is then a principal $L\,\mathrm{Spin}(d)$-bundle. The Fréchet Lie group $L\,\mathrm{Spin}(d)$ has an interesting central extension,

$$\mathrm{U}(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d),$$

called its *basic* central extension, see [PS86]. A lift of $L\,\mathrm{Spin}(M)$ to a principal $\widetilde{L\,\mathrm{Spin}}(d)$-bundle $\widetilde{L\,\mathrm{Spin}}(M)$ is called a *spin structure on loop space*. It is worth mentioning that a spin structure on a loop space is *not* the reduction of its frame bundle to a spin frame bundle, as in the finite dimensional case. In the article [Kil87] Killingback argues that the existence of a spin structure on loop space implies the absence of world-sheet anomalies for a string moving through the manifold M. Killingback furthermore argues that if $p_1(M)/2 = 0$, then there exists a spin structure on loop space. In other words, if a manifold admits a string structure, then its loop space admits a spin structure. What's more, a string structure on M can be "transgressed" to a spin structure on LM, we review this construction in Section 7.3. This is how string geometry on M and spin geometry on LM make contact. The following question now arises naturally: "What constructions on spin manifolds can be generalized to the loop space of a string manifold?"

We give a first answer to this question by constructing a spinor bundle on loop space from a spin structure on that loop space. The general outline of this construction was known, see for example [McL92, Section 3]. However, when working out the details one encounters some difficulties related to the fact that the loop space of a manifold is an infinite-dimensional manifold. Our construction of the spinor bundle on loop space as a rigged Hilbert space bundle resolves these issues, and appears to be new. Next, we assume that this spin structure on loop space comes from a string structure on the underlying manifold. Using this string structure we then construct a "fusion product" on the spinor bundle on loop space, thus proving the aforementioned conjecture by Stolz and Teichner, [ST, Theorem 1]. This fusion product does not have an analogue in finite dimensions. From a physics perspective, the spinor bundle on loop space is an interesting object because it should be to strings what the spinor bundle on a manifold is to point particles. One of the reasons that the spinor bundle on loop space is interesting from a mathematics standpoint is that it should come equipped with a Dirac operator on loop space, an object whose construction is a long open problem. The Dirac operator on loop space has often been the subject of formal considerations, which in some way bypass its full construction, see among many others [Wit86, Tau89, Bry90, Lan99]. Its rigorous construction might shed new insights on the Stolz-Höhn conjecture [Sto96].

Before we outline the construction of the spinor bundle on loop space, we recall the construction of the spinor bundle of a spin manifold M of dimension d. This starts with the construction of the Clifford algebra $\mathrm{Cl}(V)$ for any finite dimensional inner product space V. Next, one constructs an irreducible representation Δ_d of $\mathrm{Cl}(\mathbb{R}^d)$, called the space of spinors. It turns out that $\mathrm{Spin}(d)$ acts in Δ_d. We obtain a bundle of Clifford algebras by associating the Clifford algebra $\mathrm{Cl}(\mathbb{R}^d)$ to the oriented orthogonal frame bundle $\mathrm{SO}(M)$ of the tangent bundle of M, i.e. $\mathrm{Cl}(M) \coloneqq \mathrm{SO}(M) \times_{\mathrm{SO}(d)} \mathrm{Cl}(\mathbb{R}^d)$. The spinor bundle is then defined to be $\mathbb{S}(M) \coloneqq \mathrm{Spin}(M) \times_{\mathrm{Spin}(d)} \Delta_d$. Finally, using the action of $\mathrm{Cl}(\mathbb{R}^d)$ on Δ_d, one constructs an action of $\mathrm{Cl}(M)$ on $\mathbb{S}(M)$.

In this work we carry out an analogous construction where the base manifold M is replaced by its loop space LM, and where the spin group is replaced by the basic central extension of $L\,\mathrm{Spin}(d)$. The result is called the spinor bundle on loop space, and denoted $\mathcal{F}(LM)$, see Section 8.2. We moreover equip the spinor bundle on loop space with a *fusion product*, see Section 8.3. A fusion product gives us a relation between fibres over loops that can be glued together as follows. We parametrize the circle by the interval $[0, 2\pi]$. Let γ_{12} and γ_{23} be smooth loops in M with the property that the second half of γ_{12} coincides with the first half of γ_{23}, or more precisely, $\gamma_{12}(2\pi - t) = \gamma_{23}(t)$ for $t \in [0, \pi]$. If the loop γ_{13} defined by $\gamma_{13}(t) = \gamma_{12}(t)$ for $t \in [0, \pi]$ and $\gamma_{13}(t) = \gamma_{23}(t)$ for $t \in [\pi, 2\pi]$ is smooth, then we say that the pair of loops $(\gamma_{12}, \gamma_{23})$ is a *compatible pair of loops.* A fusion product is now a family of isomorphisms

$$\mathcal{F}(LM)_{\gamma_{12}} \otimes \mathcal{F}(LM)_{\gamma_{23}} \to \mathcal{F}(LM)_{\gamma_{13}}.$$

indexed by compatible pairs of loops $(\gamma_{12}, \gamma_{23})$, which moreover satisfy an associativity condition over compatible triples of loops, (a compatible triple of loops is a triple of loops, $(\gamma_{12}, \gamma_{23}, \gamma_{34})$ with the property that $(\gamma_{12}, \gamma_{23})$ and $(\gamma_{23}, \gamma_{34})$ are compatible pairs of loops). We shall give an outline of our construction of such a fusion product momentarily. This fusion product fits into a general pattern, which could be described by the slogan: "A geometric structure on a space M induces a geometric structure of one degree lower on its loop space LM, which comes equipped with a fusion product." This process of turning a geometric structure on a space into a geometric structure on its loop space is called *transgression.* We shall not endeavour to make this slogan precise, instead, we illustrate it with some examples. A first example is the statement that a spin structure on a manifold induces an orientation on its loop space, see [McL92], equipped with fusion, see [ST], (note that in this reference, Stolz and Teichner absorb the fusion product into the definition of an orientation on loop space). A second example is the statement that a multiplicative bundle gerbe over a group G transgresses to a central $\mathrm{U}(1)$-extension $\widetilde{LG}$ over its loop group LG, equipped with a fusion product, i.e. a family of isomorphisms $\widetilde{LG}_{\gamma_{12}} \otimes \widetilde{LG}_{\gamma_{23}} \to \widetilde{LG}_{\gamma_{13}}$, see [Wal17]. We consider bundle gerbes in Appendix D, for now, we just mention that bundle gerbes are higher versions of line bundles. As a third example we note that a string structure on a manifold transgresses to a *fusive* spin structure on its loop space, i.e. a spin structure on loop space, equipped with a fusion product. These second and third examples play an important role in our construction of the fusion product on the spinor bundle on loop space, which is why we discuss them in some detail in Chapters 6 and 7. These results on transgression and fusion were not known at the time [ST] was written, and using them to resolve the conjecture is one of the central parts of this work. We should remark that, for technical reasons, it turns out to be convenient to work with compatible triples of paths instead of compatible pairs of loops. This is what we shall do in the main text.

We now give an overview of our construction of the spinor bundle on loop space and its fusion product. By analogy with the spin case, we see that we should find a vector space $\mathcal{F}_L$, analogous to Δ_d, which is both a module for $\widetilde{L\,\mathrm{Spin}}(d)$ and for some appropriate Clifford algebra. One of the main difficulties here is that it is not immediately clear what the "appropriate" Clifford algebra is, which is aggravated by the fact that there are many slightly different approaches to defining infinite dimensional Clifford algebras, our approach, described in Chapter 3, is consistent with [PR94]. The Clifford algebra and the module that we end up working with are known as the "Majorana free fermions on the circle", (from now on simply the free fermions). These also appear in [DH], where Douglas and Henriques use them to construct a model for the string group. The Clifford algebra is the Clifford C*-algebra of the Hilbert space $V := L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d)$, where $\mathbb{S}$ is the odd spinor bundle on the circle. In string theory terminology this means that we are working with Neveu-Schwarz boundary conditions. We explain the construction of the Clifford algebra $\mathrm{Cl}(V)$ and the construction of its "Fock module" $\mathcal{F}_L$ in Chapter 3. Suffice to say for now that it depends crucially on the choice of a Lagrangian subspace $L \subset V$, whence the notation $\mathcal{F}_L$. The group $L\,\mathrm{SO}(d)$ acts in the Clifford algebra by algebra automorphisms. This means we can already define a bundle of Clifford algebras $\mathrm{Cl}(LM) := L\,\mathrm{SO}(M) \times_{L\,\mathrm{SO}(d)} \mathrm{Cl}(V) \to LM$. For each loop γ there is a natural isomorphism of C*-algebras $\mathrm{Cl}(LM)_\gamma \to \mathrm{Cl}(L^2(S^1, \mathbb{S} \otimes \gamma^* TM_{\mathbb{C}}))$, these Clifford algebras were also considered in [ST, Example 15]. The next step is to equip $\mathcal{F}_L$ with a representation of $\widetilde{L\,\mathrm{Spin}}(d)$, which is done as follows. There is a subgroup $\mathrm{O}_L(V) \subset \mathrm{O}(V)$, called the restricted orthogonal group, which acts projectively in $\mathcal{F}_L$. The deprojectivization then yields a central extension $\mathrm{U}(1) \to \mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$, the group $\mathrm{Imp}_L(V)$ is known as the group of implementers. The group $L\,\mathrm{Spin}(d)$ acts in V through $\mathrm{O}_L(V)$, so that it is possible to pull back the $\mathrm{U}(1)$-extension $\mathrm{Imp}_L(V)$ to a central extension of $L\,\mathrm{Spin}(d)$. Using techniques from [PS86] we prove that this pullback is in fact $\widetilde{L\,\mathrm{Spin}}(d)$. We can then complete the first part of our construction by defining the spinor bundle on loop space to be

$$\mathcal{F}(LM) := \widetilde{L\,\mathrm{Spin}}(M) \times_{\widetilde{L\,\mathrm{Spin}}(d)} \mathcal{F}_L.$$

Using the action of $\mathrm{Cl}(V)$ on $\mathcal{F}_L$ we equip the bundle $\mathcal{F}(LM)$ with an action of the bundle $\mathrm{Cl}(LM)$.

The above construction works well in the setting of topological manifolds. However, an important feature of the way we set up the construction is that, with care, it allows us to adjust it so that we obtain Fréchet manifolds instead of topological manifolds. The main challenge in defining a smooth structure on the associated bundle $\mathcal{F}(LM)$ is a typical one for representations of infinite dimensional Lie groups: the map $\widetilde{L\,\mathrm{Spin}}(d) \times \mathcal{F}_L \to \mathcal{F}_L$ is not smooth. In Chapter 3 we prove, using tools from [Nee10a, Nee10b], that there is a dense subspace $\mathcal{F}_L^{\mathrm{s}} \subset \mathcal{F}_L$, which can be equipped with the structure of Fréchet space, such that the map $\widetilde{L\,\mathrm{Spin}}(d) \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_L^{\mathrm{s}}$ is smooth. And similarly there is a dense subspace $\mathrm{Cl}(V)^{\mathrm{s}} \subset \mathrm{Cl}(V)$ which can be equipped with the structure of Fréchet algebra such that the map $L\,\mathrm{Spin}(d) \times \mathrm{Cl}(V)^{\mathrm{s}} \to \mathrm{Cl}(V)^{\mathrm{s}}$ is smooth. We prove that, equipped with these Fréchet structures, the action $\mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_L^{\mathrm{s}}$ is smooth. This is sufficient to conclude that the bundles

$$\mathcal{F}^{\mathrm{s}}(LM) := \widetilde{L\,\mathrm{Spin}}(M) \times_{\widetilde{L\,\mathrm{Spin}(d)}} \mathcal{F}_L^{\mathrm{s}} \qquad \text{and} \qquad \mathrm{Cl}^{\mathrm{s}}(LM) := L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{Cl}(V)^{\mathrm{s}}$$

are Fréchet vector bundles over LM, and that the map $\mathrm{Cl}^{\mathrm{s}}(LM) \times_{LM} \mathcal{F}^{\mathrm{s}}(LM) \to \mathcal{F}^{\mathrm{s}}(LM)$ is smooth. The fact that $\mathcal{F}^{\mathrm{s}}(LM)$ is a Fréchet vector bundle is important because the Dirac operator $D : \Gamma(\mathcal{F}^{\mathrm{s}}(LM)) \to \Gamma(\mathcal{F}^{\mathrm{s}}(LM))$ should be a differential operator, and thus in particular an endomorphism of the *smooth* sections. The bundle $\mathcal{F}^{\mathrm{s}}(LM)$ comes equipped with a Hermitian metric, such that for each $\gamma \in LM$ the completion of $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ is $\mathcal{F}(LM)_\gamma$. This means that $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ is a rigged Hilbert space, and that the bundle $\mathcal{F}^{\mathrm{s}}(LM)$ is what we call a *rigged Hilbert space bundle*, see Section 8.1. This extra structure will be important when we consider the fusion product.

At this point it is appropriate to compare the construction we have described so far to the work done in [Amb12]. There, Ambler supposes given an oriented rank k vector bundle $E \to M$, with k even, and a trivialization of a certain bundle gerbe over LM. From this, Ambler constructs a Clifford algebra bundle, denoted $\mathrm{Cl}(LE)$, and a module bundle, denoted S, over LM. Our construction is a specialization of the work by Ambler, in the sense that we have set $E = TM$, however our work is more general in the following two ways. By twisting with the odd spinor bundle $\mathbb{S}$ we are able to drop the condition that the rank of E (hence the dimension of M) is even. We have moreover constructed a Fréchet vector bundle $\mathcal{F}^{\mathrm{s}}(LM)$, which is a fibrewise dense subbundle of Ambler's purely topological bundle S. The connection between our work and Ambler's is examined more closely in Section 9.1.

We proceed with our discussion of the fusion product on $\mathcal{F}(LM)$. We first observe that the fibres of $\mathcal{F}(LM)$ can be viewed as bimodules over certain Clifford algebras as follows. Recall that we have a bundle of Clifford algebras $\mathrm{Cl}(LM)$ over the loop space, whose fibre at a loop γ may be identified with the Clifford algebra of $\mathcal{H}_\gamma := L^2(S^1, \mathbb{S} \otimes \gamma^* TM_{\mathbb{C}})$. We may thus consider the Clifford algebras corresponding to the subspaces $\mathcal{H}_\gamma^+$ and $\mathcal{H}_\gamma^-$ defined to consist of those sections supported in the top or bottom part of the circle, respectively. The Clifford algebras $\mathrm{Cl}(\mathcal{H}_\gamma^\pm)$ include into the Clifford algebra $\mathrm{Cl}(\mathcal{H}_\gamma)$, and in fact $\mathcal{F}(LM)_\gamma$ becomes a $\mathrm{Cl}(\mathcal{H}_\gamma^+)$-$\mathrm{Cl}(\mathcal{H}_\gamma^-)^{\mathrm{opp}}$-bimodule, where $\mathrm{Cl}(\mathcal{H}_\gamma^-)^{\mathrm{opp}}$ stands for the opposite algebra of $\mathrm{Cl}(\mathcal{H}_\gamma^-)$. Let $(\gamma_{12}, \gamma_{23})$ be a compatible pair of loops in M, and define γ_{13} by $\gamma_{13}(t) = \gamma_{12}(t)$ for $t \in [0, \pi]$ and $\gamma_{13}(t) = \gamma_{23}(t)$ for $t \in [\pi, 2\pi]$. We can then identify $\mathrm{Cl}(\mathcal{H}_{\gamma_{12}}^+)$ with $\mathrm{Cl}(\mathcal{H}_{\gamma_{13}}^+)$, and $\mathrm{Cl}(\mathcal{H}_{\gamma_{23}}^-)^{\mathrm{opp}}$ with $\mathrm{Cl}(\mathcal{H}_{\gamma_{13}}^-)^{\mathrm{opp}}$, and finally we can identify $\mathrm{Cl}(\mathcal{H}_{\gamma_{12}}^-)^{\mathrm{opp}}$ with $\mathrm{Cl}(\mathcal{H}_{\gamma_{23}}^+)$. This last identification allows us to consider the relative tensor product

$$\mathcal{F}(LM)_{\gamma_{12}} \otimes_{\mathrm{Cl}(\mathcal{H}_{\gamma_{23}}^+)} \mathcal{F}(LM)_{\gamma_{23}},$$

which is then, using the other two identifications, a $\mathrm{Cl}(\mathcal{H}_{\gamma_{13}}^+)$-$\mathrm{Cl}(\mathcal{H}_{\gamma_{13}}^-)^{\mathrm{opp}}$-bimodule, just like $\mathcal{F}(LM)_{\gamma_{13}}$. The conjecture of Stolz and Teichner was then that there exists a family of isomorphisms of bimodules, indexed by pairs of compatible loops, $(\gamma_{12}, \gamma_{23})$,

$$\mathcal{F}(LM)_{\gamma_{12}} \otimes_{\mathrm{Cl}(\mathcal{H}_{\gamma_{23}}^+)} \mathcal{F}(LM)_{\gamma_{23}} \to \mathcal{F}(LM)_{\gamma_{13}}, \tag{1.2}$$

which satisfy an associativity condition for triples of compatible loops. These isomorphisms are called *fusion isomorphisms*. Our main result is the construction of such a family of isomorphisms. At this point two remarks are in order. First, the identification of $\mathrm{Cl}(\mathcal{H}_{\gamma_{12}}^-)^{\mathrm{opp}}$ with $\mathrm{Cl}(\mathcal{H}_{\gamma_{23}}^+)$ is non-trivial, care must be taken to get this right. Second, the algebras $\mathrm{Cl}(\mathcal{H}_{\gamma_{ij}}^\pm)$ must be completed to von Neumann algebras, and the relative tensor product we use is the Connes fusion product, which we shall denote by $\boxtimes$ from now on. We recall the definition of the Connes fusion product and some of its elementary properties in Chapter 4.

Let us now give some of the main steps in this construction. First, we construct the fusion isomorphism, (1.2), for the case that the base manifold M is a point. We write $V^+ \subset V$ and $V^- \subset V$

for the subspaces consisting of sections supported in the top or bottom part of the circle, respectively. The Hilbert space $\mathcal{F}_L$ is then a $\mathrm{Cl}(V^+)$-$\mathrm{Cl}(V^-)^{\mathrm{opp}}$-bimodule, and there is an isomorphism $\mathrm{Cl}(V^-)^{\mathrm{opp}} \to \mathrm{Cl}(V^+)$. We construct an isomorphism of $\mathrm{Cl}(V^+)$-$\mathrm{Cl}(V^-)^{\mathrm{opp}}$-bimodules

$$\mathcal{F}_L \boxtimes_{\mathrm{Cl}(V^+)} \mathcal{F}_L \to \mathcal{F}_L,$$

and then prove that this isomorphism is associative. This is done in Chapter 5. We now return to the case that M is an arbitrary string manifold. Picking local trivializations $\mathcal{F}(LM)_{\gamma_{ij}} \to \mathcal{F}_L$ would then allow us to define a fusion isomorphism through the diagram

$$\begin{array}{ccc} \mathcal{F}(LM)_{\gamma_{12}} \boxtimes_{\mathrm{Cl}(\mathcal{H}^+_{\gamma_{23}})} \mathcal{F}(LM)_{\gamma_{23}} & \dashrightarrow & \mathcal{F}(LM)_{\gamma_{13}} \\ \downarrow & & \uparrow \\ \mathcal{F}_L \boxtimes_{\mathrm{Cl}(V^+)} \mathcal{F}_L & \longrightarrow & \mathcal{F}_L \end{array}$$

However, if we pick these trivializations arbitrarily, then there is no way to tell if the induced fusion isomorphisms satisfy the associativity condition over triples of compatible loops. To get around this, we find a way to construct a trivialization over γ_{13} from trivializations over γ_{12} and γ_{23} as follows. We observe that, because $\mathcal{F}(LM)$ is a bundle associated to $\widetilde{L\,\mathrm{Spin}}(M)$, a trivialization of $\mathcal{F}(LM)_\gamma$ is the same thing as an element of $\widetilde{L\,\mathrm{Spin}}(M)_\gamma$. As mentioned above, the assumption that M is string tells us that the spin structure on loop space is *fusive*, which we exploit as follows. Pick a pair of compatible loops $(\hat{\gamma}_{12}, \hat{\gamma}_{23})$ in $\mathrm{Spin}(M)$ such that $\hat{\gamma}_{12}$ lifts γ_{12} and $\hat{\gamma}_{23}$ lifts γ_{23}, this is possible because $\mathrm{Spin}(d)$ is simply connected. Finally pick elements $\varphi_{12}, \varphi_{23} \in \widetilde{L\,\mathrm{Spin}}(M)$ such that $\varphi_{12} \mapsto \hat{\gamma}_{12}$ and $\varphi_{23} \mapsto \hat{\gamma}_{23}$. The fusion product on $\widetilde{L\,\mathrm{Spin}}(M)$ then produces an element $\varphi_{13} \in \widetilde{L\,\mathrm{Spin}}(M)$ which lies over $\hat{\gamma}_{13}$. All of this conspires in such a way that the isomorphism

$$\mathcal{F}(LM)_{\gamma_{12}} \boxtimes_{\mathrm{Cl}(\mathcal{H}^+_{\gamma_{23}})} \mathcal{F}(LM)_{\gamma_{23}} \to \mathcal{F}(LM)_{\gamma_{13}},$$

induced by $\varphi_{12}, \varphi_{23}$ and φ_{13} does not depend on any of the choices made. Moreover, these isomorphisms satisfy the required associativity condition over triples of compatible loops.

Proving that the fusion isomorphisms described above really do not depend on any of the choices made is one of the main technical contributions of this work. The reason that this works out is that the fusion products on $\widetilde{L\,\mathrm{Spin}}(M)$, and on $\widetilde{L\,\mathrm{Spin}}(d)$ and the fusion isomorphism $\mathcal{F}_L \boxtimes_{\mathrm{Cl}(V^+)} \mathcal{F}_L \to \mathcal{F}_L$ all work together. To prove this, we view elements of $\widetilde{L\,\mathrm{Spin}}(d)$ as bimodule automorphisms of $\mathcal{F}_L$ and prove that the fusion product on $\widetilde{L\,\mathrm{Spin}}(d)$ is actually induced by the relative tensor product of compatible bimodule automorphisms of $\mathcal{F}_L$, see Chapters 5 to 7.

In [ST, Theorem 4] Stolz and Teichner moreover conjecture that the spinor bundle on loop space comes equipped with a *conformal connection*, see [ST, Definition 3], which is expected to play an important role in the construction of a Dirac operator on loop space. Essentially, a conformal connection is a notion of parallel transport along surfaces in M, which is compatible with the fusion isomorphisms described above. In Section 9.1 we give an equivalent, more geometric, description of the spinor bundle on loop space that we hope will be amenable to the construction of such a conformal connection.

Part of this work has appeared as [KW19]. More precisely, most of Chapters 2, 3 and 5 to 7 and Appendices A and C appeared in that work, the only exceptions are Sections 3.6, 5.4 and 7.3, which have not appeared before. Chapters 4, 8 and 9 and Appendices B and D are completely new.

2. Odd spinors on the circle

One of the main goals of this work is to define a spinor bundle on loop space. As mentioned in the introduction, this should be a vector bundle on loop space, with an action of a certain Clifford algebra. If our manifold is just a point, then its loop space is just a point, and the spinor bundle on the point should be a vector space with the action of a Clifford algebra. As we recall in Chapter 3, to construct a Clifford algebra, we require a Hilbert space as input data. In this section we define the Hilbert space of which we take the Clifford algebra. It will be the Hilbert space of square integrable sections of the odd spinor bundle.

2.1. The odd spinor bundle on the circle

We equip the disk $D^2 \subset \mathbb{C}$ and the circle $S^1 = \partial D^2$ with induced orientations and metrics. We have $\mathrm{SO}(1) = \{1\}$ and define $\mathrm{Spin}(1) := \mathbb{Z}_2 = \{\pm 1\}$. Thus, a spin structure on S^1 is the same as a principal $\mathbb{Z}_2$-bundle over S^1, i.e., a double cover.

There are, up to isomorphism, only two spin structures on S^1, namely, the connected double cover, called *odd*, and the non-connected double cover, called *even*. We will be interested in the odd spin structure, for which we write $\mathrm{Spin}(S^1)$, and which we realize as the submanifold

$$\mathrm{Spin}(S^1) := \{(e^{i\varphi}, \pm e^{i\varphi/2}) \in S^1 \times S^1 \mid \varphi \in \mathbb{R}\},$$

equipped with the projection onto the first component. The $\mathbb{Z}_2$-action is on the second component. The *odd spinor bundle* $\mathbb{S}$ is the associated complex line bundle

$$\mathbb{S} := \mathrm{Spin}(S^1) \times_{\mathbb{Z}_2} \mathbb{C}.$$

Remark 2.1.1. This definition of the odd spinor bundle fits in the general theory of spin structures, as $\mathbb{C}$ is a model for the Clifford algebra of $\mathbb{R}$.

Next we show that the bundles $\mathrm{Spin}(S^1)$ and $\mathbb{S}$ can be related to appropriate bundles over the disk D^2, see [LM89] and the Mathoverflow question [MO1] for a motivation of this discussion. We define $\mathrm{Spin}(2)$ to be the group $\mathrm{SO}(2)$ equipped with the map

$$\mathrm{Spin}(2) := \mathrm{SO}(2) \to \mathrm{SO}(2)\ ,\ A \mapsto A^2.$$

Since all principal bundles over D^2 are trivializable, we may use the trivial bundle as a model for both the oriented orthonormal frame bundle $\mathrm{SO}(D^2)$ and the (unique) spin structure $\mathrm{Spin}(D^2)$, with the projection map given by

$$\mathrm{Spin}(D^2) := D^2 \times \mathrm{SO}(2) \to D^2 \times \mathrm{SO}(2) := \mathrm{SO}(D^2)\ ,\ (z, A) \mapsto (z, A^2).$$

We write $R_\varphi \in \mathrm{SO}(2)$ for the rotation of the plane by an angle φ. The inclusion $S^1 \subset D^2$ and the isomorphism $S^1 \cong \mathrm{SO}(2)$ induce an embedding

$$\iota : \mathrm{Spin}(S^1) \to \mathrm{Spin}(D^2)\ ,\ (e^{i\varphi}, \pm e^{i\varphi/2}) \mapsto (e^{i\varphi}, \pm R_{\varphi/2}),$$

which yields a commutative square:

$$\begin{array}{ccc} \mathrm{Spin}(S^1) & \xrightarrow{\iota} & \mathrm{Spin}(D^2) \\ \big\downarrow & & \big\downarrow \\ S^1 & \longrightarrow & D^2 \end{array}$$

In this sense, the spin structure on the disk restricts to the odd spin structure on the circle.

Next, we consider the complex line bundle

$$\mathbb{D} = \mathrm{Spin}(D^2) \times_{\mathrm{SO}(2)} \mathbb{C}.$$

over D^2, by letting $A \in \mathrm{SO}(2)$ act on $\mathrm{Spin}(D^2) = D^2 \times \mathrm{SO}(2)$ by multiplication, and on $\lambda \in \mathbb{C}$ via $(A, \lambda) \mapsto A \triangleright \lambda := A^{-1}\lambda$. This action was chosen such that we obtain a minus sign in the exponent in (2.1), which in turn determines the identification between the spaces $L^2(S^1)$ and $L^2_{-2\pi}$ later on. The bundle $\mathbb{D}$ is not the spinor bundle of the disk, but it has the following interesting property:

Lemma 2.1.2. *The restriction of $\mathbb{D}$ to S^1 is the odd spinor bundle $\mathbb{S}$.*

Proof. It is easy to check that

$$\iota \times \mathrm{id}_{\mathbb{C}} : \mathrm{Spin}(S^1) \times \mathbb{C} \to \mathrm{Spin}(D^2) \times \mathbb{C}$$

descends to a vector bundle morphism $\mathbb{S} \to \mathbb{D}$ over the inclusion $S^1 \subset D^2$. It has trivial kernel in each fibre; hence, it induces an isomorphism. □

From Lemma 2.1.2 it follows that a trivialization of $\mathbb{D}$ induces a trivialization of $\mathbb{S}$. A trivialization of $\mathbb{D}$ may be given by the following pair of inverse maps:

$$\begin{aligned} D^2 \times \mathbb{C} &\to \mathbb{D} \\ (z, \lambda) &\to [(z, \mathbb{1}), \lambda]_{\mathrm{SO}(2)} \end{aligned} \qquad\qquad \begin{aligned} \mathbb{D} &\to D^2 \times \mathbb{C} \\ [(z, A), \lambda]_{\mathrm{SO}(2)} &\mapsto (z, A^{-1}\lambda) \end{aligned}$$

The induced trivialization of $\mathbb{S}$ is then given by

$$\mathbb{S} \to S^1 \times \mathbb{C}\ ,\ [(e^{i\varphi}, \pm e^{i\varphi/2}), \lambda]_{\mathbb{Z}_2} \mapsto (e^{i\varphi}, \pm e^{-i\varphi/2}\lambda). \tag{2.1}$$

2.2. Smooth and square-integrable odd spinors

We consider the space of smooth sections of $\mathbb{S}$, which we denote by $\Gamma(\mathbb{S})$. The trivialization (2.1) of the spinor bundle induces an isomorphism

$$t : \Gamma(\mathbb{S}) \to C^\infty(S^1, \mathbb{C}).$$

However, it turns out to be more natural to identify $\Gamma(\mathbb{S})$ with the space $C^\infty_{-2\pi} = C^\infty_{-2\pi}(\mathbb{R}, \mathbb{C})$ of 2π anti-periodic smooth maps from the real numbers into the complex numbers. The reason is that in Example 3.1.2 and Chapter 5 we will equip the space $\Gamma(\mathbb{S})$ with a real structure that is easily described when $\Gamma(\mathbb{S})$ is identified with $C^\infty_{-2\pi}$, but takes an awkward form on $C^\infty(S^1, \mathbb{C})$.

Lemma 2.2.1. *If $f \in C^\infty_{-2\pi}$, then the map*

$$\mathbb{R} \to \mathbb{S}\ ,\ \varphi \mapsto [(e^{i\varphi}, e^{i\varphi/2}), f(\varphi)]_{\mathbb{Z}_2}$$

descends along $\mathbb{R} \to S^1 : \varphi \mapsto e^{i\varphi}$ *to a smooth section* $\sigma_f : S^1 \to \mathbb{S}$*. Furthermore, the map*

$$\sigma : C^\infty_{-2\pi} \to \Gamma(\mathbb{S}) \, , \; f \mapsto \sigma_f \, ,$$

is an isomorphism.

Proof. It suffices to show that, for all $\varphi \in \mathbb{R}$, it holds that $\sigma_f(\varphi) = \sigma_f(\varphi + 2\pi)$. Indeed,

$$\begin{aligned}
\sigma_f(\varphi) &= [(e^{i\varphi}, e^{i\varphi/2}), f(\varphi)]_{\mathbb{Z}_2} \\
&= [(e^{i\varphi}, -e^{i\varphi/2}), -f(\varphi)]_{\mathbb{Z}_2} \\
&= [(e^{i(\varphi+2\pi)}, e^{i(\varphi+2\pi)/2}), f(\varphi+2\pi)]_{\mathbb{Z}_2} \\
&= \sigma_f(\varphi + 2\pi).
\end{aligned}$$

Let us now consider the converse; to that end, let $\sigma : S^1 \to \mathbb{S}$ be a smooth section. There exists a unique function $f : \mathbb{R} \to \mathbb{C}$ such that, for all $\varphi \in \mathbb{R}$,

$$\sigma(e^{i\varphi}) = [(e^{i\varphi}, e^{i\varphi/2}), f(\varphi)]_{\mathbb{Z}_2}.$$

It is not hard to see that f is smooth and satisfies $f(\varphi) = -f(\varphi + 2\pi)$; finally, one observes that $\sigma = \sigma_f$. □

Remark 2.2.2. The composition of the isomorphisms σ and t results in an isomorphism

$$\tilde{\sigma} := t \circ \sigma : C^\infty_{-2\pi} \to C^\infty(S^1, \mathbb{C}),$$

such that $(\tilde{\sigma} f)(e^{i\varphi}) = e^{-i\varphi/2} f(\varphi)$, for all $f \in C^\infty_{-2\pi}$ and $\varphi \in \mathbb{R}$.

The three vector spaces $\Gamma(\mathbb{S})$, $C^\infty(S^1, \mathbb{C})$, and $C^\infty_{-2\pi}$ are equipped with L^2-inner products, for instance if $f, g \in C^\infty_{-2\pi}$ then

$$\langle f, g \rangle := \int_0^{2\pi} f(\varphi)\overline{g(\varphi)} \mathrm{d}\varphi.$$

The isomorphisms σ, t, and $\tilde{\sigma}$ are isometries. We denote by $L^2_{-2\pi}$ the Hilbert completion of $C^\infty_{-2\pi}$, by $L^2(S^1)$ the Hilbert completion of $C^\infty(S^1, \mathbb{C})$, and by $L^2(\mathbb{S})$ the Hilbert completion of $\Gamma(\mathbb{S})$. The isomorphisms σ, t, and $\tilde{\sigma}$ extend to isometric isomorphisms

$$L^2_{-2\pi} \cong L^2(\mathbb{S}) \cong L^2(S^1).$$

The Hilbert space $L^2_{-2\pi}$ has a basis $\{\xi_n\}_{n \in \mathbb{Z}}$ given by $\xi_n(\varphi) = e^{-i\left(n + \frac{1}{2}\right)\varphi}$, which is identified under $\tilde{\sigma}$ with the standard basis $\{z^{-n-1}\}_{n \in \mathbb{Z}}$ of $L^2(S^1)$.

3. Clifford algebras, Fock spaces, and implementers

In this chapter we introduce infinite dimensional Clifford algebras, their representations on Fock spaces, and implementers. Most of the results in this section are well-known, see among many others [Ara87, PR94, Ott95]; unfortunately, there exist many variations of the basic setting, and many competing conventions, and we have not been able to find a consistent treatment of all aspects of the theory we will need later.

Throughout this section, we let V be a complex Hilbert space equipped with a real structure α, i.e. an anti-unitary map $\alpha : V \to V$ with the property that $\alpha^2 = \mathbb{1}$. Using the real structure we equip V with a non-degenerate symmetric complex bilinear form $b : V \times V \to \mathbb{C}$ defined by

$$b(v, w) = \langle v, \alpha(w)\rangle.$$

Our discussion in the present Chapter 3 will be fairly general; later we will restrict to the situation described below as Example 3.1.2, and variations thereof.

3.1. Lagrangians in Hilbert spaces

Definition 3.1.1. A linear subspace $L \subset V$ is a *Lagrangian* if it is a closed b-isotropic subspace, such that V decomposes as $V = L \oplus \alpha(L)$.

Example 3.1.2. We consider the Hilbert space $V := L^2(\mathbb{S}) \otimes \mathbb{C}^d$, where $\mathbb{C}^d$ is equipped with the standard inner product. Under the identification $L^2(\mathbb{S}) \cong L^2_{-2\pi}$, this Hilbert space has an orthonormal basis indexed by two numbers, $n \in \mathbb{N}$ and $j = 1, ..., d$, namely, the functions

$$\xi_{n,j}(t) = e^{-i(n+\frac{1}{2})t} \otimes e_j, \quad t \in \mathbb{R}$$

where $\{e_j\}_{j=1,...,d}$ is the standard basis for $\mathbb{C}^d$ (see Section 2.2). We equip V with the real structure α, defined to be pointwise complex conjugation. On the basis elements $\xi_{n,j}$ it takes the form

$$\alpha(\xi_{n,j})(t) := e^{i(n+\frac{1}{2})t} \otimes e_j = \xi_{-(n+1),j}(t).$$

We define the Lagrangian $L \subset V$ to be the closed complex linear span

$$L := \langle \xi_{n,j} \mid n \geqslant 0, j = 1, ..., d\rangle.$$

We remark that the corresponding subspace of $L^2(S^1)$ spanned by $\{\tilde{\sigma}(\xi_n)\}_{n\in\mathbb{N}_0} = \{z^{-n-1}\}_{n\in\mathbb{N}_0}$ is the space of functions $S^1 \to \mathbb{C}$ that extend to anti-holomorphic functions on the disk. In precisely that sense, we consider the Lagrangian of spinors on the circle that extend to anti-holomorphic functions on the disk.

We proceed with the general theory of Lagrangians in Hilbert spaces. We denote by $\mathrm{U}(V)$ the usual unitary group of V, i.e.,

$$\mathrm{U}(V) := \{T \in \mathrm{Gl}(V) \mid T^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\}.$$

The *orthogonal group* $\mathrm{O}(V)$ is the subgroup of $\mathrm{U}(V)$ of transformations that commute with the real structure α, i.e.,

$$\mathrm{O}(V) := \{T \in \mathrm{U}(V) \mid \alpha T = T\alpha\}.$$

Note that if $L \subset V$ is a Lagrangian, and $T \in \mathrm{O}(V)$, then $T(L)$ is Lagrangian; this property does not generally hold for $T \in \mathrm{U}(V)$.

A *unitary structure* on V is a map $\mathcal{J} \in \mathrm{O}(V)$ with the property that $\mathcal{J}^2 = -\mathbb{1}$, see [PR94, Chapter 2, Section 1]. If L is a subspace of V, let us write P_L for the orthogonal projection onto L, and $P_L^\perp$ for the orthogonal projection onto the orthogonal complement of L. A straightforward verification [PR94, Chapter 2, Section 1] shows the following.

Lemma 3.1.3. *There is a one-to-one correspondence between unitary structures $\mathcal{J} \in \mathrm{O}(V)$ and Lagrangian subspaces $L \subset V$, established by the assignments*

$$L \mapsto \mathcal{J}_L := i(P_L - P_L^\perp) \quad \textit{and} \quad \mathcal{J} \mapsto \ker(\mathcal{J} - i\mathbb{1}).$$

3.2. Clifford algebras

If A is a unital C*-algebra, then a complex linear map $f : V \to A$ is called *Clifford map* if it satisfies the properties

$$f(v)f(w) + f(w)f(v) = 2b(v,w)\mathbb{1} \quad \text{and} \quad f(v)^* = f(\alpha(v))$$

for all $v, w \in V$. We note that every Clifford map is bounded and injective. The Clifford C*-algebra is the C*-algebra through which any Clifford map factors:

Definition 3.2.1. A *Clifford C*-algebra* of V is a unital C*-algebra $\mathrm{Cl}(V)$ equipped with a Clifford map $\iota : V \to \mathrm{Cl}(V)$, such that for every unital C*-algebra A and every Clifford map $f : V \to A$ there exists a unique unital isometric $*$-homomorphism $\mathrm{Cl}(f)$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ \Big\downarrow{\scriptstyle \iota} & \nearrow_{\mathrm{Cl}(f)} & \\ \mathrm{Cl}(V) & & \end{array}$$

The Clifford C*-algebra $\mathrm{Cl}(V)$ is unique up to unique unital $*$-isomorphism. Its existence is proved in [PR94, Sections 1.1 & 1.2]. The construction is based on the purely algebraic Clifford algebra, which we denote by $\mathrm{Cl}_{\mathrm{alg}}(V)$, which is then completed to a C*-algebra under a certain norm. We list some elementary properties of the algebraic and of the Clifford C*-algebra that will be required in the sequel.

- The algebraic Clifford algebra is algebraically generated by V.
- The map $\iota : V \to \mathrm{Cl}(V)$ is a linear isometry, and hence injective and smooth.
- The Clifford C*-algebra is generated by V as a C*-algebra.
- Denote by $\overline{V}$ the conjugate Hilbert space. Then, there is a unique conjugate-linear unital $*$-isomorphism $\mathrm{Cl}(V) \to \mathrm{Cl}(\overline{V})$ fixing V pointwise.

In the following we will fix a Clifford C*-algebra $\mathrm{Cl}(V)$ and its Clifford map $\iota : V \to \mathrm{Cl}(V)$. If $g \in \mathrm{O}(V)$, then the map $\iota \circ g : V \to \mathrm{Cl}(V)$ is a Clifford map, and hence there is a unique unital isometric $*$-homomorphism $\theta_g : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$, such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad g \quad} & V \\ \downarrow & & \downarrow \\ \mathrm{Cl}(V) & \xrightarrow[\theta_g]{\qquad} & \mathrm{Cl}(V) \end{array}$$

If $g, g' \in \mathrm{O}(V)$, then we have $\theta_g \theta_{g'} = \theta_{gg'}$; moreover, $\theta_{\mathbb{1}} = \mathrm{id}_{\mathrm{Cl}(V)}$. It follows that θ is a group homomorphism $\theta : \mathrm{O}(V) \to \mathrm{Aut}(\mathrm{Cl}(V))$ into the group of unital $*$-automorphisms of $\mathrm{Cl}(V)$. The automorphism θ_g is called the *Bogoliubov automorphism* associated to g. Since θ_g restricts to g on V, the homomorphism θ is injective; in other words, it is a faithful representation of $\mathrm{O}(V)$ on $\mathrm{Cl}(V)$ by unital $*$-automorphisms.

Remark 3.2.2. The Bogoliubov automorphism $\theta_{-\mathbb{1}}$ is involutive and hence induces a $\mathbb{Z}_2$-grading on $\mathrm{Cl}(V)$, see [PR94, p. 27]. The image of $\iota : V \to \mathrm{Cl}(V)$ is odd. If A is a $\mathbb{Z}_2$-graded C*-algebra and the image of a Clifford map $f : V \to A$ is odd, then the induced $*$-homomorphism $\mathrm{Cl}(f) : \mathrm{Cl}(V) \to A$ is even, i.e., it preserves the gradings.

In the remainder of this section we investigate some of the regularity properties of the action of $\mathrm{O}(V)$ on $\mathrm{Cl}(V)$ by algebra automorphisms. The results will be used in Chapter 8, where we will define a particular bundle of Clifford algebras.

Lemma 3.2.3. *For each $a \in \mathrm{Cl}(V)$, the map $\mathrm{O}(V) \to \mathrm{Cl}(V), g \mapsto \theta_g(a)$ is continuous.*

Proof. First, suppose that $a \in \mathrm{Cl}_{\mathrm{alg}}(V)$, we claim that the map $g \mapsto \theta_g(a)$ is continuous. Because the algebraic Clifford algebra is generated by V, it suffices to prove that $g \mapsto \theta_g(a)$ is continuous for $a \in V$, but this is obvious.

Now, let $a \in \mathrm{Cl}(V)$ be arbitrary. Because $\mathrm{Cl}(V)$ is the closure of $\mathrm{Cl}_{\mathrm{alg}}(V)$ there exists a Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathrm{Cl}_{\mathrm{alg}}(V)$ which converges to a. We have already proven that for each $n \in \mathbb{N}$, the map $f_n : g \mapsto \theta_g(a_n)$ is continuous. Clearly, the sequence of maps $\{f_n\}_{n\in\mathbb{N}}$ converges to the map $g \mapsto \theta_g(a)$ pointwise. Hence, it is sufficient to prove that the sequence of maps $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly. To that end we compute for arbitrary $g \in \mathrm{O}(V)$, using that θ_g is an isometry,

$$\|f_n(g) - \theta_g(a)\| = \|\theta_g(a_n) - \theta_g(a)\| = \|a_n - a\|. \qquad \square$$

The following improvement of Lemma 3.2.3 follows immediately from an application of [Nee10a, Proposition 5.1].

Lemma 3.2.4. *The map $\mathrm{O}(V) \times \mathrm{Cl}(V) \to \mathrm{Cl}(V), (g, a) \mapsto \theta_g(a)$ is continuous.*

For an alternative proof see [Amb12, Proposition 4.35]

We view $\mathrm{O}(V)$ as Banach Lie group in the usual way. We define $\mathrm{Cl}(V)^{\mathrm{s}}$ to be the space of elements $a \in \mathrm{Cl}(V)$ such that the map $\mathrm{O}(V) \to \mathrm{Cl}(V), g \mapsto \theta_g(a)$ is smooth. We then equip $\mathrm{Cl}(V)^{\mathrm{s}}$ with the structure of Fréchet space as described in [Nee10a, Section 4 & Proposition 5.4]. By [Nee10a, Theorem 6.2] we then have that $\mathrm{Cl}(V)^{\mathrm{s}}$ is a Fréchet algebra, i.e. multiplication is continuous, that the group of units $(\mathrm{Cl}(V)^{\mathrm{s}})^{\times}$ is open and that the map $(\mathrm{Cl}(V)^{\mathrm{s}})^{\times} \to \mathrm{Cl}(V)^{\mathrm{s}}, a \mapsto a^{-1}$ is continuous, that is, $\mathrm{Cl}(V)^{\mathrm{s}}$ is a *continuous inverse algebra.* Moreover, the map $\mathrm{O}(V) \times \mathrm{Cl}(V)^{\mathrm{s}} \to \mathrm{Cl}(V)^{\mathrm{s}}$ is smooth.

Because multiplication $\mathrm{Cl}(V)^{\mathrm{s}} \times \mathrm{Cl}(V)^{\mathrm{s}} \to \mathrm{Cl}(V)^{\mathrm{s}}$ is continuous and bilinear, it is also smooth.

Lemma 3.2.5. *The algebra* $\mathrm{Cl}(V)^{\mathrm{s}}$ *contains the algebraic Clifford algebra, in particular* $\mathrm{Cl}(V)^{\mathrm{s}}$ *is dense in* $\mathrm{Cl}(V)$.

Proof. Let $a \in \mathrm{Cl}_{\mathrm{alg}}(V)$ be arbitrary, then we need to show that the map $\mathrm{O}(V) \to \mathrm{Cl}(V), g \mapsto \theta_g(a)$ is smooth. Any element $a \in \mathrm{Cl}_{\mathrm{alg}}(V)$ is finitely generated by elements of $V \subset \mathrm{Cl}_{\mathrm{alg}}(V)$. Now, since multiplication and addition in the Clifford algebra are smooth, we may assume that $a \in V \subset \mathrm{Cl}(V)$. In this case we have $g \mapsto \theta_g(a) = g(a) \in V$, which is clearly smooth. □

Lemma 3.2.6. *Let* V' *be a second complex Hilbert space with real structure, and let* $\nu : V \to V'$ *be a unitary map intertwining the real structures. Then, the map* $\mathrm{Cl}(\nu) : \mathrm{Cl}(V) \to \mathrm{Cl}(V')$ *restricts to an isomorphism of Fréchet algebras* $\mathrm{Cl}(\nu) : \mathrm{Cl}(V)^{\mathrm{s}} \to \mathrm{Cl}(V')^{\mathrm{s}}$.

Proof. Let us show that if $a \in \mathrm{Cl}(V)^{\mathrm{s}}$ then $\mathrm{Cl}(\nu)(a) \in \mathrm{Cl}(V')^{\mathrm{s}}$. To prove that, we need to show that, for all $a \in \mathrm{Cl}(V)^{\mathrm{s}}$, the map $\mathrm{O}(V') \to \mathrm{Cl}(V')^{\mathrm{s}}, g \mapsto \theta_g \mathrm{Cl}(\nu)(a)$ is smooth. We decompose that map as follows

$$\mathrm{O}(V') \longrightarrow \mathrm{O}(V) \longrightarrow \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(V')$$

$$g \longrightarrow \nu^{-1} g \nu \longrightarrow \theta_{\nu^{-1} g \nu}(a) \longrightarrow \theta_g \mathrm{Cl}(\nu)(a)$$

The first map is obviously smooth. The second map is smooth because $a \in \mathrm{Cl}(V)^{\mathrm{s}}$. Writing

$$\theta_{\nu^{-1} g \nu} = \mathrm{Cl}(\nu^{-1}) \theta_g \mathrm{Cl}(\nu),$$

we see that the last map is simply $\mathrm{Cl}(\nu)$, which is smooth because $\mathrm{Cl}(\nu)$ is a linear isometry, (see Definition 3.2.1). □

3.3. Fock spaces

Let $L \subset V$ be a Lagrangian subspace. We define the *Fock space* $\mathcal{F}_L$ to be the Hilbert completion of the exterior algebra

$$\Lambda L := \bigoplus_{n=0}^{\infty} \Lambda^n L.$$

We equip $\mathcal{F}_L$ with an action of $\mathrm{Cl}(V)$ as follows. Let $(v, w) \in L \oplus \alpha(L) = V$. We use the bilinear form $b : V \times V \to \mathbb{C}$ to identify $\alpha(L)$ with the dual of L, denoted by L^*, and write $w^* \in L^*$ for the element corresponding to $w \in \alpha(L)$. We denote by $a(w^*) : \mathcal{F}_L \to \mathcal{F}_L$ contraction with w^*, and we write $c(v) : \mathcal{F}_L \to \mathcal{F}_L$ for left multiplication. One may now verify that the assignment $(v, w) \mapsto c(v) + a(w^*)$ is a Clifford map $V \to \mathcal{B}(\mathcal{F}_L)$. This means that this map extends to a unital isometric $*$-homomorphism $\pi_L : \mathrm{Cl}(V) \to \mathcal{B}(\mathcal{F}_L)$. We shall adopt the notation $a \triangleright v := \pi_L(a)(v)$ for $a \in \mathrm{Cl}(V)$ and $v \in \mathcal{F}_L$. Finally, one may verify that the vector $\Omega := 1 \in \Lambda^0 L$ is annihilated by $\alpha(L) \subset V \subset \mathrm{Cl}(V)$. In the terminology of [PR94] this means that Ω is a *vacuum vector* for this representation. A basic result ([PR94, Theorem 2.4.2]) is the following.

Proposition 3.3.1. *The Fock space* $\mathcal{F}_L$ *is an irreducible* $\mathrm{Cl}(V)$*-representation.*

As a corollary to Proposition 3.3.1 we have that the von Neumann algebra $\mathrm{Cl}(V)'' \subseteq \mathcal{B}(\mathcal{F}_L)$ is in fact equal to $\mathcal{B}(\mathcal{F}_L)$, and hence a factor of type I. This is because irreducibility of $\mathrm{Cl}(V)$ means that $\mathrm{Cl}(V)' = \mathbb{C}\mathbb{1}$ and hence that $\mathrm{Cl}(V)'' = \mathcal{B}(\mathcal{F}_L)$.

Remark 3.3.2. Just like the Clifford algebra, the Fock space $\mathcal{F}_L$ is $\mathbb{Z}_2$-graded. The graded components are the completions of even and odd exterior products of L. The grading on $\mathcal{F}_L$ induces a grading on $\mathcal{B}(\mathcal{F}_L)$, for which the image of $V \to \mathcal{B}(\mathcal{F}_L)$ is odd. Hence, the representation $\pi_L : \mathrm{Cl}(V) \to \mathcal{B}(\mathcal{F}_L)$ preserves this grading (see Remark 3.2.2).

Remark 3.3.3. Our definition of the Clifford algebra and its representation on Fock space is consistent with [Ara87] and [BJL02]. However, there are some competing conventions. For example, one might start with a complex Hilbert space H, which will play the role of our Lagrangian L. In this case, the Fock space will be the completion of ΛH, see for example [Ott95] and [Nee10b]. Information on the relationship between these approaches is given in Chapter 2.6 of [Ott95]. In [PR94] yet another approach is taken. There, it is assumed that V is a real Hilbert space, equipped with a unitary structure $\mathcal{J} : V \to V$. In this case, $V \otimes \mathbb{C}$ is a complex Hilbert space, naturally equipped with both a real structure and a unitary structure, which puts us in the setting we have described so far. In [PR94], the real Hilbert space is then equipped with a complex structure by setting $iv := \mathcal{J}(v)$ for all $v \in V$, and one writes $V_{\mathcal{J}}$ for the complex Hilbert space obtained in this way. There is then an isometric isomorphism $V_{\mathcal{J}} \to L_{\mathcal{J}}, v \mapsto 2^{-1/2}(v - i\mathcal{J}v)$, see [PR94] Section 2.1.

3.4. Implementable operators

Let $L \subset V$ be an arbitrary Lagrangian. We consider $\mathrm{Cl}(V) \subset \mathcal{B}(\mathcal{F}_L)$ via the faithful representation π_L. Given an element $g \in \mathrm{O}(V)$, one might wonder if there exists a unitary operator $U \in \mathrm{U}(\mathcal{F}_L)$, such that the equation

$$\theta_g(a) = UaU^*$$

in $\mathcal{B}(\mathcal{F}_L)$ holds for all $a \in \mathrm{Cl}(V)$. In this case the operator $g \in \mathrm{O}(V)$ is said to be *implementable* in $\mathcal{F}_L$, and the unitary operator U is called an *implementer* that *implements* g. We recall below the criterion for $g \in \mathrm{O}(V)$ to be implementable and then discuss the structure of the set of all implementers.

For a bounded operator $A \in \mathcal{B}(V)$ we write $\|A\|$ for the usual operator norm, and $\|A\|_2$ for the Hilbert-Schmidt norm, i.e. $\|A\|_2 := \mathrm{trace}\,(A^*A)$. We recall that A is called a *Hilbert-Schmidt operator* if $\|A\|_2$ is finite. We have the following important result, see [PR94, Theorem 3.3.5] or [Ara87, Theorem 6.3].

Theorem 3.4.1. *An orthogonal operator $g \in \mathrm{O}(V)$ is implementable if and only if $[g, \mathcal{J}_L]$ is a Hilbert-Schmidt operator.*

Suppose $g, h \in \mathrm{O}(V)$ are implementable. Since the Hilbert-Schmidt operators form an ideal in the algebra of bounded operators, the identity $[gh, \mathcal{J}_L] = g[h, \mathcal{J}_L] + [g, \mathcal{J}_L]h$ implies that $gh \in \mathrm{O}(V)$ is again implementable. Similarly, the equation $[g^{-1}, \mathcal{J}_L] = -g^{-1}[g, \mathcal{J}_L]g^{-1}$ tells us that if g is implementable, then g^{-1} is implementable. Thus, the implementable operators form a subgroup, which we call the *restricted orthogonal group* and denote by $\mathrm{O}_L(V)$.

Remark 3.4.2. It is common that one is interested in a fixed Lagrangian $L \subset V$ in which case one might suppress the dependence of the group $\mathrm{O}_L(V)$ on L and write $\mathrm{O}_{\mathrm{res}}(V)$ or something similar instead.

Remark 3.4.3. Let $P_L : V \to L$ be the projection onto L. We recall from Lemma 3.1.3 that $\mathcal{J}_L = i(P_L - P_L^{\perp})$. Let $A \in \mathcal{B}(V)$. Then, we may write A in block form with respect to the decomposition $V = L \oplus \alpha(L)$ as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have the relations

$$a = P_L A P_L, \qquad b = P_L A P_L^\perp,$$
$$c = P_L^\perp A P_L, \qquad d = P_L^\perp A P_L^\perp.$$

From which it follows that the condition that $[A, \mathcal{J}_L]$ is Hilbert-Schmidt is equivalent to the statement that both b and c are Hilbert-Schmidt. If A is unitary, then we have that if b is Hilbert-Schmidt, then c is Hilbert-Schmidt and vice-versa.

We claim that $\mathrm{O}_L(V)$ is a Banach Lie group with the underlying topology induced from the so-called $\mathcal{J}$-norm. In the following we will describe the Banach Lie group structure explicitly. We let $\mathcal{B}_L(V)$ be the unital algebra

$$\mathcal{B}_L(V) := \{A \in \mathcal{B}(V) \mid \|[A, \mathcal{J}]\|_2 < \infty\}.$$

On the algebra $\mathcal{B}_L(V)$, the $\mathcal{J}$-norm is defined by

$$\|A\|_{\mathcal{J}} := \|A\| + \|[\mathcal{J}, A]\|_2.$$

It is elementary to check that the $\mathcal{J}$-norm turns $\mathcal{B}_L(V)$ into a Banach algebra. We see that $\mathrm{O}_L(V) = \mathcal{B}_L(V)^\times \cap \mathrm{O}(V)$ is a closed subgroup of the Banach Lie group $\mathcal{B}_L(V)^\times$. Unlike in finite dimensions, this is not sufficient for being a Banach Lie group itself, making further considerations necessary. First, we equip $\mathrm{O}_L(V) \subset \mathcal{B}_L(V)^\times$ with the induced topology, so that it becomes a topological group. We note the following result about this topology, which is part of Theorem 6.3 in [Ara87].

Proposition 3.4.4. *The topological group $\mathrm{O}_L(V)$ has two connected components.*

Remark 3.4.5. We consider the following norms on $\mathcal{B}_L(V)$:

$$A \mapsto \|A\| + \|[A, \mathcal{J}]\|_2, \qquad A \mapsto \|P_L A P_L\| + \|[A, \mathcal{J}]\|_2,$$
$$A \mapsto \|A\| + \|P_L A P_L^\perp\|_2, \qquad A \mapsto \|P_L A P_L\| + \|P_L A P_L^\perp\|_2.$$

Their restrictions to $\mathrm{O}_L(V)$ are all equivalent. We mention this because some sources use one of the other norms to define the topology on $\mathrm{O}_L(V)$.

Next we construct explicitly a Banach Lie group structure on $\mathrm{O}_L(V)$. As usual, in unital Banach algebras the exponential map

$$\exp : \mathcal{B}_L(V) \to \mathcal{B}_L(V)^\times \ , \ A \mapsto e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

is smooth and a local diffeomorphism at 0. A local inverse can be defined on the $\mathcal{J}$-norm open ball $B_1(\mathbb{1})$ around the unit $\mathbb{1}$ of radius 1, where it is given by the logarithm

$$B_1(\mathbb{1}) \to \mathcal{B}_L(V),$$
$$A \mapsto \ln(A) := -\sum_{n=1}^{\infty} \frac{1}{n} (1 - A)^n.$$

We define the Banach Lie algebra $\mathfrak{o}_L(V)$ to be the Lie subalgebra

$$\mathfrak{o}_L(V) := \{A \in \mathcal{B}_L(V) \mid A^* = -A \text{ and } [\alpha, A] = 0\}.$$

It will indeed turn out to be the Lie algebra of $\mathrm{O}_L(V)$.

Lemma 3.4.6. *The exponential map is a local homeomorphism at* 0 *from* $\mathfrak{o}_L(V)$ *to* $\mathrm{O}_L(V)$.

Proof. First, we prove that $\exp(\mathfrak{o}_L(V)) \subseteq \mathrm{O}_L(V)$. We have that $A^* = -A$ implies that $\exp(A) \in \mathrm{U}(V)$, then $[\alpha, A] = 0$ implies $\exp(A) \in \mathrm{O}_L(V)$. Now, let $W' \subseteq \mathcal{B}_L(V)$ and $U' \subseteq \mathcal{B}_L(V)^\times$ be open neighbourhoods of 0 and 1 respectively with the property that $\exp : W' \to U'$ is a diffeomorphism. It follows that $W := W' \cap \mathfrak{o}_L(V)$ is an open neighbourhood of $0 \in \mathfrak{o}_L(V)$ and $U := U' \cap \mathrm{O}_L(V)$ is an open neighbourhood of $1 \in \mathrm{O}_L(V)$. We claim that exp is a homeomorphism from W to U. Clearly exp is a homeomorphism from W to $\exp(W)$. It remains to show that $\exp(W) = U = U' \cap \mathrm{O}_L(V)$. The inclusion $\exp(W) \subseteq U' \cap \mathrm{O}_L(V)$ is clear. For the other inclusion it suffices to show $U' \cap \mathrm{O}_L(V) \subseteq \exp(\mathfrak{o}_L(V))$. Or, in other words, that the logarithm of $A \in U' \cap \mathrm{O}_L(V)$ is in $\mathfrak{o}_L(V)$. This is easily verified using the series expansion of the logarithm around 1. □

With Lemma 3.4.6 at hand it is standard to equip $\mathrm{O}_L(V)$ with the structure of a Banach Lie group with Lie algebra $\mathfrak{o}_L(V)$. Moreover, $\mathrm{O}_L(V)$ is an embedded submanifold of $\mathcal{B}_L(V)^\times$.

3.5. Implementers

We define $\mathrm{Imp}_L(V) \subset \mathrm{U}(\mathcal{F})$ to be the set of all implementers of operators $g \in \mathrm{O}_L(V)$. Suppose that $U \in \mathrm{Imp}_L(V)$, then since $\theta : \mathrm{O}(V) \to \mathrm{Aut}(\mathrm{Cl}(V))$ is injective, the operator g that is implemented by U is determined uniquely; in other words, we have a well-defined map $q : \mathrm{Imp}(V) \to \mathrm{O}_L(V)$. If U and U' implement operators g and g', respectively, then UU' implements gg'. Likewise, U^{-1} implements g^{-1}. Hence, $\mathrm{Imp}(V)$ is a subgroup of $\mathrm{U}(\mathcal{F})$, and q is a group homomorphism. If $U, U' \in \mathrm{Imp}_L(V)$ implement the same operator g, then we have

$$UaU^* = U'aU'^*$$

for all $a \in \mathrm{Cl}(V)$, which implies $UU'^{-1} \in \mathrm{Cl}(V)'$ and hence $UU'^{-1} \in \mathrm{U}(1)\mathbb{1}$. Thus, we have a central extension

$$\mathrm{U}(1) \to \mathrm{Imp}_L(V) \to \mathrm{O}_L(V) \tag{3.1}$$

of groups. Our next goal is to equip $\mathrm{Imp}_L(V)$ with the structure of a Banach Lie group, such that (3.1) is a central extension of Banach Lie groups.

For this purpose, we infer the existence of a local section $\sigma : U \to \mathrm{Imp}_L(V)$, defined on an open neighbourhood U of $\mathbb{1} \in \mathrm{O}_L(V)$ on which the exponential map is injective. We refer to [Ara87, PR94, Ott95, Nee10b] for constructions of this section, and recall some steps in the following. Let $\mathcal{L}(\mathcal{F}_L)$ be the algebra of unbounded skew-symmetric operators on $\mathcal{F}_L$, with invariant dense domain equal to the algebraic Fock space ΛL. We shall outline how to produce for each $A \in \mathfrak{o}_L(V)$ an element $\tilde{A} \in \mathcal{L}(\mathcal{F})$, such that $\exp(\tilde{A})$ is unitary and implements $\exp(A)$. Then, $\sigma(\exp(A)) := \exp(\tilde{A})$. In order to define $\tilde{A}$, we first require the following extension of Remark 3.4.3, which can be proved by an explicit computation of $[\alpha, A]$.

Lemma 3.5.1. *With respect to the decomposition* $V = L \oplus \alpha(L)$, *an element* $A \in \mathfrak{o}_L(V)$ *can uniquely be written as*

$$A = \begin{pmatrix} a & a'\alpha \\ \alpha a' & \alpha a \alpha \end{pmatrix},$$

with $a : L \to L$ *a bounded linear skew-symmetric transformation, and* $a' : L \to L$ *a Hilbert-Schmidt conjugate-linear skew-symmetric transformation.*

Given a decomposition of A as in Lemma 3.5.1, we define a skew-symmetric unbounded operator $\tilde{A}_0$ on $\mathcal{F}_L$ by

$$\tilde{A}_0 : \Lambda^n L \to \Lambda^n L,$$
$$f_1 \wedge ... \wedge f_n \mapsto \sum_{i=1}^{n} f_1 \wedge ... \wedge af_i \wedge ... \wedge f_n;$$

this operator has invariant dense domain ΛL, see [Ott95, Section 2.3]. Next, since a' is Hilbert-Schmidt and conjugate-linear, there exists a unique element $\hat{a}' \in \Lambda^2 L$, such that

$$\langle \hat{a}', f_1 \wedge f_2 \rangle = \langle a'(f_1), f_2 \rangle, \tag{3.2}$$

for all $f_1, f_2 \in L$, see [PR94, Theorem 2.2.2], [Nee10b, Lemma D3] or [Ott95, Section 2.4]. Moreover, the following equation then holds

$$\|\hat{a}'\|^2 = \frac{1}{2}\|a'\|_2^2.$$

We then obtain a skew-symmetric unbounded operator $\tilde{A}_1$ with invariant dense domain ΛL by setting

$$\tilde{A}_1 : \Lambda^n L \to \Lambda^{n-2} L \oplus \Lambda^{n+2} L\ ,\ \xi \mapsto \frac{1}{2}\left(\iota_{\hat{a}'}\xi - \hat{a}' \wedge \xi\right)$$

where $\iota_{\hat{a}'}$ stands for the adjoint of the map $\xi \mapsto \hat{a}' \wedge \xi$. We now set $\tilde{A} := \tilde{A}_0 + \tilde{A}_1 \in \mathcal{L}(\mathcal{F})$. It is straightforward to see that $A \mapsto \tilde{A}$ is linear. That $\exp(\tilde{A})$ implements $\exp(A)$ is proved in [Ott95, page 44]. This completes our recollection of the construction of σ. We are now in position to state the main result of this subsection, which is well-known and appears, e.g., in [Ara87, Nee10b].

Theorem 3.5.2. *There exists a unique Banach Lie group structure on* $\mathrm{Imp}_L(V)$ *such that the section* σ *is smooth in an open neighbourhood of* $\mathbb{1}$*. Moreover, when equipped with this Banach Lie group structure,*

$$\mathrm{U}(1) \to \mathrm{Imp}_L(V) \xrightarrow{q} \mathrm{O}_L(V)$$

is a central extension of Banach Lie groups.

Proof. We choose an open neighbourhood $V \subset U$ of $\mathbb{1}$ such that $V^2 \subset U$. We let $f_\sigma : V \times V \to \mathrm{U}(1)$ be the 2-cocycle associated to the section σ via the formula

$$\sigma(g_1)\sigma(g_2) = f_\sigma(g_1, g_2)\sigma(g_1 g_2).$$

The goal is to show that f_σ is smooth in an open neighbourhood of $(\mathbb{1}, \mathbb{1})$. That this implies all statements of the theorem is a standard result; for the convenience of the reader we have described a proof in Appendix A, see Proposition A.1.

In order to show that f_σ is smooth in an open neighbourhood of $(\mathbb{1}, \mathbb{1})$, we consider the map

$$\psi_\sigma : U \times U \to \mathbb{C}\ ,\ (g, h) \mapsto \langle \sigma(g)\sigma(h)\Omega, \Omega \rangle\ .$$

It is proved in [Nee10b, Section 10] that ψ_σ is smooth in an open neighbourhood of $(\mathbb{1}, \mathbb{1})$. Since ψ_σ is non-zero at $(\mathbb{1}, \mathbb{1})$ and smooth, there exists an open neighbourhood $U' \subset U$ of $\mathbb{1}$ such that $\psi_\sigma(g, h) \neq 0$ for all $g, h \in U'$. Let V' be an open neighbourhood such that $V'^2 \subset U'$. We note from the definition of σ that $\sigma(\mathbb{1}) = \mathbb{1}$. We now compute, for $g_1, g_2 \in V'$,

$$\begin{aligned} f_\sigma(g_1, g_2)\psi_\sigma(1, g_1 g_2) &= f_\sigma(g_1, g_2)\langle \sigma(g_1 g_2)\Omega, \Omega \rangle \\ &= \langle f_\sigma(g_1, g_2)\sigma(g_1 g_2)\Omega, \Omega \rangle \\ &= \langle \sigma(g_1)\sigma(g_2)\Omega, \Omega \rangle \\ &= \psi_\sigma(g_1, g_2). \end{aligned}$$

It follows that

$$f_\sigma(g_1, g_2) = \frac{\psi_\sigma(g_1, g_2)}{\psi_\sigma(1, g_1 g_2)};$$

hence, we see that f_σ is smooth on $V' \times V'$. □

Remark 3.5.3. The section σ is not continuous when $\mathrm{Imp}_L(V) \subset \mathrm{U}(\mathcal{F}_L)$ is equipped with the norm-topology; as a consequence, the inclusion $\mathrm{Imp}_L(V) \to \mathrm{U}(\mathcal{F})$ is not a homomorphism of Banach Lie groups.

With the Banach Lie group structure on $\mathrm{Imp}_L(V)$ at hand, we can now use its Banach Lie algebra, which is a central extension

$$\mathbb{R} \to \mathfrak{imp}(V) \to \mathfrak{o}_L(V).$$

Here, we have identified the Lie algebra of U(1) with $\mathbb{R}$. The section σ induces a section $\sigma_* : \mathfrak{o}_L(V) \to \mathfrak{imp}(V)$, which in turn determines a Lie algebra 2-cocycle $\omega_\sigma : \mathfrak{o}_L(V) \times \mathfrak{o}_L(V) \to \mathbb{R}$ by

$$\omega_\sigma(X, Y) := [\sigma_*(X), \sigma_*(Y)] - \sigma([X, Y]). \tag{3.3}$$

This cocycle is computed in [Ara87, Theorem 6.10], resulting in

$$\omega_\sigma(A_1, A_2) = \frac{1}{8} \operatorname{trace}(\mathcal{J}[\mathcal{J}, A_1][\mathcal{J}, A_2]). \tag{3.4}$$

The same cocycle can be described in two further ways. If we put $A_3 := [A_1, A_2]$ and write

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

with respect to the decomposition $V = L \oplus \alpha(L)$, then it is shown in [Ara87, Theorem 6.10] that

$$\omega_\sigma(A_1, A_2) = \frac{1}{2} \operatorname{trace}\left([a_1, a_2] - a_3\right). \tag{3.5}$$

Finally, according to [Nee10b, Theorem 10.2] we may write

$$A_i = \begin{pmatrix} a_i & a_i'\alpha \\ \alpha a_i' & \alpha a_i \alpha \end{pmatrix}$$

for the decomposition of A_i according to Lemma 3.5.1, and then obtain

$$\omega_\sigma(A_1, A_2) = -\frac{1}{2i} \operatorname{trace}([a_1', a_2']). \tag{3.6}$$

To see that (3.6) and (3.4) coincide, one may use Lemma 3.1.3 and Remark 3.4.3.

Remark 3.5.4. The Lie algebra cocycle given above in (3.4) is 1/2 times the Lie algebra cocycle given in [PS86, Proposition 6.6.5], we also note that their operator J differs from our operator $\mathcal{J}$ by a factor of i.

Because $\mathfrak{imp}(V)$ is a central extension of $\mathfrak{o}_L(V)$ by $\mathbb{R}$, it follows that, as a vector space, the Banach Lie algebra of $\mathrm{Imp}_L(V)$ is $\mathfrak{imp}(V) := \mathfrak{o}_L(V) \oplus \mathbb{R}$. The bracket is then given by

$$[(A_1, \lambda_1), (A_2, \lambda_2)] = ([A_1, A_2], \omega_\sigma(A_1, A_2)),$$

and the norm is given by

$$\|(A, \lambda)\|^2 = \|A\|_{\mathcal{J}}^2 + |\lambda|^2.$$

Remark 3.5.5. The exponential map $\exp : \mathfrak{imp}(V) \to \mathrm{Imp}_L(V)$ is $(A, \lambda) \mapsto \exp(\tilde{A} + i\lambda\mathbb{1})$. We may relate $\mathfrak{imp}(V)$ to the algebra $\mathcal{L}(\mathcal{F})$ used in the definition of the section σ by considering the injective linear map

$$\mathfrak{imp}(V) \to \mathcal{L}(\mathcal{F}_L)\ ,\ (A, \lambda) \mapsto \tilde{A} + i\lambda\mathbb{1}.$$

That way, the exponential map of $\mathrm{Imp}_L(V)$ factors through the exponential of $\mathcal{L}(\mathcal{F})$.

For later purpose, we consider the unitary representation of the Banach Lie group $\mathrm{Imp}_L(V)$ of implementers on the Fock space $\mathcal{F}$, obtained from the inclusion $\mathrm{Imp}_L(V) \subset \mathrm{U}(\mathcal{F})$.

Proposition 3.5.6. *The set of smooth vectors*

$$\mathcal{F}_L^{\mathrm{s}} := \{v \in \mathcal{F}_L \mid \mathrm{Imp}_L(V) \to \mathcal{F}_L, U \mapsto Uv \text{ is smooth}\}$$

contains the algebraic Fock space ΛL; in particular, $\mathcal{F}_L^{\mathrm{s}}$ is dense in $\mathcal{F}_L$. Moreover, if $a \in \mathrm{Cl}(V)^{\mathrm{s}}$ and $v \in \mathcal{F}_L^{\mathrm{s}}$, then $a \triangleright v \in \mathcal{F}_L^{\mathrm{s}}$.

Proof. We claim that the vacuum vector Ω is a smooth vector for $\mathrm{Imp}_L(V)$. A general theorem for unitary representations [Nee10a, Theorem 7.2] implies that Ω is a smooth vector if the map $U \mapsto \langle \Omega, U\Omega \rangle$ is smooth in an open neighbourhood of $\mathbb{1}$. Let $W \subset \mathrm{O}_L(V)$ be the domain of the local section σ from Theorem 3.5.2. By our characterization of the Lie group structure on $\mathrm{Imp}_L(V)$, the map $W \times \mathrm{U}(1) \to \mathrm{Imp}_L(V) : (g, \lambda) \mapsto \lambda\sigma(g)$ is a local diffeomorphism at $(\mathbb{1}, 1)$. Now, it suffices to show that the map $(g, \lambda) \mapsto \langle \Omega, \lambda g(\sigma)\Omega \rangle$ is smooth in a neighbourhood of $(\mathbb{1}, 1)$. But the latter expression is equal to $\lambda\ \overline{\psi_\sigma(1, g)}$, where ψ_σ appeared in the proof of Theorem 3.5.2, and is smooth. This proves that Ω is a smooth vector.

Next, we claim that if $v \in \Lambda L$, then the map $\psi_v : \mathrm{Imp}_L(V) \to \mathcal{F}_L, U \mapsto Uv$ is smooth. Because v is in the algebraic Fock space by assumption, there exists an element $a \in \mathrm{Cl}_{\mathrm{alg}}(V)$ such that $a \triangleright \Omega = v$. Now, ψ_v may be decomposed as

$$\begin{array}{ccccccc} \mathrm{Imp}_L(V) & \longrightarrow & \mathrm{O}_L(V) \times \mathrm{Imp}_L(V) & \longrightarrow & \mathrm{Cl}(V) \times \mathcal{F}_L & \longrightarrow & \mathcal{F}_L \\ U & \longmapsto & (g, U) & \longmapsto & (\theta_g(a), U\Omega) & \longmapsto & \theta_g(a) \triangleright U\Omega \end{array}$$

The first map is clearly smooth. The second map is smooth because Ω is a smooth vector and because $a \in \mathrm{Cl}_{\mathrm{alg}}(V) \subseteq \mathrm{Cl}(V)^{\mathrm{s}}$. The last map is smooth because it is simply the evaluation map, $\mathcal{B}(\mathcal{F}_L) \times \mathcal{F}_L \to \mathcal{F}_L$. This proves that $\Lambda L \subseteq \mathcal{F}^{\mathrm{s}}$.

If we replace Ω by an arbitrary smooth vector $v \in \mathcal{F}_L^{\mathrm{s}}$ and if we assume that $a \in \mathrm{Cl}(V)^{\mathrm{s}}$, then the same argument proves that the map $U \mapsto U(a \triangleright v) = \theta_g(a) \triangleright Uv$ is smooth, and hence that the vector $a \triangleright v$ is a smooth vector, whence $\mathrm{Cl}(V)^{\mathrm{s}} \triangleright \mathcal{F}^{\mathrm{s}} = \mathcal{F}^{\mathrm{s}}$. □

The notation $\mathcal{F}_L^{\infty}$ is often used for the set of smooth vectors. We prefer $\mathcal{F}_L^{\mathrm{s}}$ for aesthetic reasons.

We equip $\mathcal{F}_L^{\mathrm{s}}$ with the structure of Fréchet space as described in [Nee10a, Section 4], with respect to which the map $\mathrm{Imp}_L(V) \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_L^{\mathrm{s}}$ is smooth.

Lemma 3.5.7. *The action map $\mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_L^{\mathrm{s}}$ is smooth.*

Proof. First, one proves that for a bilinear map between Fréchet spaces to be smooth it is sufficient that it is continuous. We prove this in Appendix B.

To prove that the action map is continuous it suffices to prove that it is continuous in $(0,0) \in \mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}_L^{\mathrm{s}}$. This is proved by an explicit consideration of the semi-norms that define the topology on $\mathcal{F}_L^{\mathrm{s}}$ in Appendix B. □

Let V' be a second complex Hilbert space with real structure.

Lemma 3.5.8. *For any orthogonal map $\nu : V \to V'$, the unitary map $\Lambda_\nu : \mathcal{F}_L \to \mathcal{F}_{\nu(L)}$ restricts to an isomorphism*

$$\Lambda_\nu : \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_{\nu(L)}^{\mathrm{s}}.$$

The proof is completely analogous to the proof of Lemma 3.2.6.

We recall from Proposition 3.4.4 that the topological group $\mathrm{O}_L(V)$ has two connected components; this implies that $\mathrm{Imp}_L(V)$ has two connected components as well. We recall from Remark 3.3.2 that the Fock space is graded, hence so is $\mathrm{Imp}_L(V) \subset \mathcal{B}(\mathcal{F}_L)$. We now have the following result, which is [Ara87, Theorem 6.7] and [Nee10b, Remark 10.8].

Proposition 3.5.9. *All elements of $\mathrm{Imp}_L(V)$ are homogeneous, and the connected component of the identity in $\mathrm{Imp}_L(V)$ consists of the even implementers.*

Remark 3.5.10. The central extension $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ can be considered in the setting of Fréchet-Lie groups. A topological version of this is considered in [Ara87] and in [Ott95]. In these sources the group $\mathrm{O}_L(V)$ is equipped with the $\mathcal{J}$-strong topology, which is strictly weaker than the $\mathcal{J}$-norm topology we have considered. In this topology it has two components as well [Ara87, Theorem 6.3]. In [Car84] the coarsest topology on $\mathrm{O}_L(V)$ is determined in which the projective representation on $\mathcal{F}$ is continuous, but this topology is not the one of any manifold. We refer to Chapter 2.4, Theorem 6 in [Ott95], and [PR94] pages 109 and 110 for more information on the connection between the different treatments of the groups $\mathrm{O}_L(V)$ and $\mathrm{Imp}_L(V)$.

We should note that for the various results on the spaces of smooth vectors $\mathcal{F}_L^{\mathrm{s}}$ and $\mathrm{Cl}(V)^{\mathrm{s}}$ it was important that we treated the groups $\mathrm{Imp}_L(V)$ and $\mathrm{O}_L(V)$ as Banach Lie groups, and not Fréchet Lie groups, see [Nee10a, Example 4.8].

Remark 3.5.11. The Lie algebra section $\sigma_* : \mathfrak{o}_L(V) \to \mathfrak{imp}(V)$ determines a connection ν_σ on the Banach principal $\mathrm{U}(1)$-bundle $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$, whose horizontal subspaces are the left-translates of the image of σ_*. As a 1-form on $\mathrm{Imp}_L(V)$, it is given by $\nu_\sigma := \theta^{\mathrm{Imp}_L(V)} - \sigma_*(q^*\theta^{\mathrm{O}_L(V)})$, where θ^G denotes the left-invariant Maurer-Cartan form on a Lie group G. The curvature of ν_σ is $-\frac{1}{2}\omega_\sigma(\theta \wedge \theta) \in \Omega^2(\mathrm{O}_L(V))$. We will further discuss this connection in Section 7.2.

3.6. The equivalence problem & and the restricted Lagrangian Grassmannian

In Section 3.4 we discussed the implementability problem. It is well-known that this is closely related to the so-called equivalence problem [PR94, Section 3.4], which we discuss now. This discussion will be relevant in Section 9.1.

Let us write $\mathrm{Lag}(V)$ for the set of Lagrangians in V. Let $L \in \mathrm{Lag}(V)$ be an arbitrary Lagrangian. We first define the *restricted unitary group*:

$$\mathrm{U}_L(V) := \{ g \in \mathrm{U}(V) \mid \|[g, \mathcal{J}_L]\|_2 < \infty \}.$$

Suppose now that $L' \subset V$ is a second Lagrangian. We may now ask if there exists a unitary operator $T : \mathcal{F}_L \to \mathcal{F}_{L'}$ that intertwines the $\mathrm{Cl}(V)$ representations, i.e. has the property that $a \triangleright v = T^*(a \triangleright (Tv))$ for all $a \in \mathrm{Cl}(V)$ and all $v \in \mathcal{F}_L$. In this case we say that the representations $\mathcal{F}_L$ and $\mathcal{F}_{L'}$ are *unitarily equivalent.*

Lemma 3.6.1. *The following are equivalent:*

1. *The representations $\mathcal{F}_L$ and $\mathcal{F}_{L'}$ are unitarily equivalent.*
2. *The operator $P_L^\perp P_{L'} : V \to V$ is Hilbert-Schmidt.*
3. *The operator $P_L - P_{L'} : V \to V$ is Hilbert-Schmidt.*
4. *There exists an element $g \in \mathrm{O}_L(V)$ such that $g(L) = L'$.*
5. *There exists an element $g \in \mathrm{U}_L(V)$ such that $g(L) = L'$.*

Proof. That 1-4 are equivalent is proven in [PR94, Theorem 3.4.1 & Theorem 3.4.2]. That 4 implies 5 is obvious, after all $\mathrm{O}_L(V) \subset \mathrm{U}_L(V)$. Finally, assume that 5 holds. We shall prove that 3 then holds. First, we compute

$$[g, \mathcal{J}_L] = i[g, P_L - P_L^\perp] = i[g, 2P_L - \mathbb{1}] = 2i[g, P_L],$$

whence we conclude that $[P_L, g]$ is Hilbert-Schmidt. By assumption, we have that $g(L) = L'$, hence

$$P_L - P_{L'} = P_L - gP_L g^{-1} = (P_L g - gP_L)g^{-1} = [P_L, g]g^{-1}. \qquad \square$$

If a pair of Lagrangians satisfies any of the equivalent conditions of Lemma 3.6.1 we say that the Lagrangians are *equivalent.* It is clear that this defines an equivalence relation on the set of Lagrangians. We write $\mathrm{Lag}_L(V)$ for the set of Lagrangians equivalent to L. We embed the unitary group $\mathrm{U}(L)$ into $\mathrm{O}_L(V)$ by sending the operator $T \in \mathrm{U}(L)$ to the operator that acts by T on L and by $\alpha T \alpha$ on $\alpha(L)$.

Lemma 3.6.2. *When equipped with its usual Banach Lie group structure,* $\mathrm{U}(L)$ *is a submanifold of* $\mathrm{O}_L(V)$.

Proof. First of all, as topological spaces, both $\mathrm{U}(L)$ and $\mathrm{O}_L(V)$ are metric spaces, where the metric on $\mathrm{U}(L)$ is given by $d(T_1, T_2) = \|T_1 - T_2\|$ and the metric on $\mathrm{O}_L(V)$ is given by $d(g_1, g_2) = \|g_1 - g_2\| + \|[g_1 - g_2, \mathcal{J}_L]\|_2$. It is clear that the inclusion $\mathrm{U}(L) \to \mathrm{O}_L(V)$ is an isometry and hence a topological embedding. It remains to show that the inclusion is an immersion, this can be checked by computing the derivative of the inclusion at the identity. $\square$

Lemma 3.6.1 tells us that $\mathrm{O}_L(V)/\mathrm{U}(L) = \mathrm{Lag}_L(V)$. A direct consequence of Lemma 3.6.2 is the following result.

Lemma 3.6.3. *There exists a unique structure of Banach manifold on $\mathrm{O}_L(V)/\mathrm{U}(L)$ such that the projection $\mathrm{O}_L(V) \to \mathrm{O}_L(V)/\mathrm{U}(L)$ is a surjective submersion.*

Proof. According to [Bou98, Proposition 1.6.11 of Chapter III] it is sufficient that the subgroup $\mathrm{U}(L)$ is a submanifold of $\mathrm{O}_L(V)$. $\square$

An interesting observation that we will use momentarily is the following.

Lemma 3.6.4. *The homomorphism $\sigma : \mathrm{U}(L) \to \mathrm{Imp}_L(V), g \mapsto \Lambda_g$ is a section of $\mathrm{Imp}_L(V)|_{\mathrm{U}(L)}$.*

Proof. It is clear that σ is a group homomorphism. That it is a section follows from the identity

$$\theta_g(a) \triangleright v = \Lambda_g a \triangleright \Lambda_g^{-1} v, \quad a \in \mathrm{Cl}(V), v \in \mathcal{F}.$$

The map $\sigma : \mathrm{U}(L) \to \mathrm{Imp}_L(V)$ coincides with the local section σ defined in Section 3.5 on the intersection of their domains, hence, it is smooth in a neighbourhood of the identity, whence it is smooth everywhere. □

We view $\mathrm{Lag}_L(V)$ as a Banach manifold through the identification with $\mathrm{O}_L(V)/\,\mathrm{U}(L)$.

Remark 3.6.5. The manifold $\mathrm{Lag}_L(V)$ turns out to be a rich geometric object, and appears frequently in the literature, see among others [PS86, Chapter 7], [Bor92, Fur04, PW09]. In particular, it supports a *Pfaffian line bundle* (see [SW07]).

The description of the Banach manifold structure on $\mathrm{Lag}_L(V)$ given in [PS86, Bor92] is, at first glance, different from the one we just gave. We will prove that they in fact agree. To that end we first recall the definition from [PS86, Bor92]. For proofs of the claims we make here we refer to [Bor92, Section 2 & Section 4]. Let $W \in \mathrm{Lag}_L(V)$ be arbitrary. Then, following [Bor92], we write $I_2^a(W, W^\perp)$ for the Hilbert-Schmidt operators $A : W \to W^\perp$ that are skew in the sense that $\alpha A = -A^*\alpha$. It then follows that for all $A \in I_2^a(W, W^\perp)$ we have that $W_A := \mathrm{graph}(A) \in \mathrm{Lag}_L(V)$. For each $W \in \mathrm{Lag}_L(V)$ we define a neighbourhood $U_W \subset \mathrm{Lag}_L(V)$ of W by

$$U_W := \{\mathrm{graph}(A) \in \mathrm{Lag}_L(V) \mid A \in I_2^a(W, W^\perp)\} = I_2^a(W, W^\perp).$$

By letting W range over $\mathrm{Lag}_L(V)$ we obtain a cover of $\mathrm{Lag}_L(V)$ by U_W. Moreover $I_2^a(W, W^\perp)$ is isomorphic to $I_2^a(L, L^\perp)$ for each $W \in \mathrm{Lag}_L(V)$. This defines an atlas for $\mathrm{Lag}_L(V)$. That the transition functions are smooth is proved in [PS86, Proposition 7.1.2]. Let us write $\mathrm{Lag}_L^\vee(V)$ for $\mathrm{Lag}_L(V)$ equipped with this smooth structure. The following result then implies that we may discard this extra piece of notation immediately.

Lemma 3.6.6. *The projection map $\pi : \mathrm{O}_L(V) \to \mathrm{Lag}_L^\vee(V)$ is a surjective submersion. Hence $\mathrm{Lag}_L^\vee(V) = \mathrm{Lag}_L(V)$.*

Proof. Clearly, the projection map is surjective. To prove submersivity we start by proving submersivity at the identity of $\mathrm{O}_L(V)$. Let $\mathcal{O} \subset \mathrm{O}_L(V)$ be the open ball of radius 1 with centre the identity element. First of all, we shall prove that $\pi(\mathcal{O})$ is contained in the open neighbourhood $U_L \subset \mathrm{Lag}_L(V)$. Let $g \in \mathcal{O}$ be arbitrary. We decompose g with respect to the splitting $V = L \oplus \alpha(L)$:

$$g = \begin{pmatrix} g_{11} & \alpha g_{21} \alpha \\ g_{21} & \alpha g_{11} \alpha \end{pmatrix}.$$

Because $g \in \mathcal{O}$ we have $\|\mathbb{1} - g\| < 1$ and hence $\|\mathbb{1}_L - g_{11}\| < 1$ and hence g_{11} is invertible. We now compute

$$\begin{aligned} \pi(g) = g(L) &= \{(g_{11}v, g_{21}v) \in L \oplus L^\perp \mid v \in L\} \\ &= \{(v, g_{21}g_{11}^{-1}v) \in L \oplus L^\perp \mid v \in L\} \\ &= \mathrm{graph}(g_{21}g_{11}^{-1}) \in U_L. \end{aligned}$$

Hence, using the chart $U_L \simeq I_2^a(L, L^\perp)$, we see that the projection map π is given by $\pi(g) = g_{21}g_{11}^{-1} \in I_2^a(L, L^\perp)$. The map π factors as

$$\begin{array}{ccccccc} \mathcal{O} & \longrightarrow & I_2(L, L^\perp) \times \mathcal{B}(L)^\times & \longrightarrow & I_2(L, L^\perp) \times \mathcal{B}(L) & \longrightarrow & I_2(L, L^\perp), \\ g & \longmapsto & (g_{21}, g_{11}) & \longmapsto & (g_{21}, g_{11}^{-1}) & \longmapsto & g_{21}g_{11}^{-1} \end{array}$$

We equip $I_2(L, L^\perp)$ with the Banach space structure induced by the Hilbert-Schmidt norm, and $\mathcal{B}(L)$ with the Banach space structure induced by the operator norm, and finally we equip $\mathcal{B}(L)^\times$ with the Banach Lie group structure inherited from $\mathcal{B}(L)$. It is then clear that this first map is smooth. The second map is inversion on the second factor, which is also smooth. The fact that the last map is continuous can be seen from the usual proof that the Hilbert-Schmidt operators form an ideal in the bounded operators. It is moreover bilinear, and hence smooth.

Let us show that it is also submersive at the identity. Because the map $\exp : \mathfrak{o}_L(V) \to \mathrm{O}_L(V)$ is a diffeomorphism at the identity it suffices to check that the map $\pi \exp : \mathfrak{o}_L(V) \to I_2^a(L, L^\perp)$ is a submersion at 0. A straightforward computation using the chain rule then shows that the derivative of $\pi \exp$ at 0 is given by the projection onto the first factor with respect to the splitting $\mathfrak{o}_L(V) = I_2^a(L, L^\perp) \oplus \mathcal{B}^a(L)$, where $\mathcal{B}^a(L)$ is the space of skew operators on L.

Now, we would like to use this to prove that π is submersive everywhere. We write d_g for the operator that takes the derivative at g. Next, for each $g \in \mathrm{O}_L(V)$ we write $\rho(g)$ for the map $\mathrm{Lag}_L^\vee(V) \to \mathrm{Lag}_L^\vee(V), W \mapsto g(W)$. We first prove that the map $\rho(g)$ is a smooth submersion at L. A chart around L is given by (U_L, φ) where

$$U_L = \{\mathrm{graph}(T) \subset V \mid T \in I_2^a(L, L^\perp)\}$$

and $\varphi : U_L \to I_2^a(L, L^\perp)$ sends $\mathrm{graph}(T)$ to the operator T. A chart around $\rho(g)(L) = g(L)$ is given by $(U_{g(L)}, \varphi_g)$ where

$$U_{g(L)} = \{\mathrm{graph}(T) \subset V \mid T \in I_2^a(g(L), g(L^\perp))\}$$

and φ_g sends $\mathrm{graph}(T)$ to the operator $g^{-1}Tg$. A straightforward computation shows that composite $\varphi_g \rho(g) \varphi : I_2^a(L, L^\perp) \to I_2^a(L, L^\perp)$ is the identity, hence $\rho(g)$ is a smooth submersion (and in fact an immersion as well). We then obtain $\pi = \rho(g) \circ \pi \circ l_{g^{-1}}$, and hence $\mathrm{d}_g \pi = \mathrm{d}_L \rho(g) \circ \mathrm{d}_e \pi \circ \mathrm{d}_g l_{g^{-1}}$, from which we conclude that π is a smooth submersion at g. □

Suppose that L' is equivalent to L, then we define $\mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_L, \mathcal{F}_{L'})$ to be the set of unitary equivalences from $\mathcal{F}_L$ to $\mathcal{F}_{L'}$. It is clear that $\mathrm{U}(1)$ acts on this space, simply by multiplication. In fact, this defines a principal $\mathrm{U}(1)$-bundle $\mathcal{Q}$ on $\mathrm{Lag}_L(V)$ with total space

$$\mathcal{Q} = \coprod_{W \in \mathrm{Lag}_L(V)} \mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_W, \mathcal{F}_L).$$

We shall equip $\mathcal{Q}$ with the structure of Banach manifold such that $\mathcal{Q} \to \mathrm{Lag}_L(V)$ is a principal $\mathrm{U}(1)$-bundle momentarily.

A topological version of this bundle is considered in [Amb12, Section 11.1].

First, we make the relationship between the implementation problem and the equivalence problem more explicit:

Lemma 3.6.7. *Let $g \in \mathrm{O}_L(V)$ be arbitrary. Then the map*

$$\mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L) \to \mathrm{Imp}_L(V)_g, \qquad T \mapsto T\Lambda_g,$$

is a bijection, with inverse

$$\mathrm{Imp}_L(V)_g \to \mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L), \qquad U \mapsto U\Lambda_g^{-1}.$$

Moreover, these maps are isomorphisms of $\mathrm{U}(1)$*-torsors.*

Proof. It is clear that the maps are mutually inverse, hence we only need to check that the codomain is correctly identified. Hence, let $T \in \mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L)$ then we compute, for $v \in \mathcal{F}_L$ and $a \in \mathrm{Cl}(V)$ arbitrary

$$(T\Lambda_g)a \triangleright (T\Lambda_g)^* v = T\Lambda_g a \triangleright \Lambda_g^{-1} T^* v = \theta_g(a) \triangleright v,$$

as desired. That $U\Lambda_g^{-1} \in \mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L)$ if $U \in \mathrm{Imp}_L(V)_g$ is checked in a similar manner. It is obvious that these maps intertwine the U(1) actions. ☐

We identify $\mathrm{U}(L)$ as a Lie subgroup of $\mathrm{Imp}_L(V)$ through the immersion $\sigma : \mathrm{U}(L) \to \mathrm{Imp}_L(V)$. We define $Q := \mathrm{Imp}_L(V)/\mathrm{U}(L)$. We then obtain a commutative square

$$\begin{array}{ccc} \mathrm{Imp}_L(V) & \longrightarrow & Q \\ \big\downarrow q & & \big\downarrow \\ \mathrm{O}_L(V) & \longrightarrow & \mathrm{Lag}_L(V) \end{array}$$

It follows that Q is a principal U(1)-bundle over $\mathrm{Lag}_L(V)$, with the property that $\pi^* Q = \mathrm{Imp}_L(V)$.

Lemma 3.6.8. *For each $W \in \mathrm{Lag}_L(V)$ there is an isomorphism of* U(1)*-torsors $Q_W \to \mathcal{Q}_W$.*

Proof. Let $W \in \mathrm{Lag}_L(V)$ be arbitrary. Let $x \in Q_W$ be arbitrary. Choose a representative $U_1 \in \mathrm{Imp}_L(V)$ for x. Set $g_1 = q(U_1)$. Then Lemma 3.6.7 tells us that $U_1\Lambda_{g_1^{-1}} \in \mathcal{Q}_W$. Now, suppose that U_2 is a second representative for x. Set $g_2 = q(U_2)$. Because $U_1 \sim U_2$ we have $U_2 = U_1\sigma(g_1^{-1}g_2) = U_1\Lambda_{g_1^{-1}g_2}$. We now compute

$$U_2\Lambda_{g_2^{-1}} = U_1\Lambda_{g_1^{-1}g_2}\Lambda_{g_2^{-1}} = U_1\Lambda_{g_1^{-1}}.$$

This implies that the assignment $x \mapsto U_1\Lambda_{g_1^{-1}}$ is a well-defined map from Q_W to $\mathcal{Q}_W$. This map clearly intertwines the U(1) actions, and hence is an isomorphism of U(1)-torsors. ☐

Hence, we identify $\mathcal{Q} = Q$, which turns $\mathcal{Q}$ into a principal U(1)-bundle over $\mathrm{Lag}_L(V)$ with the property that $\pi^*\mathcal{Q} = \mathrm{Imp}_L(V)$.

The following result will be used in Section 9.1.

Lemma 3.6.9. $\mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L) = \mathrm{U}_{\mathrm{Cl}(V)^{\mathrm{s}}}(\mathcal{F}^{\mathrm{s}}_{g(L)}, \mathcal{F}^{\mathrm{s}}_L)$ *for all $g \in \mathrm{O}_L(V)$.*

Proof. The inclusion $\mathrm{U}_{\mathrm{Cl}(V)^{\mathrm{s}}}(\mathcal{F}^{\mathrm{s}}_{g(L)}, \mathcal{F}^{\mathrm{s}}_L) \subseteq \mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L)$ is routine. We consider the reverse inclusion. To that end, let $T \in \mathrm{U}_{\mathrm{Cl}(V)}(\mathcal{F}_{g(L)}, \mathcal{F}_L)$. All that we need to show is that $T(\mathcal{F}^{\mathrm{s}}_{g(L)}) \subseteq \mathcal{F}^{\mathrm{s}}_L$. Hence, let $v \in \mathcal{F}^{\mathrm{s}}_{g(L)}$. We claim that $Tv \in \mathcal{F}_L$ is a smooth vector. By [Nee10a, Theorem 7.2] it is sufficient to prove that the map $\mathrm{Imp}_L(V) \to \mathbb{C}, U \mapsto \langle UTv, Tv\rangle = \langle T^*UTv, v\rangle$ is smooth in an open neighbourhood of the identity. In turn, it suffices to show that the map $\mathrm{Imp}_L(V) \to \mathrm{Imp}_{g(L)}(V), U \mapsto T^*UT$ is smooth in an open neighbourhood of the identity. By Lemma 3.6.7 we may assume that $T = U_0\Lambda_g^{-1}$ for some $U_0 \in \mathrm{Imp}_L(V)$. Because $\mathrm{Imp}_L(V)$ is a Banach Lie group, the map $U \mapsto U_0^* U U_0$ is smooth. It remains to be shown that the map $C_{\Lambda_g} : \mathrm{Imp}_L(V) \to \mathrm{Imp}_{g(L)}(V), U \mapsto \Lambda_g U \Lambda_g^{-1}$ is smooth in an open neighbourhood of the identity. To that end, we use the fact that the Banach Lie group $\mathrm{Imp}_L(V)$ is modelled on the Banach Lie algebra $\mathfrak{imp}_L(V)$ as follows. Let $C_g : \mathfrak{o}_L(V) \to \mathfrak{o}_{g(L)}(V)$ stand for conjugation by g, this map is clearly smooth. We claim that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{o}_L(V) \oplus \mathbb{R} & \xrightarrow{C_g \oplus \mathbb{1}} & \mathfrak{o}_{g(L)}(V) \oplus \mathbb{R} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Imp}_L(V) & \xrightarrow{C_{\Lambda_g}} & \mathrm{Imp}_{g(L)}(V) \end{array}$$

From which follows that C_{Λ_g} is smooth, which would complete the proof.

To prove that this diagram indeed commutes we use the fact that the exponential of $\mathfrak{o}_L(V) \oplus \mathbb{R}$ factors through the exponential of $\mathcal{L}(\mathcal{F}_L)$ as explained in Remark 3.5.5 and proceed as follows. Let $(A, \lambda) \in \mathfrak{o}_L(V)$ be arbitrary. Now, let $f_1 \wedge \cdots \wedge f_n \in \Lambda^n g(L)$ be arbitrary, then we compute

$$\begin{aligned}\Lambda_g \tilde{A}_0 \Lambda_g^{-1}(f_1 \wedge \cdots \wedge f_n) &= \Lambda_g \tilde{A}_0(g^{-1} f_1 \wedge \cdots \wedge g^{-1} f_n) \\ &= \Lambda_g \sum_{j=1}^{n} (g^{-1} \wedge \cdots \wedge A_0 g^{-1} f_j \wedge \cdots \wedge g^{-1} f_n) \\ &= \widetilde{g A_0 g^{-1}}(f_1 \wedge \cdots \wedge f_n).\end{aligned}$$

Hence $\Lambda_g \tilde{A}_0 \Lambda_g^{-1} = \widetilde{g A_0 g^{-1}}$. Now let $\xi \in \Lambda^n g(L)$ be arbitrary, we compute

$$\Lambda_g \tilde{A}_1 \Lambda_g^{-1}(\xi) = \frac{1}{2}\left(\iota_{\Lambda_g(\hat{a}')} - \Lambda_g(\hat{a}') \wedge \xi\right).$$

A straightforward computation using (3.2) shows that $\Lambda_g \hat{a}' = \widehat{g a' g^{-1}}$, whence $\Lambda_g \tilde{A}_1 \Lambda_g^{-1} = \widetilde{g A_1 g^{-1}}$. Which in turn shows that $\Lambda_g \tilde{A} \Lambda_g^{-1} = \widetilde{g A g^{-1}}$. Exponentiating this relationship one obtains

$$\Lambda_g e^{\tilde{A}} \Lambda_g^{-1} = e^{\Lambda_g \tilde{A} \Lambda_g^{-1}} = e^{\widetilde{g A g^{-1}}}.$$

From this result it follows that the diagram above commutes, explicitly, we have:

$$\begin{array}{ccc} (A, \lambda) & \xrightarrow{C_g \oplus \mathbb{1}} & (g A g^{-1}, \lambda) \\ \downarrow & & \downarrow \\ e^{\tilde{A} + i\lambda} & \xrightarrow{C_{\Lambda_g}} & \Lambda_g e^{\tilde{A} + i\lambda} \Lambda_{g^{-1}} = e^{\widetilde{g A g^{-1}} + i\lambda} \end{array} \qquad \square$$

Let $\mathcal{Q}^* \to \mathrm{Lag}_L(V)$ be the dual of $\mathcal{Q}$. The associated bundle $\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L$ is a Banach vector bundle over $\mathrm{Lag}_L(V)$. The map $\mathrm{Cl}(V) \times \mathcal{Q}^* \times \mathcal{F}_L \to \mathcal{Q}^* \times \mathcal{F}_L, (a, u, v) \mapsto (u, a \triangleright v)$ descends to a fibrewise $\mathrm{Cl}(V)$ action on $\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L$. We observe that the map

$$\mathcal{Q}_W^* \times \mathcal{F}_L \to \mathcal{F}_W, (u, v) \mapsto u^{-1}(v),$$

descends to an isomorphism $(\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L)_W \to \mathcal{F}_W$ of left $\mathrm{Cl}(V)$-modules, for each $W \in \mathrm{Lag}_L(V)$. Using this isomorphism we equip the set $E := \coprod_{W \in \mathrm{Lag}_L(V)} \mathcal{F}_W$ with the structure of Banach vector bundle over $\mathrm{Lag}_L(V)$. The group $\mathrm{O}_L(V)$ acts on $\mathrm{Lag}_L(V)$ through the map $(g, W) \mapsto g(W)$. The bundle $E \to \mathrm{Lag}_L(V)$ can be equipped with the structure of $\mathrm{O}_L(V)$-equivariant bundle by defining the action $\mathrm{O}_L(V) \times E \to E, (g, v) \mapsto \Lambda_g(v)$.

Finally, we remark that the bundle $D := \mathrm{Imp}_L^*(V) \times_{\mathrm{U}(1)} \mathcal{F}_L \to \mathrm{O}_L(V)$ is equal to the pullback of E along $\pi : \mathrm{O}_L(V) \to \mathrm{Lag}_L(V)$. For each $g \in \mathrm{O}_L(V)$ we obtain a unitary map $D_g \to \mathcal{F}_{g(L)}, [U, v] \mapsto \Lambda_g U^{-1}(v)$. Using this identification we equip D with the structure of $\mathrm{O}_L(V)$-equivariant bundle by defining the action $\mathrm{O}_L(V) \times D \to D, (g, v) \mapsto \Lambda_g(v)$.

Remark 3.6.10. The bundle $E \to \mathrm{Lag}_L(V)$, including its $\mathrm{O}_L(V)$-equivariant structure, is also considered in [Amb12, Chapter 8] using a different construction.

We could also consider the associated bundle $\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L^{\mathrm{s}}$, however, this would not be a Banach vector bundle, because $\mathcal{F}_L^{\mathrm{s}}$ is not a Banach space. The space $\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L^{\mathrm{s}}$ does come equipped with an inclusion map $\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L^{\mathrm{s}} \to \mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L$. The appropriate tools to handle such a situation are developed in Section 8.1.

We note that, using Lemma 3.6.9 we obtain that the map

$$\mathcal{Q}_W^* \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_W^{\mathrm{s}}, (u,v) \mapsto u^{-1}(v),$$

descends to a linear bijection $(\mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L^{\mathrm{s}})_W \to \mathcal{F}_W^{\mathrm{s}}$, for each $W \in \mathrm{Lag}_L(V)$.

3.7. The basic central extension

We consider the Banach Lie group central extension $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ of Section 3.5, with respect to the data specified in Example 3.1.2. That is, $V = L^2(\mathbb{S}) \otimes \mathbb{C}^d$, the real structure α is pointwise complex conjugation, the Lagrangian L consists of those spinors that extend to anti-holomorphic functions on the disk, and the unitary structure $\mathcal{J}$ corresponds to L under the bijection of Lemma 3.1.3.

Our goal is to give an operator-algebraic construction of the basic central extension

$$U(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$$

of the loop group of $\mathrm{Spin}(d)$. The existence of such models using implementers on Fock space is well-known, see, e.g. [PS86, Nee02, SW07], but we have not found a complete treatment of all aspects and in our specific setting. Before we start, we briefly recall how the group $L\,\mathrm{Spin}(d) := C^\infty(S^1, \mathrm{Spin}(d))$ can be equipped with the structure of Fréchet Lie group, see [PS86, Section 3.2] for more details. We will then downgrade all Banach Lie groups to Fréchet Lie groups, and handle all smoothness issues within the Fréchet setting.

The vector space $L\mathfrak{spin}(d)$ is a Fréchet space when equipped with the topology of uniform convergence of functions and all partial derivatives. Equipped with the pointwise bracket, $L\mathfrak{spin}(d)$ is then a Fréchet Lie algebra. The pointwise exponential $\exp : L\mathfrak{spin}(d) \to L\,\mathrm{Spin}(d)$ may then be used to define charts for a Fréchet Lie group structure on $L\,\mathrm{Spin}(d)$. We write $\mathrm{End}(d)$ for the algebra of endomorphism of $\mathbb{C}^d$. The algebra $L\,\mathrm{End}(d) := C^\infty(S^1, \mathrm{End}(d))$ is a Fréchet algebra in the same way as $L\mathfrak{spin}(d)$. It acts through bounded operators on $V = L^2(\mathbb{S}) \otimes \mathbb{C}^d$ via pointwise multiplication on $\mathbb{C}^d$, we denote this representation by $m : L\,\mathrm{End}(d) \to \mathcal{B}(V)$.

Lemma 3.7.1. *The image of m is contained in the Banach algebra $\mathcal{B}_L(V)$. Furthermore, m is a continuous homomorphism of Fréchet algebras.*

Proof. Let $f \in L\,\mathrm{End}(d)$. Then we may write f as

$$f(z) = \sum_{n\in\mathbb{Z}} A_n z^n, \quad z \in S^1,$$

where A_n are elements of $\mathrm{End}(d)$. In the proof of Theorem 6.3.1 in [PS86] it is shown that we now have

$$\|[m(f), \mathcal{J}]\|_2 = \left(\sum_{n\in\mathbb{Z}} |n| \|A_n\|^2 \right)^{1/2}. \tag{3.7}$$

To see that this is finite we proceed as follows. If f is smooth, then its derivative f' is square-integrable, with L^2-norm given by

$$\|f'\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} n^2 \|A_n\|^2 \geqslant \sum_{n\in\mathbb{Z}} |n| \|A_n\|^2.$$

This shows that $m(f) \in \mathcal{B}_L(V)$. Furthermore, because S^1 is compact, we have

$$|f|_1 = \sup_{t \in S^1} \|f'\| \geqslant \|f'\|_{L^2} \geqslant \|[m(f), \mathcal{J}]\|_2,$$

which will be useful in the next step. It is easy to see that m is an algebra homomorphism, and in particular linear. Thus, all that remains to be shown is that m is continuous. A simple calculation shows $\|m(f)\|^2 \leqslant |f|_0$ for all $f \in L\,\mathrm{End}(d)$; hence,

$$\|m(f)\|_{\mathcal{J}} = \|m(f)\| + \|[m(f), \mathcal{J}]\|_2 \leqslant |f|_0 + |f|_1,$$

which implies that m is continuous. □

Both $L\,\mathrm{End}(d)$ and $\mathcal{B}_L(V)$ are equipped with an exponential map. We have already investigated the properties of the exponential map on $\mathcal{B}_L(V)$ in Section 3.4; the well-definedness of the exponential on $L\,\mathrm{End}(d)$ is a standard fact. Since m is a continuous homomorphism of Fréchet algebras by Lemma 3.7.1, we have a commutative diagram

$$\begin{array}{ccc} L\,\mathrm{End}(d) & \xrightarrow{m} & \mathcal{B}_L(V) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ L\,\mathrm{End}(d) & \xrightarrow[m]{} & \mathcal{B}_L(V). \end{array}$$

One sees readily that m restricts to a map $m : L\,\mathrm{SO}(d) \to \mathrm{O}_L(V)$. Using the natural projection map $\mathrm{Spin}(d) \to \mathrm{SO}(d)$ we obtain a projection map $\pi : L\,\mathrm{Spin}(d) \to L\,\mathrm{SO}(d)$. Let us define $M := m \circ \pi : L\,\mathrm{Spin}(d) \to \mathrm{O}_L(V)$. Lemma 3.7.1 now implies:

Proposition 3.7.2. *The map $M : L\,\mathrm{Spin}(d) \to \mathrm{O}_L(V)$ is a homomorphism of Fréchet Lie groups.*

We may now pull the central extension $\mathrm{U}(1) \to \mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ back along the map M to obtain a central extension

$$\mathrm{U}(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$$

of Fréchet Lie groups, fitting into a commutative diagram

$$\begin{array}{ccc} \widetilde{L\,\mathrm{Spin}}(d) & \xrightarrow{\widetilde{M}} & \mathrm{Imp}_L(V) \\ \downarrow & & \downarrow \\ L\,\mathrm{Spin}(d) & \xrightarrow[M]{} & \mathrm{O}_L(V). \end{array}$$

In particular, we see that the elements of $\widetilde{L\,\mathrm{Spin}}(d)$ act through $\widetilde{M}$ on the Fock space $\mathcal{F}$.

Proposition 3.7.3. *As operators on Fock space, all elements of $\widetilde{L\,\mathrm{Spin}}(d)$ are even.*

Proof. Because $\mathrm{Spin}(d)$ is simply connected, $L\,\mathrm{Spin}(d)$ is connected; hence $M(L\,\mathrm{Spin}(d))$ is contained in the connected component of the identity of $\mathrm{O}_L(V)$. Proposition 3.5.9 then tells us that the elements of $\widetilde{L\,\mathrm{Spin}}(d)$ are even. □

In Section 3.5 we considered a local section σ of the projection $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$, and we considered the corresponding 2-cocycle ω_σ of (3.4). Next, we pull ω_σ back to a cocycle on $L\mathfrak{so}(d)$, and from there to a cocycle on $L\mathfrak{spin}(d)$.

Lemma 3.7.4. *The pullback of the 2-cocycle ω_σ on $\mathfrak{o}_L(V)$ to $L\mathfrak{so}(d)$ is given by*

$$L\mathfrak{so}(d) \times L\mathfrak{so}(d) \to \mathbb{R}\ ,\ (f,g) \mapsto -\frac{1}{4\pi i}\int_0^{2\pi} \operatorname{trace}(f(t)g'(t))\,dt$$

Proof. This result is well-known, see for example [PS86, Proposition 6.7.1] or [Bor92, Proposition 8.2]. For the convenience of the reader we present a proof adapted to our notation. Let $f, g \in L\mathfrak{so}(d)$ and let A_1, A_2 and A_3 be the operators corresponding to f, g and $[f,g]$ respectively; and write

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

with respect to the decomposition $V = L \oplus \alpha(L)$. From the formula (3.5) for ω we see that we need to prove that

$$\frac{1}{2}\operatorname{trace}\left([a_1,a_2] - a_3\right) = -\frac{1}{4\pi i}\int_0^{2\pi} \operatorname{trace}(f(t)g'(t))\mathrm{d}t.$$

By linearity it suffices to consider $f(t) = Xe^{ikt}$ and $g(t) = Ye^{imt}$ with $X, Y \in \mathfrak{gl}_d(\mathbb{R})$ and $k, m \in \mathbb{Z}$. We now distinguish two cases:

$k + m \neq 0$: In this case one immediately sees that $\int_0^{2\pi}(f(t)g'(t))\mathrm{d}t = 0$. On the other hand, the operators $[a_1, a_2]$ and a_3 have no diagonal components, and hence $\operatorname{trace}([a_1,a_2] - a_3) = 0$.

$k + m = 0$: In this case we have

$$-\frac{1}{4\pi i}\int_0^{2\pi} \operatorname{trace}(f(t)g'(t))\mathrm{d}t = -\frac{k}{2}\operatorname{trace}(XY).$$

On the other hand we note that the operators $[a_1, a_2]$ and a_3 preserve the subspaces $\mathbb{C}^d \cdot z^q$ for $q \in \mathbb{N}_{\geqslant 0}$. Let $v \in \mathbb{C}^d$ be arbitrary. If $q \geqslant k$, then $[a_1,a_2]vz^q = a_3vz^q$. If $q < k$ then $[a_1,a_2]vz^q = -YXvz^q$, while $a_3vz^q = [X,Y]vz^q$, hence

$$\frac{1}{2}\operatorname{trace}([a_1,a_2] - a_3) = -\frac{1}{2}\sum_{j=0}^{k-1}\operatorname{trace}(XY) = -\frac{k}{2}\operatorname{trace}(XY).$$

This concludes the proof. ☐

The last link in our argument is the following lemma, which is well-known and easy to check using any explicit description of the root lattice of $\mathfrak{spin}(d)$.

Lemma 3.7.5. *For all $d > 4$, the bilinear form*

$$\langle\cdot,\cdot\rangle : \mathfrak{spin}(d) \times \mathfrak{spin}(d) \to \mathbb{R}\ ,\ (X,Y) \mapsto -\frac{1}{2}\operatorname{trace}(XY),$$

is the basic one, i.e., it is the smallest bilinear form such that $\langle h_\alpha, h_\alpha\rangle$ is even for every coroot h_α.

Theorem 3.7.6. *If $d > 4$, then the pullback of the central extension* $\mathrm{U}(1) \to \mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ *along M is the basic central extension of* $L\,\mathrm{Spin}(d)$.

Proof. According to [PS86, Theorem 4.4.1 (iv) & Proposition 4.4.6] and the preceding discussion, a central extension of $L\,\mathrm{Spin}(d)$, coming from a Lie algebra cocycle ω is basic if

$$\omega(f,g) = \frac{1}{2\pi i}\int_0^{2\pi} \langle f(t), g'(t)\rangle \mathrm{d}t,$$

where $\langle\cdot,\cdot\rangle$ is the basic inner product. Now, Lemmas 3.7.4 and 3.7.5 complete the proof. ☐

4. Tomita-Takesaki theory and Connes fusion

One of the main goals of this work is to define a spinor bundle on loop space equipped with *fusion* over compatible loops. This notion of fusion involves the notion of tensor product for bimodules of von Neumann algebras, which is called Connes fusion. In this section we give an exposition of the notion of Connes fusion in a manner specifically geared to our application. A comprehensive reference is [Tak13], a more concise review is given in [Tho11].

Let $\mathcal{A}_1$, $\mathcal{A}_2$ be von Neumann algebras, then, an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule is a Hilbert space H equipped with normal $*$-homomorphisms $\mathcal{A}_1 \to \mathcal{B}(H)$ and $\mathcal{A}_2^{\mathrm{op}} \to \mathcal{B}(H)$ such that the left and right actions commute. Let $\mathcal{A}_3$ be another von Neumann algebra, and let K be an $\mathcal{A}_2$-$\mathcal{A}_3$-bimodule. In Section 4.2 we will construct an $\mathcal{A}_1$-$\mathcal{A}_3$-bimodule $H \boxtimes_{\mathcal{A}_2} K$. This construction is associative (see Lemma 4.2.7) and natural (see Lemma 4.2.5).

This tensor product, $\boxtimes_{\mathcal{A}_2}$, should be seen as an adaptation of the relative tensor product of bimodules of algebras to the von Neumann setting. An important observation in the setting of algebras, is that an algebra is always a bimodule over itself, and moreover, that it is neutral with respect to the relative tensor product. The analogous statement does not directly hold in the von Neumann setting; a von Neumann algebra is not in a natural way a Hilbert space, hence it cannot be seen as a bimodule over itself. It turns out, however, that each von Neumann algebra has a completion to a Hilbert space, which is a bimodule for the von Neumann algebra that we started with. This bimodule is called the *standard form* of the von Neumann algebra. We will recall this construction in Section 4.1, and we recall the result that this indeed yields a unit for the Connes fusion product in Lemma 4.2.4.

The notion of Connes fusion was famously used in [Was98] to equip the collection of positive energy representations of $L\,\mathrm{SU}(n)$ with a product. The relative tensor product for bimodules of algebras allows one to define the notion of Morita equivalence of algebras. The Connes fusion product allows one to define this notion for von Neumann algebras, this was investigated in [Bro03].

4.1. Standard forms of a von Neumann algebra and Tomita-Takesaki theory

We first define the usual notions of modules of von Neumann algebras to fix notation. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be von Neumann algebras.

Definition 4.1.1. A left $\mathcal{A}_1$-module is a Hilbert space H equipped with a normal $*$-homomorphism $\mathcal{A}_1 \to \mathcal{B}(H)$. We adopt the notation $a \triangleright v$ for the element $a \in \mathcal{A}_1$ acting on the vector $v \in H$ from the left. A right $\mathcal{A}_2$-module is a Hilbert space H equipped with a normal $*$-homomorphism $\mathcal{A}_2^{\mathrm{opp}} \to \mathcal{B}(H)$. We adopt the notation $v \triangleleft a$ for the right action of $a \in \mathcal{A}_2$ on $v \in H$. An $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule is a Hilbert space H which is a left $\mathcal{A}_1$-module and at the same time a right $\mathcal{A}_2$-module, such that the left and right actions commute, i.e. $(a_1 \triangleright v) \triangleleft a_2 = a_1 \triangleright (v \triangleleft a_2)$ for all $a_1 \in \mathcal{A}_1$, all $a_2 \in \mathcal{A}_2$, and all $v \in H$. When we say that H is an $\mathcal{A}_1$-module, we mean that it a left $\mathcal{A}_1$-module.

First we recall the notion of a standard form, see [Tak13, Chapter IX, Definition 1.13]) or [Haa75].

Definition 4.1.2. A *standard form* of a von Neumann algebra $\mathcal{A}$ is a quadruple $(\mathcal{A}, H, J, P)$, where H is an $\mathcal{A}$-module, J is a conjugate-linear isometry with $J^2 = 1$, and P is a closed self-dual cone in H, subject to the following conditions:

1. $J\mathcal{A}J = \mathcal{A}'$
2. $JaJ = a^*$ for all $a \in \mathcal{A} \cap \mathcal{A}'$
3. $Jv = v$ for all $v \in P$
4. $aJaJP \subseteq P$ for all $a \in \mathcal{A}$

Any standard form $(\mathcal{A}, H, J, P)$ of a von Neumann algebra $\mathcal{A}$ can be equipped with the structure of $\mathcal{A}$-$\mathcal{A}$-bimodule by defining the right action by

$$\begin{aligned} H \otimes \mathcal{A} &\to H, \\ v \otimes a &\mapsto Ja^*J \triangleright v. \end{aligned}$$

We shall write $v \triangleleft a$ for the right action of a on v.

The following result, proved in [Haa75, Theorem 2.3], tells us that standard forms are unique up to unique isomorphism.

Theorem 4.1.3. *Suppose that $\{\mathcal{A}_1, H_1, J_1, P_1\}$ and $\{\mathcal{A}_2, H_2, J_2, P_2\}$ are standard forms. Suppose furthermore that π is an isomorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$, then there exists a unique unitary u from H_1 onto H_2 such that*

1. $\pi(a) = uau^*$ *for all* $a \in \mathcal{A}_1$
2. $J_2 = uJ_1u^*$
3. $P_2 = uP_1$

Let H be a left $\mathcal{A}$-module. A vector $\xi \in H$ is called *cyclic* if $\mathcal{A} \triangleright \xi$ is dense in H, and it is called *separating* if the map $\mathcal{A} \to H, a \mapsto a \triangleright \xi$ is injective. A vector $\xi \in H$ is separating for $\mathcal{A}$ if and only if it is cyclic for $\mathcal{A}'$. If a cyclic and separating vector $\xi \in H$ is given, then one can equip H with the structure of standard form of $\mathcal{A}$ as follows.

First, we consider the densely defined Tomita operator $S : H \to H$. It is defined to be the closure of the operator

$$\mathcal{A} \triangleright \xi \to \mathcal{A} \triangleright \xi, \quad a \triangleright \xi \mapsto a^* \triangleright \xi.$$

We then write $S = J\Delta^{1/2}$ for the polar decomposition of S, where J is an anti-unitary map called the *modular conjugation* and $\Delta^{1/2}$ is a positive unbounded map called the *modular operator*. These operators depend on the choice of cyclic and separating vector ξ. Where confusion may arise we will write S_ξ, J_ξ and $\Delta_\xi^{1/2}$. A fundamental result of Tomita-Takesaki theory is then the following, proven in, for example [Tak13, Chapter IX]

Theorem 4.1.4. *The assignment $\mathcal{A} \ni a \mapsto Ja^*J$ is an anti-isomorphism of von Neumann algebras from $\mathcal{A}$ onto its commutant $\mathcal{A}'$.*

It is proved in [Ara74, Lemma 3] that if $a \in \mathcal{A} \cap \mathcal{A}'$, then $JaJ = a^*$. We define $P \subset H$ to be the closure of $\{JaJa \triangleright \xi \in H \mid a \in \mathcal{A}\}$. Then P is a closed self-dual cone in H. It is proved in [Ara74, Theorem 4] that $Jv = v$ for all $v \in P$ and that $aJaJP \subseteq P$ for all $a \in \mathcal{A}$. In conclusion, we have the following result.

Lemma 4.1.5. *The quadruple $(\mathcal{A}, H, J, P)$ is a standard form of $\mathcal{A}$.*

It is well-known that, through the GNS construction, any normal state on a von Neumann algebra, $\mathcal{A}$, produces a representation of $\mathcal{A}$. This representation always has a cyclic vector. If the state is faithful, then it has a cyclic and separating vector. This construction will be useful later, so we review the main steps now. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a faithful and normal state on a von Neumann algebra $\mathcal{A}$. Then the assignment $\mathcal{A} \times \mathcal{A} \to \mathbb{C}, (a, b) \mapsto \phi(b^* a)$ is a non-degenerate sesquilinear form on $\mathcal{A}$. We write $L^2_\phi(\mathcal{A})$ for the completion of $\mathcal{A}$ with respect to this sesquilinear form. The algebra $\mathcal{A}$ acts from the left on $L^2_\phi(\mathcal{A})$ by (the extension of) left multiplication. Clearly the identity $\mathbb{1} \in \mathcal{A} \subseteq L^2_\phi(\mathcal{A})$ is a cyclic and separating vector for this left action. It follows that the quadruple $(\mathcal{A}, L^2_\phi(\mathcal{A}), J_{\mathbb{1}}, P)$ is a standard form of $\mathcal{A}$.

Lemma 4.1.6. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space H, and let $\xi \in H$ be a cyclic and separating vector. Let $\phi : \mathcal{A} \to \mathbb{C}$ be the faithful and normal state $\phi(a) = \langle a \triangleright \xi, \xi \rangle$. The unitary map $f : H \to L^2_\phi(\mathcal{A})$ from Theorem 4.1.3 is given by the closure of the map*

$$f : a \triangleright \xi \mapsto a.$$

Proof. A straightforward verification shows that the map f satisfies properties 1-3 from Theorem 4.1.3 □

4.2. Connes fusion of bimodules

Let $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ be von Neumann algebras. Let H be an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule and let K be an $\mathcal{A}_2$-$\mathcal{A}_3$-bimodule. In the sequel our goal is to explain the definition of the Connes fusion product of H with K. In short, the Connes fusion of H with K is an $\mathcal{A}_1$-$\mathcal{A}_3$-bimodule $H \boxtimes_\phi K$, which is defined with respect to some faithful and normal state $\phi : \mathcal{A}_2 \to \mathbb{C}$. The result will be, up to unique isomorphism, independent of the state ϕ. Our main references are [Tho11] and [Tak13, Chapter IX].

Remark 4.2.1. There are ways to define the Connes fusion product even when the von Neumann algebra $\mathcal{A}_2$ does not admit a faithful and normal state. We shall not need this more involved construction, so we shall not mention it in the sequel.

Following the preceding discussion we equip the standard form $L^2_\phi(\mathcal{A}_2)$ of $\mathcal{A}_2$ with the right action given by

$$\begin{aligned} L^2_\phi(\mathcal{A}_2) \otimes \mathcal{A}_2 &\to L^2_\phi(\mathcal{A}_2), \\ \xi \otimes a &\mapsto Ja^* J \triangleright \xi. \end{aligned}$$

From Theorem 4.1.4 we see that $Ja^*J \in \mathcal{A}'$, and hence that the left and right action commute.

We write $\mathcal{D}(H, \phi) := \mathrm{Hom}_{-\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), H)$ for the space of right module maps from $L^2_\phi(\mathcal{A}_2)$ into H. This space is canonically an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule, explicitly, if $x \in \mathcal{D}(H, \phi)$, $v \in L^2_\phi(\mathcal{A}_2)$, $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, then we have

$$(a_1 \triangleright x)(v) := a_1 \triangleright x(v), \qquad \text{and} \qquad (x \triangleleft a_2)(v) := x(a_2 \triangleright v).$$

We note that $\mathcal{D}(H,\phi)$ includes into H as a dense subspace through the map $x \mapsto x(\mathbb{1})$. This map is, however, not a bimodule map on the nose, in fact, one may verify that the right action is twisted by conjugation by the modular operator $\Delta^{1/2}$. There is a canonical $\mathcal{A}_2$-valued inner product on $\mathcal{D}(H,\phi)$, defined as follows. If $x \in \mathcal{D}(H,\phi)$, then its adjoint, written x^*, is an element of $\mathrm{Hom}_{-\mathcal{A}_2}(H, L^2_\phi(\mathcal{A}_2))$. Hence if $x, y \in \mathcal{D}(H,\phi)$, then

$$y^* x \in \mathrm{Hom}_{-\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2)).$$

There is a canonical isomorphism $p_\phi : \mathrm{Hom}_{-\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2)) \to \mathcal{A}_2$, which is determined by the relation

$$p_\phi(x) \triangleright v = x(v),$$

for $x \in \mathrm{Hom}_{-\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2))$ and $v \in L^2_\phi(\mathcal{A}_2)$. The aforementioned $\mathcal{A}_2$-valued inner product on $\mathcal{D}(H,\phi)$ is given by

$$(x, y) = p_\phi(y^* x).$$

On the algebraic tensor product $\mathcal{D}(H,\phi) \otimes K$ we define a (degenerate) sesquilinear form by

$$\langle (x \otimes v), (y \otimes w) \rangle_\phi = \langle p_\phi(y^* x) \triangleright v, w \rangle_K.$$

Definition 4.2.2. The *Connes fusion product $H \boxtimes_\phi K$ of H with K relative to the faithful and normal state ϕ* of $\mathcal{A}_2$ is the completion of

$$\mathcal{D}(H,\phi) \otimes K / \ker\langle \cdot, \cdot \rangle_\phi$$

with respect to the inner product $\langle \cdot, \cdot \rangle_\phi$, with the left $\mathcal{A}_1$ action obtained from the $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule structure of $\mathcal{D}(H,\phi)$, and the right $\mathcal{A}_3$ action obtained from the $\mathcal{A}_2$-$\mathcal{A}_3$ bimodule structure on K.

The definition above does not treat H and K on equal footing, the following observation tells us that this is just an artifact of our description. We define $\mathcal{D}'(K,\phi) = \mathrm{Hom}_{\mathcal{A}_2-}(L^2_\phi(\mathcal{A}_2), K)$ to be the space of left module maps from $L^2_\phi(\mathcal{A}_2)$ into K. We then have that $\mathcal{D}'(K,\phi)$ includes into K as a dense subspace, through the map $x \mapsto x(\mathbb{1})$. Using the canonical isomorphism p'_ϕ : $\mathrm{Hom}_{\mathcal{A}_2-}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2)) \to \mathcal{A}_2$ we define sesquilinear forms on $H \otimes \mathcal{D}'(K,\phi)$ and on $\mathcal{D}(H,\phi) \otimes \mathcal{D}'(K,\phi)$ respectively

$$\begin{aligned}
\langle (v \otimes x'), (w \otimes y') \rangle'_\phi &= \langle v \triangleleft p'_\phi((y')^* x'), w \rangle_H, && v, w \in H, \quad x', y' \in \mathcal{D}'(K,\phi),\\
\langle (x \otimes x'), (y \otimes y') \rangle''_\phi &= \langle p_\phi(y^* x) \triangleright \mathbb{1} \triangleleft p'_\phi((y')^* x'), \mathbb{1} \rangle_{L^2_\phi(\mathcal{A}_2)} && x, y \in \mathcal{D}(H,\phi) \quad x', y' \in \mathcal{D}'(K,\phi).
\end{aligned}$$

Lemma 4.2.3. *The spaces $H \otimes \mathcal{D}'(K,\phi)/\ker\langle \cdot, \cdot \rangle'_\phi$ and $\mathcal{D}(H,\phi) \otimes \mathcal{D}'(K,\phi)/\ker\langle \cdot, \cdot \rangle''_\phi$ are dense in $H \boxtimes_\phi K$.*

For a proof, see [Tho11, Section 5.2] or [Tak13, Proposition 3.15 & Definition 3.16]. Lemma 4.2.3 allows us to identify $H \boxtimes_\phi K$ with the completion of $H \otimes \mathcal{D}'(K,\phi)/\ker\langle \cdot, \cdot \rangle'_\phi$, which will be used in Lemma 4.2.7.

Lemma 4.2.4. *The $\mathcal{A}_2$ bimodule $L^2_\phi(\mathcal{A}_2)$ is neutral with respect to Connes fusion.*

Proof. One verifies that the map

$$\begin{aligned}
\mathcal{D}(L^2_\phi(\mathcal{A}_2), \phi) \otimes K &\to K,\\
x \otimes v &\mapsto p_\phi(x) \triangleright v,
\end{aligned}$$

induces an isomorphism of $\mathcal{A}_2$-$\mathcal{A}_3$ bimodules $L^2_\phi(\mathcal{A}_2) \boxtimes_\phi K$ and K. A similar map exists for the Connes fusion $H \boxtimes_\phi L^2_\phi(\mathcal{A}_2)$. □

Using Theorem 4.1.3 it follows that Lemma 4.2.4 generalizes immediately to the statement that any standard form of $\mathcal{A}_2$ is neutral with respect to Connes fusion.

Now we come to the functoriality of the Connes fusion product, see also [BDH14]. Let $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ be von Neumann algebras. Let H' be a $\mathcal{B}_1$-$\mathcal{B}_2$ bimodule and let K' be a $\mathcal{B}_2$-$\mathcal{B}_3$ bimodule. Let

$$\nu_i : \mathcal{A}_i \to \mathcal{B}_i, \text{ for } i = 1, 2, 3$$

be $*$-isomorphisms. Let $\nu_H : H \to H'$ and $\nu_K : K \to K'$ be unitary intertwiners along the isomorphisms ν_i, i.e.

$$\nu_1(a_1) \triangleright \nu_H(v) \triangleleft \nu_2(a_2) = \nu_H(a_1 \triangleright v \triangleleft a_2),$$

and similarly for ν_K. Let $\phi' : \mathcal{B}_2 \to \mathbb{C}$ be a faithful and normal state. Finally, denote by $\psi : \mathcal{A}_2 \to \mathbb{C}$ the faithful and normal state $\phi' \circ \nu_2$. Let $u : L^2_\phi(\mathcal{A}_2) \to L^2_\psi(\mathcal{A}_2)$ be the unitary given by Theorem 4.1.3, with $\pi = \mathbb{1}$.

Lemma 4.2.5. *The map $\nu_2 : \mathcal{A}_2 \to \mathcal{B}_2$ extends to a unitary map $\overline{\nu}_2 : L^2_\psi(\mathcal{A}_2) \to L^2_{\phi'}(\mathcal{B}_2)$ which intertwines both the left and the right action along ν_2. The map*

$$\begin{aligned} \mathcal{D}(H, \phi) \otimes K &\to \mathcal{D}(H', \phi') \otimes K', \\ x \otimes v &\mapsto \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v), \end{aligned}$$

descends to an isomorphism

$$\nu_H \boxtimes \nu_K : H \boxtimes_\phi K \to H' \boxtimes_{\phi'} K',$$

which intertwines the actions along ν_1 and ν_3. Moreover, the assignment $(\nu_H, \nu_K) \mapsto \nu_H \boxtimes \nu_K$ is compatible with composition of isomorphisms. To be precise, suppose we are given the following data: von Neumann algebras $\mathcal{C}_i$ for $i = 1, 2, 3$, a $\mathcal{C}_1$-$\mathcal{C}_2$ bimodule H'' and a $\mathcal{C}_2$-$\mathcal{C}_3$ bimodule K'', equipped with isomorphisms $\nu_{H'} : H' \to H''$ and $\nu_{K'} : K' \to K''$, which intertwine the actions along von Neumann algebra isomorphisms $\nu_i' : \mathcal{A}_i' \to \mathcal{A}_i''$. Then the following diagram commutes

$$\begin{array}{ccccc} H \boxtimes_\phi K & \xrightarrow{\nu_H \boxtimes \nu_K} & H' \boxtimes_{\phi'} K' & \xrightarrow{\nu_{H'} \boxtimes \nu_{K'}} & H'' \boxtimes_{\phi''} K'' \end{array}$$
$$H \boxtimes_\phi K \xrightarrow{\nu_{H'}\nu_H \boxtimes \nu_{K'}\nu_K} H'' \boxtimes_{\phi''} K'' \qquad (4.1)$$

or, equivalently $(\nu_{H'} \boxtimes \nu_{K'})(\nu_H \boxtimes \nu_K) = \nu_{H'}\nu_H \boxtimes \nu_{K'}\nu_K$.

Proof. The fact that ν_2 extends to a unitary map follows from the fact that it intertwines the inner product $\mathcal{A}_2 \times \mathcal{A}_2 \to \mathbb{C}, (a_1, a_2) \mapsto \psi(a_2^* a_1)$ with the inner product $\mathcal{B}_2 \times \mathcal{B}_2 \to \mathbb{C}, (a_1, a_2) \mapsto \phi'(a_2^* a_1)$. The fact that this extension intertwines the left and right actions along ν_2 follows from the fact that ν_2 is a *-isomorphism, which implies that $J_{\phi'} = \overline{\nu}_2 J_\phi \overline{\nu}_2^*$.

To prove that the map $x \otimes v \mapsto \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v)$ descends to an isomorphism it suffices to show that it intertwines the inner products. Explicitly, we need to prove that for all $x \otimes v$ and $y \otimes w$ in $\mathcal{D}(H, \phi) \otimes K$ we have

$$\langle \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v), \nu_H y u^* \overline{\nu}_2^* \otimes \nu_K(w) \rangle \stackrel{?}{=} \langle x \otimes v, y \otimes w \rangle.$$

We start from the left-hand side

$$\begin{aligned} \langle \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v), \nu_H y u^* \overline{\nu}_2^* \otimes \nu_K(w) \rangle &= \langle p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*) \nu_K(v), \nu_K(w) \rangle \\ &= \langle \nu_K^* p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*) \nu_K(v), w \rangle. \end{aligned}$$

Hence it suffices to show that

$$\nu_K^* p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\nu_K(v) \stackrel{?}{=} p_\phi(y^*x)v. \tag{4.2}$$

Using the fact that ν_K is an intertwiner along ν_2 we obtain

$$\nu_K^* p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\nu_K = \nu_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)) \tag{4.3}$$

Now, we compute the action of the right hand side of (4.3) on an element $a \in L^2_\psi(\mathcal{B}_2)$

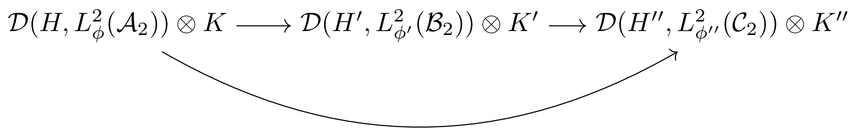
$$\begin{aligned} \nu_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*))a &= \overline{\nu}_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\overline{\nu}_2(a)) \\ &= u y^* x u^* a && \text{(Defining property of } p_{\phi'}) \\ &= u p_\phi(y^*x) u^*(a) && \text{(Defining property of } p_\phi) \\ &= p_\phi(y^*x)a, \end{aligned}$$

since this holds for all a in $L^2_\psi(\mathcal{B}_2)$ we obtain

$$\nu_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)) = p_\phi(y^*x).$$

Together with (4.3), this implies (4.2).

Finally, to prove that the diagram (4.1) commutes we shall prove that the following diagram commutes:



The composition of the arrows on top is given by the map

$$x \otimes v \mapsto \nu_{H'}\nu_H x u^* \overline{\nu}_2^* (u')^* (\overline{\nu}_2')^* \otimes \nu_{K'}\nu_K(v), \tag{4.4}$$

where $u' : L^2_{\phi'}(\mathcal{B}_2) \to L^2_{\phi''\nu_2'}(\mathcal{B}_2)$ is the unitary given by Theorem 4.1.3, with $\pi = \mathbb{1}$. On the other hand, the bottom map is given by

$$x \otimes v \mapsto \nu_{H'}\nu_H x (u'')^* \overline{\nu}_2^* (\overline{\nu}_2')^* \otimes \nu_{K'}\nu_K(v), \tag{4.5}$$

where $u'' : L^2_\phi(\mathcal{A}_2) \to L^2_{\phi''\nu_2'\nu_2}(\mathcal{A}_2)$ is the unitary given by Theorem 4.1.3, with $\pi = \mathbb{1}$. Note that in this expression, $\overline{\nu}_2$ is viewed as an isomorphism from $L^2_{\phi''\nu_2'\nu_2}(\mathcal{A}_2)$ into $L^2_{\phi''\nu_2'}(\mathcal{B}_2)$, (instead of from $L^2_{\phi'\nu_2}(\mathcal{B}_2)$ into $L^2_{\phi'}(\mathcal{B}_2)$ as before). Comparing (4.4) with (4.5), we see that it is sufficient to prove that

$$u^*\overline{\nu}_2^*(u')^*(\overline{\nu}_2')^* \stackrel{?}{=} (u'')^*\overline{\nu}_2^*(\overline{\nu}_2')^*.$$

Taking the adjoint on both sides of the equation and cancelling $\overline{\nu}_2'$, we see that this is equivalent to

$$u'\overline{\nu}_2 u \stackrel{?}{=} \overline{\nu}_2 u''.$$

One checks that both $u'\overline{\nu}_2 u$ and $\overline{\nu}_2 u''$ are isomorphisms from $L^2_\phi(\mathcal{A}_2)$ into $L^2_{\phi''\nu_2'}(\mathcal{B}_2)$. We claim that they both satisfy properties (1-3) from Theorem 4.1.3, hence that by the uniqueness statement in that theorem we conclude they must be equal. Let us check that they do in fact satisfy properties (1-3):

1. Both maps intertwine the left action along ν_2.

2. By construction we have $\overline{\nu_2} u'' J_\phi (u'')^* \overline{\nu_2}^* = J_{\phi''\nu_2'} = u'\overline{\nu}_2 u J_\phi u^* \overline{\nu}_2^* (u')^*$.

3. From Theorem 4.1.3 we have that $uP_\phi = P_{\phi'\nu_2}$. Now we argue that $\overline{\nu}_2 P_{\phi'\nu_2} = P_{\phi'}$. Let $J_{\phi'\nu_2} a J_{\phi'\nu_2} a \in P_{\phi'\nu_2}$ be arbitrary. Then we compute

$$\overline{\nu}_2 J_{\phi'\nu_2} a J_{\phi'\nu_2} a = \overline{\nu}_2 J_{\phi'\nu_2} \overline{\nu}_2^* \overline{\nu}_2 a \overline{\nu}_2^* \overline{\nu}_2 J_{\phi'\nu_2} \overline{\nu}_2^* \overline{\nu}_2 a = J_{\phi'} \nu_2(a) J_{\phi'} \overline{\nu}_2 a,$$

from which the claim follows. Then, another application of Theorem 4.1.3 yields $u' P_{\phi'} = P_{\phi''\nu_2'}$, hence $u'\overline{\nu_2} u P_\phi = P_{\phi''\nu_2'}$. A similar argument then shows that $\overline{\nu}_2 u'' P_\phi = P_{\phi''\nu_2'}$.

This completes the proof. □

Remark 4.2.6. We have defined the map $\nu_H \boxtimes \nu_K$ by specifying it on the dense subspace $\mathcal{D}(H,\phi) \otimes K$. Of course we could have just as well defined the map $\nu_H \boxtimes \nu_K$ to be the extension of a similarly defined map on the dense subspace $H \otimes \mathcal{D}'(K,\phi)$. One may check that this procedure has the same result.

We may specialize Lemma 4.2.5 to the case that all isomorphisms ν are identities to conclude that for any two faithful and normal states ϕ, ϕ' on $\mathcal{B}_2$, there exists a natural isomorphism

$$H \boxtimes_\phi K \to H \boxtimes_{\phi'} K.$$

The fact that the diagram (4.1) commutes then tells us that these isomorphisms are coherent, which allows us to define the Connes fusion product $H \boxtimes_{\mathcal{A}_2} K$ as the colimit of $H \boxtimes_\phi K$, where ϕ ranges over all faithful and normal states of $\mathcal{A}_2$. In practice, however, it will often be advantageous to pick a fixed faithful and normal state ϕ. In [Tak13, Exercise IX.3.8 (p.210)] another approach to defining a tensor product of bimodules without reference to a state is given.

Lemma 4.2.7. *The Connes fusion product is coherently associative. That is, if H, K, L are $\mathcal{A}_1$-$\mathcal{A}_2$, $\mathcal{A}_2$-$\mathcal{A}_3$, and $\mathcal{A}_3$-$\mathcal{A}_4$ bimodules, respectively, ϕ is a faithful and normal state on $\mathcal{A}_2$, and ψ is a faithful and normal state on $\mathcal{A}_3$, then the map*

$$\begin{gathered}(\mathcal{D}(H,\phi) \otimes K) \otimes \mathcal{D}'(L,\psi) \to \mathcal{D}(H,\phi) \otimes (K \otimes \mathcal{D}'(L,\psi)) \\ ((x \otimes v) \otimes y) \mapsto (x \otimes (v \otimes y)),\end{gathered}$$

extends to an isomorphism of bimodules, $(H \boxtimes_\phi K) \boxtimes_\psi L \to H \boxtimes_\phi (K \boxtimes_\psi L)$. Moreover, these isomorphisms satisfy the pentagon identity.

Proof. That the above map indeed extends to an isomorphism of bimodules is [Tak13, Theorem 3.20]. The coherence can be found in [Bro03]. □

Lemma 4.2.8. *The associator from Lemma 4.2.7 is natural: Let H, K, L be $\mathcal{A}_1$-$\mathcal{A}_2$, $\mathcal{A}_2$-$\mathcal{A}_3$, and $\mathcal{A}_3$-$\mathcal{A}_4$ bimodules respectively, and let H', K', L' be $\mathcal{B}_1$-$\mathcal{B}_2$, $\mathcal{B}_2$-$\mathcal{B}_3$, and $\mathcal{B}_3$-$\mathcal{B}_4$ bimodules respectively. Suppose that we are moreover given isomorphisms $\nu_i : \mathcal{A}_i \to \mathcal{B}_i$ for $i = 1,2,3,4$, and unitary maps $\nu_H : H \to H'$, $\nu_K : K \to K'$ and $\nu_L : L \to L'$ that intertwine along the isomorphisms ν_i. Then, the following diagram commutes*

$$\begin{array}{ccc}(H \boxtimes_{\mathcal{A}_2} K) \boxtimes_{\mathcal{A}_3} L & \longrightarrow & H \boxtimes_{\mathcal{A}_2} (K \boxtimes_{\mathcal{A}_3} L) \\ {\scriptstyle (\nu_H \boxtimes \nu_K) \boxtimes \nu_L} \downarrow & & \downarrow {\scriptstyle \nu_H \boxtimes (\nu_K \boxtimes \nu_L)} \\ (H' \boxtimes_{\mathcal{A}_2} K') \boxtimes_{\mathcal{A}_3} L' & \longrightarrow & H' \boxtimes_{\mathcal{A}_2} (K' \boxtimes_{\mathcal{A}_3} L')\end{array}$$

Proof. This follows from a straightforward computation using the explicit forms of the isomorphisms on the appropriate dense subspaces from Lemmas 4.2.5 and 4.2.7. □

5. The free fermions on the circle and their Tomita-Takesaki theory

In this chapter we will further study the representation of the Clifford C*-algebra $\mathrm{Cl}(V)$ on the Fock space $\mathcal{F}_L$ in the case of our main Example 3.1.2. Thus, $V = L^2(\mathbb{S}) \otimes \mathbb{C}^d$, where the real structure α is pointwise complex conjugation, and the Lagrangian L consists of those spinors that extend to anti-holomorphic functions on the disk. We write $\mathcal{F} := \mathcal{F}_L$. We will apply the theory of Chapter 4 to certain von Neumann algebras associated to specific subalgebras of $\mathrm{Cl}(V)$. The results of this section will be used in Section 6.2 for the construction of a fusion factorization on the basic central extension of $L\,\mathrm{Spin}(d)$ that we constructed in Section 3.7, and again in Section 8.3 to define fusion on the spinor bundle on loop space, which is constructed in Section 8.2.

Various aspects of this representation are studied in the literature. For example, in [Ten17] it is studied in the context of functorial field theory. We point out more connections to the literature in the text.

5.1. Reflection of free fermions

Let us write $I_+ \subset S^1$ for the open upper semicircle and $I_- \subset S^1$ for the open lower semicircle:

$$I_+ := \{e^{i\varphi} \in S^1 \mid \varphi \in (0,\pi)\}, \qquad I_- := \{e^{i\varphi} \in S^1 \mid \varphi \in (\pi, 2\pi)\}.$$

If $f \in C^\infty_{-2\pi}$ (see Section 2.2), then we write $\mathrm{Supp}(f)$ for the support of f. We consider the subspaces

$$C^\infty_{-2\pi}|_\pm := \{f \in C^\infty_{-2\pi} \mid \exp(i\,\mathrm{Supp}(f)) \subseteq I_\pm\}$$

and denote their Hilbert completions by $V_\pm$. One sees immediately that V decomposes as $V = V_- \oplus V_+$, and that α restricts to real structures on $V_\pm$. The Clifford algebras $\mathrm{Cl}(V_-)$ and $\mathrm{Cl}(V_+)$ can be considered as subalgebras of $\mathrm{Cl}(V)$, and the algebra product

$$\mathrm{Cl}(V_-) \times \mathrm{Cl}(V_+) \subset \mathrm{Cl}(V) \times \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

induces a unital $*$-isomorphism $\mathrm{Cl}(V_-) \otimes \mathrm{Cl}(V_+) \cong \mathrm{Cl}(V)$ of $\mathbb{Z}_2$-graded C*-algebras. (Here, $\otimes$ stands for any choice of $\mathbb{Z}_2$-graded tensor product of C*-algebras. The choice is immaterial, because the Clifford C*-algebra is uniformly hyperfinite, and hence in particular nuclear.)

Remark 5.1.1. More generally, if I is an open connected non-dense subset of the circle S^1 we can write $V_I \subset V$ for the functions with support I. The assignment $I \mapsto \mathrm{Cl}(V_I)'' \subset \mathcal{B}(\mathcal{F})$ is then an isotonic net of von Neumann algebras on the circle. This net can be equipped with the structure of local Möbius covariant net (conformal net) see, for example, [GF93, Bis12, Hen14]. This net is called the *free fermions on the circle*. Here, we use that name to address the decomposition $V = V_- \oplus V_+$.

Lemma 5.1.2. *The decompositions $V = V_- \oplus V_+$ and $V = L \oplus \alpha(L)$ are in general position, that is,*

$$V_+ \cap L = \{0\} = V_+ \cap \alpha(L), \qquad V_- \cap L = \{0\} = V_- \cap \alpha(L).$$

Proof. This can be proven using the fact that the elements of L are (limits of) anti-holomorphic functions and that the elements of $V_\pm$ are (limits of) functions that vanish on an open segment of S^1; see for example [Was98, Section 14]. □

We denote the restriction of complex conjugation to the circle $S^1 \subset \mathbb{C}$ by v, that is, we write

$$v : S^1 \to S^1 \ , \ z \mapsto \bar{z}.$$

Note that v exchanges I_- with I_+. We denote by τ the map

$$\tau : V \to V \ , \ f \mapsto f \circ v.$$

Note that τ exchanges V_- with V_+ and at the same time exchanges L with $\alpha(L)$. The map τ is an isometric isomorphism that preserves the real structure, i.e., $\tau \in \mathrm{O}(V)$. The associated Bogoliubov automorphism $\theta_\tau : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ exchanges the subalgebras $\mathrm{Cl}(V_-)$ and $\mathrm{Cl}(V_+)$. Since τ exchanges L with $\alpha(L)$, one can check that $[\tau, \mathcal{J}] = 2\tau\mathcal{J}$, which is not Hilbert-Schmidt. Thus, $\tau \notin \mathrm{O}_L(V)$ and hence θ_τ is not implementable, by Theorem 3.4.1. Since α interchanges L with $\alpha(L)$ as well, the map $\alpha \circ \tau$ preserves both L and $\alpha(L)$.

Remark 5.1.3. We should note that when restricted to the basis of V given in Example 3.1.2 the map $\alpha \circ \tau$ is $\mathbb{1}$. This means that $\alpha \circ \tau$ is the conjugate linear extension of the map that fixes the basis elements $\{\xi_{n,j}\}$.

Lemma 5.1.4. *The map $\alpha \circ \tau : V \to V$ extends uniquely to a complex conjugate-linear $*$-automorphism $\kappa : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ of the Clifford algebra.*

Proof. Write $\overline{V}$ for the complex conjugate of V (with the same real structure α). Write $\bar{b} : \overline{V} \times \overline{V} \to \mathbb{C}$ for the corresponding complex conjugate bilinear form. It is straightforward to check that $\iota \circ \alpha \circ \tau : V \to \mathrm{Cl}(\overline{V})$ is a Clifford map. Hence, it extends uniquely to a $*$-homomorphism $\kappa' : \mathrm{Cl}(V) \to \mathrm{Cl}(\overline{V})$. Composing κ' with the unique conjugate-linear $*$-isomorphism $\mathrm{Cl}(\overline{V}) \to \mathrm{Cl}(V)$ we obtain κ. □

If $\mathcal{A}$ is a subset of $\mathcal{B}(\mathcal{F})$ we write $\mathcal{A}^\backprime$ for the *graded commutant* of $\mathcal{A}$, that is

$$\mathcal{A}^\backprime := \{x \in \mathcal{B}(\mathcal{F}) \mid [x, a]_\pm = 0 \text{ for all } a \in \mathcal{A}\},$$

where $[\cdot,\cdot]_\pm$ stands for the graded commutator.

Let $\mathrm{k} : \mathcal{F} \to \mathcal{F}$ be the *Klein transformation*, that is, k acts on $\mathcal{F}$ as the identity on the even part, and as multiplication by i on the odd part. Note that k is unitary. The Klein transformation will be useful for us due to the following property, which is straightforward to check; see, e.g., [Hen14].

Lemma 5.1.5. *Conjugation by the Klein transformation takes the commutant to the graded commutant. Explicitly, if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{F})$, then $\mathrm{k}\,\mathcal{A}'\,\mathrm{k}^{-1} = \mathcal{A}^\backprime$.*

We note the following routine facts, just in order to fix notation. Let $T : L \to L$ be an (anti-) unitary isomorphism. Then, there exists a unique (anti-) unitary algebra isomorphism $\Lambda L \to \Lambda L$ extending T. That operator in turn extends to a unique (anti-) unitary isometric isomorphism $\Lambda_T : \mathcal{F} \to \mathcal{F}$. Moreover, if $T : V \to V$ is an (anti-) unitary isomorphism that preserves L, then we will write Λ_T as a shortcut for $\Lambda_{T|_L}$.

Lemma 5.1.6. *The anti-unitary operator $\Lambda_{\alpha\tau} : \mathcal{F} \to \mathcal{F}$ implements the conjugate-linear automorphism κ, i.e., for all $a \in \mathrm{Cl}(V)$ we have*

$$\kappa(a) = \Lambda_{\alpha\tau} a \Lambda_{\alpha\tau}.$$

In particular, $\kappa : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ extends uniquely to a conjugate-linear automorphism of $\mathcal{B}(\mathcal{F})$.

Proof. Because $\mathrm{Cl}(V)$ is generated by V, it suffices to prove that $\kappa(f) = \Lambda_{\alpha\tau} f \Lambda_{\alpha\tau}$ for all $f \in V$. Let $f \in V$ and let $g_1, \dots g_n, \in L$, then we compute

$$\begin{aligned}\Lambda_{\alpha\tau} f \triangleright \Lambda_{\alpha\tau}(g_1 \wedge \cdots \wedge g_n) &= \Lambda_{\alpha\tau} f \triangleright (\alpha\tau g_1 \wedge \cdots \wedge \alpha\tau g_n) \\ &= \alpha\tau(P_L f) \wedge g_1 \wedge \cdots \wedge g_n + \Lambda_{\alpha\tau}\iota_{\alpha(f)}(\alpha\tau g_1 \wedge \cdots \wedge \alpha\tau g_n) \\ &= P_L(\alpha\tau(f)) \wedge g_1 \wedge \cdots \wedge g_n + \iota_{\tau\alpha(\alpha(f))}(g_1 \wedge \cdots \wedge g_n) \\ &= \kappa(f) \triangleright g_1 \wedge \cdots \wedge g_n.\end{aligned}$$ □

Remark 5.1.7. In fact, the map $\Lambda_{\alpha\tau}$ is closely related to the modular conjugation that is part of Tomita-Takesaki theory of the triple $(\mathrm{Cl}(V_-)'', \mathcal{F}, \Omega)$, see Proposition 5.2.2.

If $g \in \mathrm{O}(V)$, then we write $\tau(g) := \tau \circ g \circ \tau \in \mathrm{O}(V)$. With this notation we have, for all $f \in V$ and all $g \in \mathrm{O}(V)$, that $\tau(g(f)) = \tau(g)(\tau(f))$. Because τ interchanges L with $\alpha(L)$ we have that τ maps $\mathrm{O}_L(V)$ into $\mathrm{O}_L(V)$; furthermore, it is clear that it is a group homomorphism and that it is smooth.

Lemma 5.1.8. *For all $a \in \mathrm{Cl}(V)$ and all $g \in \mathrm{O}_L(V)$ we have $\kappa(\theta_g(a)) = \theta_{\tau(g)}(\kappa(a))$.*

Proof. Let $a = f_1 \dots f_n \in \mathrm{Cl}(V)$ be arbitrary, then we compute

$$\kappa(\theta_g(a)) = \kappa((gf_1)...(gf_n)) = \tau(gf_1^*) \dots \tau(gf_n^*) = \theta_{\tau(g)}\kappa(a).$$ □

Now we are in position to prove the first main result about the conjugate-linear automorphism κ.

Proposition 5.1.9. *Let $g \in \mathrm{O}_L(V)$ and let $U \in \mathrm{Imp}_L(V)$ implement g. Then $\kappa(U)$ implements $\tau(g)$, i.e. $q(\kappa(U)) = \tau(q(U))$. In particular, κ restricts to a group homomorphism $\kappa : \mathrm{Imp}_L(V) \to \mathrm{Imp}_L(V)$.*

Proof. Let $a \in \mathrm{Cl}(V)$ be arbitrary, then we compute, using Lemma 5.1.8 and the fact that $\kappa^2 = \mathbb{1}$,

$$\kappa(U)a\kappa(U)^* = \kappa(U\kappa(a)U^*) = \kappa(\theta_g(\kappa(a))) = \theta_{\tau(g)}(a).$$ □

In order to prove our second main result about the conjugate-linear automorphism κ we require the following lemma about the relation between κ and the local section σ of the central extension $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ constructed in Section 3.5. We recall that σ was defined by the equation $\sigma(\exp(A)) = \exp(\tilde{A})$, where $\tilde{A}$ is an unbounded operator on $\mathcal{F}$ determined by $A \in \mathfrak{o}_L(V)$.

Lemma 5.1.10. *We have $\Lambda_{\alpha\tau}\widetilde{A}\Lambda_{\alpha\tau} = \widetilde{\tau(A)}$ for all $A \in \mathfrak{o}_L(V)$. Moreover, we have $\kappa \circ \sigma = \sigma \circ \tau$.*

Proof. We decompose A and $\tau(A)$ according to Lemma 3.5.1, that is, we write

$$A = \begin{pmatrix} a & a'\alpha \\ \alpha a' & \alpha a \alpha \end{pmatrix}, \qquad \tau(A) = \tau A \tau = \begin{pmatrix} \tau\alpha a\alpha\tau & \tau\alpha a'\tau \\ \tau a'\alpha\tau & \tau a\tau \end{pmatrix}.$$

Then we have $\widetilde{A} = \widetilde{A}_0 + \widetilde{A}_1$, and $\widetilde{\tau(A)} = \widetilde{\tau(A)_0} + \widetilde{\tau(A)_1}$. Now, let $f_1, \dots, f_n \in L$ be arbitrary. We compute

$$\begin{aligned}\Lambda_{\alpha\tau}\widetilde{A}_0\Lambda_{\alpha\tau}f_1 \wedge ... \wedge f_n &= \Lambda_{\alpha\tau}\sum_{j=1}^{n}\alpha\tau f_1 \wedge ... \wedge a\alpha\tau f_j \wedge ... \wedge \alpha\tau f_n \\ &= \sum_{j=1}^{n} f_1 \wedge ... \wedge \alpha\tau a\tau\alpha f_j \wedge ... \wedge f_n \\ &= \widetilde{\tau(A)_0} f_1 \wedge \cdots \wedge f_n.\end{aligned}$$

Next, $\widetilde{A}_1$ has two parts, a degree-increasing part $\widetilde{A}_1^+$, and a degree-decreasing part $\widetilde{A}_1^-$. These are related by $(\widetilde{A}_1^+)^* = -\widetilde{A}_1^-$, hence it suffices to show that $\Lambda_{\alpha\tau}\widetilde{A}_1^+\Lambda_{\alpha\tau} = \widetilde{\tau(A)}_1^+$. We compute

$$\Lambda_{\alpha\tau}\widetilde{A}_1^+\Lambda_{\alpha\tau}f_1 \wedge ... \wedge f_n = \Lambda_{\alpha\tau}\hat{a}' \wedge \Lambda_{\alpha\tau}(f_1 \wedge ... \wedge f_n) = \Lambda_{\alpha\tau}(\hat{a}') \wedge f_1 \wedge ... \wedge f_n,$$

where the notation $\hat{a}'$ was defined and characterized in (3.2). Hence, it suffices to show that $\Lambda_{\alpha\tau}(\hat{a}') = \widehat{\tau\alpha a'\alpha\tau}$. This follows from the computation

$$\langle \Lambda_{\alpha\tau}(\hat{a}'), f_1 \wedge f_2\rangle = \langle \alpha\tau f_1 \wedge \alpha\tau f_2, \hat{a}'\rangle = \langle \alpha\tau f_2, a'(\alpha\tau f_1)\rangle = \langle \tau\alpha a'\alpha\tau f_1, f_2\rangle.$$

Finally, in order to prove the equation $\kappa \circ \sigma = \sigma \circ \tau$, we compute:

$$\kappa\sigma e^A = \Lambda_{\alpha\tau}e^{\tilde{A}}\Lambda_{\alpha\tau} = e^{\Lambda_{\alpha\tau}\tilde{A}\Lambda_{\alpha\tau}} = e^{\widetilde{\tau(A)}} = \sigma\tau e^A.$$ □

Proposition 5.1.11. *The group homomorphism $\kappa : \mathrm{Imp}_L(V) \to \mathrm{Imp}_L(V)$ is smooth.*

Proof. Let $A \in \mathfrak{o}_L(V)$, then we see that $\tau(A) := \tau A\tau : V \to V$ is in $\mathfrak{o}_L(V)$ as well. We consider the real-linear map $\kappa_{\mathfrak{imp}} : \mathfrak{imp}(V) \to \mathfrak{imp}(V), (A, \lambda) \mapsto (\tau(A), -\lambda)$, which is easily checked to be an isometry, and hence bounded. Let $(A, \lambda) \in \mathfrak{imp}(V)$. We have, using Lemma 5.1.10,

$$\kappa \exp(A, \lambda) = \Lambda_{\alpha\tau}e^{\tilde{A}+i\lambda\mathbb{1}}\Lambda_{\alpha\tau} = e^{\Lambda_{\alpha\tau}\widetilde{A}\Lambda_{\alpha\tau}-i\lambda\mathbb{1}} = e^{\widetilde{\tau(A)}-i\lambda\mathbb{1}} = \exp(\kappa_{\mathfrak{imp}}(A, \lambda)),$$

this shows that the diagram

$$\begin{array}{ccc} \mathfrak{imp}(V) & \xrightarrow{\kappa_{\mathfrak{imp}}} & \mathfrak{imp}(V) \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ \mathrm{Imp}_L(V) & \xrightarrow[\kappa]{} & \mathrm{Imp}_L(V) \end{array}$$

commutes; which in turn shows that κ is smooth and that $\kappa_{\mathfrak{imp}}$ is its derivative. □

5.2. Tomita-Takesaki theory and Connes fusion of the free fermions

As mentioned before, the free fermions can be extended to a local Möbius covariant net. Many of the results of this section are consequences of this fact and the theory of such nets, see [GF93, Section II]. In our current situation, we consider the von Neumann algebras $\mathrm{Cl}(V_\pm)'' \subseteq \mathcal{B}(\mathcal{F})$ generated by $\mathrm{Cl}(V_+)$ and $\mathrm{Cl}(V_-)$. It is well known that these are type III_1 factors, [ST04, Example 4.3.2], [GF93, Lemma 2.9], [Was98, Section 16]. This statement can also be obtained as a consequence of the computation of the operator $\Delta^{1/2}$ performed in Appendix C. We note that these algebras inherit a $\mathbb{Z}_2$-grading from the Clifford algebras, which coincides with the grading of $\mathcal{B}(\mathcal{F})$.

Because the decompositions $V = V_- \oplus V_+ = L \oplus \alpha(L)$ are in general position (Lemma 5.1.2), the vacuum vector $\Omega \in \mathcal{F}$ is cyclic and separating for $\mathrm{Cl}(V_-)''$; see, for example, [BJL02, Proposition 3.4]. Thus, the left $\mathrm{Cl}(V_-)''$-module $\mathcal{F}$ can be equipped with the structure of standard form of $\mathrm{Cl}(V_-)''$ as in Section 4.1. From Theorem 4.1.3 it follows that $\mathcal{F}$ is isomorphic to $L^2_\Omega(\mathrm{Cl}(V_-)'')$, where Ω stands for the faithful and normal state $\mathrm{Cl}(V_-)'' \to \mathbb{C}, a \mapsto \langle a \triangleright \Omega, \Omega \rangle$. An explicit isomorphism $f : L^2_\Omega(\mathrm{Cl}(V_-)'') \to \mathcal{F}$ is given by the extension of the densely defined map $\mathrm{Cl}(V_-)'' \to \mathcal{F}, a \mapsto a \triangleright \Omega$.

Recall from Section 4.1 that a standard form of a von Neumann algebra $\mathcal{A}$ is in a natural way an $\mathcal{A}$-$\mathcal{A}$-bimodule, and from Lemma 4.2.4 that it is the unit with respect to Connes fusion. Hence the Fock space $\mathcal{F}$ is in a natural way a $\mathrm{Cl}(V_-)''$-$\mathrm{Cl}(V_-)''$ bimodule, and moreover, there is a natural isomorphism

$$\mu : \mathcal{F} \boxtimes_\Omega \mathcal{F} \to \mathcal{F}.$$

On the dense subspace $\mathcal{D}(\mathcal{F}, \Omega) \otimes \mathcal{F}$ the map μ is given by, (see Lemma 4.2.4)

$$x \otimes v \mapsto p_\Omega(f^{-1} \circ x) \triangleright v.$$

Similarly, on the dense subspace $\mathcal{F} \otimes \mathcal{D}'(\mathcal{F}, \Omega)$ the map μ is given by

$$v \otimes x' \mapsto v \triangleleft p'_\Omega(f^{-1} \circ x')$$

From now on, we will drop the Ω from the notation and write $\mathcal{F} \boxtimes \mathcal{F} := \mathcal{F} \boxtimes_\Omega \mathcal{F}$.

Lemma 5.2.1. *The map μ is associative, that is, the following diagram commutes*

$$\begin{array}{ccccc}
(\mathcal{F} \boxtimes \mathcal{F}) \boxtimes \mathcal{F} & & \longrightarrow & & \mathcal{F} \boxtimes (\mathcal{F} \boxtimes \mathcal{F}) \\
\downarrow{\scriptstyle \mu \boxtimes \mathbb{1}} & & & & \downarrow{\scriptstyle \mathbb{1} \boxtimes \mu} \\
\mathcal{F} \boxtimes \mathcal{F} & & & & \mathcal{F} \boxtimes \mathcal{F} \\
& \searrow^{\mu} & & \swarrow^{\mu} & \\
& & \mathcal{F} & &
\end{array}$$

Proof. Let $x \in \mathcal{D}(\mathcal{F}, \Omega)$, $v \in \mathcal{F}$ and $x' \in \mathcal{D}'(\mathcal{F}, \Omega)$, then we obtain

$$\begin{array}{ccccc}
(x \otimes v) \otimes x' & & \longmapsto & & x \otimes (v \otimes x') \\
\downarrow{\scriptstyle \mu \boxtimes \mathbb{1}} & & & & \downarrow{\scriptstyle \mathbb{1} \boxtimes \mu} \\
p_\Omega(f^{-1} \circ x) \triangleright v \otimes x' & & & & x \otimes v \triangleleft p'_\Omega(f^{-1} \circ x') \\
& \searrow^{\mu} & & \swarrow^{\mu} & \\
& & p_\Omega(f^{-1} \circ x) \triangleright v \triangleleft p'_\Omega(f^{-1} \circ x') & &
\end{array}$$

□

To understand the right action of $\mathrm{Cl}(V_-)''$ on $\mathcal{F}$ it is important to understand the modular conjugation, which in our case can be computed explicitly. The result is recorded in the following lemma.

Proposition 5.2.2. *The modular conjugation J_Ω for the triple corresponding to the cyclic and separating vector $\Omega \in \mathcal{F}$ is the operator $J = \mathrm{k}^{-1} \Lambda_{\alpha\tau}$.*

This result will be used below and then again in Section 6.3. Direct proofs can be found in, for example, [Was98, Section 15], [Hen14], and [Jan13]. Because our conventions are slightly different from the references cited, we have adapted the proof in [Jan13] to a proof of Proposition 5.2.2 in Appendix C. Another way to prove Proposition 5.2.2 is to endow the free fermions with the structure of local Möbius covariant net, and then apply the Bisognano-Wichmann theorem, see [Bis12, Section 3.2].

The following result goes under the name of *Twisted Haag duality*, [BJL02].

Proposition 5.2.3. *The graded commutant of* $\mathrm{Cl}(V_-)''$ *in* $\mathcal{B}(\mathcal{F})$ *is* $\mathrm{Cl}(V_+)''$.

Proof. This is proven in [Hen14], as follows: we use that $\mathrm{k}(\mathrm{Cl}(V_-)'')'\,\mathrm{k}^{-1}$ is the graded commutant of $\mathrm{Cl}(V_-)''$, see Lemma 5.1.5. From Theorem 4.1.4 and Proposition 5.2.2 we have $(\mathrm{Cl}(V_-)'')' = J\,\mathrm{Cl}(V_-)''J$. Substituting this one obtains that

$$\mathrm{k}\,J\,\mathrm{Cl}(V_-)''J\,\mathrm{k}^{-1} = \Lambda_{\alpha\tau}\,\mathrm{Cl}(V_-)''\Lambda_{\alpha\tau} = \theta_{\alpha\tau}(\mathrm{Cl}(V_-)'') = \mathrm{Cl}(V_+)''$$

is the graded commutant of $\mathrm{Cl}(V_-)''$. Alternatively, Proposition 5.2.3 can be proved using [BJL02, Theorem 5.8]; to apply that theorem one needs Lemma 5.1.2. □

The immediate upshot of Proposition 5.2.3 is that one may in fact view $\mathcal{F}$ as a $\mathrm{Cl}(V_-)''$-$(\mathrm{Cl}(V_+)'')^{\mathrm{op}}$ bimodule by defining a right action of $(\mathrm{Cl}(V_+)'')^{\mathrm{op}}$ as follows

$$\begin{aligned} \mathcal{F} \times (\mathrm{Cl}(V_+)'')^{\mathrm{op}} &\to \mathcal{F} \\ (\xi, b) &\mapsto \xi \triangleleft b := \mathrm{k}^{-1}\, b\, \mathrm{k} \triangleright \xi. \end{aligned}$$

Recall that $\mathcal{F}$ may also be seen as a $\mathrm{Cl}(V_-)''$-$\mathrm{Cl}(V_-)''$ bimodule by defining the right action by $\xi \triangleleft a = Ja^*J \triangleright \xi$. This means we may define an, a priori different, right action of $(\mathrm{Cl}(V_+)'')^{\mathrm{op}}$ on $\mathcal{F}$, denoted by $\tilde{\triangleleft}$, by using the isomorphism $(\mathrm{Cl}(V_+)'')^{\mathrm{op}} \to \mathrm{Cl}(V_-), b \mapsto \Lambda_{\alpha\tau} b^* \Lambda_{\alpha\tau}$. We compute, for $b \in (\mathrm{Cl}(V_+)'')^{\mathrm{op}}$ and $\xi \in \mathcal{F}$,

$$\xi \,\tilde{\triangleleft}\, b = \xi \triangleleft \Lambda_{\alpha\tau} b^* \Lambda_{\alpha\tau} = J\Lambda_{\alpha\tau} b \Lambda_{\alpha\tau} J \triangleright \xi = \mathrm{k}^{-1}\, b\, \mathrm{k} \triangleright \xi = \xi \triangleleft b,$$

and see that the two right actions coincide.

5.3. Restriction to the even part

As mentioned before, the Fock space $\mathcal{F}$ is a $\mathbb{Z}_2$-graded Hilbert space. The von Neumann algebra $\mathrm{Cl}(V_-)''$ is $\mathbb{Z}_2$-graded as well. Even though the Tomita-Takesaki theory of the triple $(\mathrm{Cl}(V_-)'', \mathcal{F}, \Omega)$ can to some extent be adapted to the $\mathbb{Z}_2$-graded case, some results are only available in the ungraded case. For this reason, we will restrict our considerations to the even parts. In this section we show in which way the results of Section 5.2 survive this process.

We write $\mathcal{F}_0$ for the even part of $\mathcal{F}$. Then, if an algebra A acts on $\mathcal{F}$, we write A_0 for the subalgebra consisting of those operators that preserve $\mathcal{F}_0$. We may now consider the commutant of $\mathrm{Cl}(V_\pm)_0$ in $\mathcal{B}(\mathcal{F}_0)$, and similarly we could consider those elements of $\mathrm{Cl}(V_\pm)' \subset \mathcal{B}(\mathcal{F})$ that preserve $\mathcal{F}_0$. It is elementary to show that both procedures have the same result, i.e., $(\mathrm{Cl}(V_\pm)_0)' = (\mathrm{Cl}(V_\pm)')_0$. Thus, in the following, we simply write $\mathrm{Cl}(V_\pm)_0'$.

We have Haag duality for these even algebras (no longer twisted because the commutant coincides with the graded commutant).

Proposition 5.3.1. *The commutant of* $\mathrm{Cl}(V_-)_0''$ *is* $\mathrm{Cl}(V_+)_0''$.

Proof. This follows directly from Proposition 5.2.3. □

Lemma 5.3.2. *The vector* $\Omega \in \mathcal{F}_0$ *is cyclic and separating for* $\mathrm{Cl}(V_-)_0''$.

Proof. The fact that Ω is separating for $\mathrm{Cl}(V_-)_0''$ is immediate from the fact that Ω is separating for the bigger algebra $\mathrm{Cl}(V_-)''$. We know that $\mathrm{Cl}(V_-)'' \triangleright \Omega$ is dense in $\mathcal{F}$, it follows that $\mathrm{Cl}(V_-)_0'' \triangleright \Omega$ is dense in $\mathcal{F}_0$. □

Let us write S_0 for the Tomita operator corresponding to the triple $(\mathrm{Cl}(V_-)_0'', \mathcal{F}_0, \Omega)$, and let J_0 be the corresponding modular conjugation.

Lemma 5.3.3. *The operator* S_0 *is the restriction of the Tomita operator* $S : \mathcal{F} \to \mathcal{F}$ *to the subspace* $\mathcal{F}_0 \subset \mathcal{F}$. *Furthermore, the modular conjugation* J_0 *is the restriction of* J *to* $\mathcal{F}_0$ *and the modular operator* $\Delta_0^{1/2}$ *is the restriction of* $\Delta^{1/2}$ *to* $\mathcal{F}_0$.

Proof. The fact that S restricts to S_0 is obvious. The remaining claims follow from the fact that J and $\Delta^{1/2}$ preserve both the even and the odd subspaces of $\mathcal{F}$. □

5.4. Fusion of implementers

Next, we consider how the decomposition $V = V_- \oplus V_+$ interacts with the central extension $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$. We define $\mathrm{O}_L^{split}(V)$ to be the subgroup of the connected component of the identity of $\mathrm{O}_L(V)$ that stabilizes the decomposition $V = V_- \oplus V_+$ i.e.,

$$\mathrm{O}_L^{split}(V) := \{g \in \mathrm{O}_L(V) \mid g|_{V_-} \in \mathrm{O}(V_-), g|_{V_+} \in \mathrm{O}(V_+),\ g \text{ is even}\}.$$

If $g \in \mathrm{O}_L^{split}(V)$ then we write $g_\pm$ for $g|_{V_\pm}$, and note that $g = g_- \oplus g_+$. We write $\mathrm{Imp}_L^{split}(V)$ for the restriction of $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ to $\mathrm{O}_L^{split}(V)$.

Remark 5.4.1. It is clear that both $\mathrm{O}_L(V)_-$ and $\mathrm{O}_L(V)_+$ are normal in $\mathrm{O}_L^{split}(V)$ (but not in $\mathrm{O}_L(V)$) and that they commute with each other. It is furthermore clear that $\mathrm{Imp}_L(V)_-$ and $\mathrm{Imp}_L(V)_+$ are normal in $\mathrm{Imp}_L^{split}(V)$.

With the following result we will ensure that the central extension $\mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ is eligible for our method of constructing fusion product from fusion factorizations explained in Section 6.3.

Proposition 5.4.2. *The groups* $\mathrm{Imp}_L(V)_-$ *and* $\mathrm{Imp}_L(V)_+$ *commute with each other.*

Proof. Let $U_- \in \mathrm{Imp}_L(V)_-$ and $U_+ \in \mathrm{Imp}_L(V)_+$, and suppose that U_- implements $g_- \oplus \mathbb{1} \in \mathrm{O}_L(V)_-$. Then, we have, for all $a \in \mathrm{Cl}(V_+)$ that $U_- a U_-^* = \theta_{g_- \oplus \mathbb{1}}(a) = a$, and hence $U_- \in (\mathrm{Cl}(V_+)_0'')'$. Similarly we see that $U_+ \in (\mathrm{Cl}(V_-)_0'')'$. Proposition 5.3.1 tells us that $(\mathrm{Cl}(V_-)_0'')' = \mathrm{Cl}(V_+)_0''$. Hence U_- commutes with U_+. □

In the remainder of this section we define a "fusion map", which takes two implementers that are compatible in a sense defined below and then produces a third implementer.

Let $U \in \mathrm{Imp}_L^{split}(V)$ implement the element $g = g_- \oplus g_+ \in \mathrm{O}_L^{split}(V)$.

Lemma 5.4.3. *The automorphism* $\theta_{g_-} : \mathrm{Cl}(V_-) \to \mathrm{Cl}(V_-)$ *extends uniquely to an automorphism of* $\mathrm{Cl}(V)''_-$*, namely* $\theta_{g_-} : a \mapsto UaU^*$.

Proof. Because $\mathrm{Cl}(V_-)$ is dense in $\mathrm{Cl}(V)''_-$, it is sufficient to prove existence. We know that $\theta_g : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ extends to an automorphism of $\mathrm{Cl}(V)'' = B(\mathcal{F})$, namely $a \mapsto UaU^*$. It is thus sufficient to prove that conjugation by U preserves $\mathrm{Cl}(V_-)''$. Let $c \in \mathrm{Cl}(V_-)$ and let $b \in \mathrm{Cl}(V_-)'$, then, because $UcU^* = \theta_{g_-} c \in \mathrm{Cl}(V_-)$, we have

$$U^*bUc = U^*bUcU^*U = U^*UcU^*bU = cU^*bU,$$

and hence $U^*bU \in \mathrm{Cl}(V_-)'$. Now let $a \in \mathrm{Cl}(V)''_-$, then we have

$$UaU^*b = UaU^*bUU^* = bUaU^*,$$

and hence $UaU^* \in \mathrm{Cl}(V_-)''$. □

It seems worth pointing out that the map $\theta_{g_-} : \mathrm{Cl}(V_-)'' \to \mathrm{Cl}(V_-)''$ does not depend on either g_+ or U. Similarly, one shows that $\theta_{g_+} : \mathrm{Cl}(V_+) \to \mathrm{Cl}(V_+)$ extends to an automorphism of $\mathrm{Cl}(V_+)''$.

Lemma 5.4.4. *The map* $\mathrm{Cl}(V_-)'' \ni a \mapsto \kappa(U)a\kappa(U^*)$ *is an automorphism of* $\mathrm{Cl}(V_-)''$.

Proof. We observe that the map $a \mapsto \kappa(U)a\kappa(U^*)$ is given by the composition of isomorphisms

$$\begin{array}{ccccccc} \mathrm{Cl}(V_-)'' & \longrightarrow & \mathrm{Cl}(V_+)'' & \xrightarrow{\theta_{g_+}} & \mathrm{Cl}(V_+)'' & \longrightarrow & \mathrm{Cl}(V_-)'' \\ a & \longmapsto & \Lambda_{\alpha\tau}a^*\Lambda_{\alpha\tau} & \longmapsto & U\Lambda_{\alpha\tau}a^*\Lambda_{\alpha\tau}U^* & \longmapsto & \Lambda_{\alpha\tau}U\Lambda_{\alpha\tau}a^*\Lambda_{\alpha\tau}U^*\Lambda_{\alpha\tau} \end{array}$$

□

We denote the map from Lemma 5.4.4 by $\tilde{\theta}_{g_+} : \mathrm{Cl}(V_-)'' \to \mathrm{Cl}(V_-)''$. Because $U \in \mathrm{Imp}_L^{split}(V)$, and hence U is in particular even, we have $\kappa(U) = JUJ$, which we use in the proof of the following lemma.

Proposition 5.4.5. *The map* $U : \mathcal{F} \to \mathcal{F}$ *has the property that for all* $a \in \mathrm{Cl}(V_-)''$ *and all* $v \in \mathcal{F}$

$$U(a \triangleright v) = \theta_{g_-}(a) \triangleright Uv,$$

and

$$U(v \triangleleft a) = (Uv) \triangleleft \tilde{\theta}_{g_+}(a).$$

Proof. We compute for $a \in \mathrm{Cl}(V_-)''$ and $v \in \mathcal{F}$

$$U(v \triangleleft a) = U(Ja^*J \triangleright v) = UJa^*JU^* \triangleright Uv = (Uv) \triangleleft (\kappa(U)a\kappa(U^*)) = (Uv) \triangleleft \tilde{\theta}_{g_+}(a).$$

The other equation follows from a similar computation. □

The preceding proposition means that U is an intertwiner along the algebra homomorphisms θ_{g_-} and $\tilde{\theta}_{g_+}$.

Now, let $U' \in \mathrm{Imp}_L^{split}(V)$ implement $g' = g'_- \oplus g'_+ \in \mathrm{O}_L^{split}(V)$ with $g'_- = \tau g_+ \tau$. Let $a \in \mathrm{Cl}(V_-)$ be arbitrary, then we compute, using Proposition 5.1.9

$$\tilde{\theta}_{g_+}(a) = \kappa(U)a\kappa(U)^* = \theta_{\tau g_+ \tau \oplus \tau g_- \tau}(a) = \theta_{\tau g_+ \tau}(a) = \theta_{g'_-}(a),$$

whence $\tilde{\theta}_{g_+} = \theta_{g'_-}$. We now have three $*$-automorphisms of $\mathcal{A}_-$, namely θ_{g_-}, $\tilde{\theta}_{g_+} = \theta_{g'_-}$, and $\tilde{\theta}_{g'_+}$. Additionally, we have the intertwiners U along θ_{g_-} and $\tilde{\theta}_{g_+}$, and U' along $\theta_{g'_-}$ and $\tilde{\theta}_{g'_+}$, according to Proposition 5.4.5. Let $U \boxtimes U'$ be the corresponding automorphism of $\mathcal{F} \boxtimes \mathcal{F}$ from Lemma 4.2.5, which is itself an intertwiner along θ_{g_-} and $\tilde{\theta}_{g'_+}$. We then define $\hat{\mu}(U, U') : \mathcal{F} \to \mathcal{F}$ through the diagram

$$\mathcal{F} \xrightarrow{\mu^{-1}} \mathcal{F} \boxtimes \mathcal{F} \xrightarrow{U \boxtimes U'} \mathcal{F} \boxtimes \mathcal{F} \xrightarrow{\mu} \mathcal{F}, \tag{5.1}$$

in other words $\hat{\mu}(U, U') := \mu \circ U \boxtimes U' \circ \mu^{-1}$. Because $\hat{\mu}(U, U')$ is an intertwiner along θ_{g_-} and $\tilde{\theta}_{g'_+}$ it implements $g_- \oplus g'_+$. Note that, in particular, this shows that if $g_- \oplus g_+ \in \mathrm{O}_L^{split}(V)$ and $\tau g_+ \tau \oplus g'_+ \in \mathrm{O}_L^{split}(V)$, then $g_- \oplus g'_+ \in \mathrm{O}_L^{split}(V)$.

The following result follows directly from Lemma 4.2.5.

Lemma 5.4.6. *The map $\hat{\mu}$ is multiplicative in the following sense. Let $U, U', V, V' \in \mathrm{Imp}_L(V)$ implement $g_- \oplus g_+, \tau g_+ \tau \oplus g'_+$, $h_- \oplus h_+$ and $\tau h_+ \tau \oplus h'_+ \in \mathrm{O}_L(V)$, respectively, then*

$$\hat{\mu}(U, U')\hat{\mu}(V, V') = \hat{\mu}(UV, U'V').$$

Remark 5.4.7. This result can be packaged nicely as follows. Let $U \in \mathrm{Imp}_L^{split}(V)$ implement $g_- \oplus g_+ \in \mathrm{O}_L^{split}(V)$. Then we set $\pi_1(U) = g_+ \in \mathrm{O}(V_+)$ and $\pi_2(U) = \tau(g_-) \in \mathrm{O}(V_+)$. We then get that $\hat{\mu}$ is a map

$$\hat{\mu} : \mathrm{Imp}_L^{split}(V) \,{}_{\pi_1}\!\times_{\pi_2} \mathrm{Imp}_L^{split}(V) \to \mathrm{Imp}_L^{split}(V).$$

Algebraically, the diagram (5.1) provides a clear understanding of the map $\hat{\mu}$. However, the group $\mathrm{Imp}_L(V)$ is a Banach Lie group, and at this point it is not clear if $\hat{\mu}$ is smooth (for some reasonable notion of smoothness). We will revisit this issue in Chapter 6, and present a solution.

Remark 5.4.8. We have constructed fusion only for *even* implementers. This is because we are using the Connes fusion product for *ungraded* bimodules, if one were to use a fusion product for $\mathbb{Z}_2$-graded bimodules one would obtain a fusion product that works equally well for both even and odd implementers.

6. Fusion on the basic central extension of the loop group

In this chapter we describe the construction of a fusion product on the operator-algebraic model for the basic central extension of $L\,\mathrm{Spin}(d)$ constructed in Section 3.7. We prove that this fusion product coincides with the fusion product from Section 5.4, whenever both are defined, in Theorem 6.3.11. We recall in Section 6.1 some generalities about fusion products on central extensions of loop groups, and introduce in Section 6.2 our new method of constructing fusion products. In Section 6.3 we apply this method in our case, using the results obtained in Chapter 5.

6.1. Fusion products

Let M be a finite dimensional smooth manifold. Consistent with our treatment of the loop group $L\,\mathrm{Spin}(d)$ we write LM for the space of smooth loops in M. We view LM as Fréchet manifold in the usual way.

We write PM for the set of smooth paths in M, with sitting instants, i.e.,

$$PM := \{\beta : [0,\pi] \to M \mid \beta \text{ is smooth, and constant around } 0 \text{ and } \pi\}, \tag{6.1}$$

which is a group under pointwise multiplication. We use sitting instants so that we are able to concatenate arbitrary paths with a common end point: the usual path concatenation $\beta_2 \star \beta_1$ is again a smooth path whenever $\beta_1(\pi) = \beta_2(0)$. Unfortunately, with sitting instants, PM can not be equipped with the structure of smooth manifold such that the inclusion map into the Fréchet manifold of all smooth paths is an embedding. Instead, we regard it as a diffeological space. A diffeology on a set X consists of a set of maps $c : U \to X$ called *plots*, where $U \subset \mathbb{R}^k$ is open and k can be arbitrary, subject to a number of axioms, see [IZ13] for details. A map $f : X \to Y$ between diffeological spaces is called smooth if its composition with any plot of X is a plot of Y. A diffeological group is a group such that multiplication and inversion are smooth. Any smooth manifold M or Fréchet manifold M becomes a diffeological space by saying that every smooth map $c : U \to M$, for every open subset $U \subset \mathbb{R}^k$ and any k, is a plot.

In case of PM, the plots are all maps $c : U \to PM$ such that the adjoint map $c^\vee : U \times [0,\pi] \to M : (u,t) \mapsto c(u)(t)$ is smooth. We remark that path concatenation $\star$ and path reversal $\beta \mapsto \bar{\beta}$ are smooth maps. The evaluation map $ev : PM \to M \times M, \beta \mapsto (\beta(0), \beta(\pi))$ is smooth, and since diffeological spaces admit arbitrary fibre products, the iterated fibre products $PM^{[k]} := PM \times_{M\times M} ... \times_{M\times M} PM$ are again diffeological spaces. Their plots are just tuples $(c_1, ..., c_k)$ of plots of PM, such that $ev \circ c_1 = ... = ev \circ c_k$. We obtain a smooth map

$$\begin{aligned} &PM^{[2]} \to LM \\ &(\beta_1, \beta_2) \mapsto \beta_1 \cup \beta_2 := \bar{\beta}_2 \star \beta_1 \end{aligned}$$

defined on the double fibre product, and three smooth maps

$$\begin{aligned}\pi_1 &: PM^{[3]} \to LM : (\beta_1, \beta_2, \beta_3) \mapsto \beta_1 \cup \beta_2,\\ \pi_2 &: PM^{[3]} \to LM : (\beta_1, \beta_2, \beta_3) \mapsto \beta_2 \cup \beta_3,\\ \mu &: PM^{[3]} \to LM : (\beta_1, \beta_2, \beta_3) \mapsto \beta_1 \cup \beta_3,\end{aligned}$$

defined on the triple fibre product.

Let $\widetilde{LM} \xrightarrow{q} LM$ be a Fréchet principal U(1)-bundle on LM.

Definition 6.1.1. A *fusion product* on $\widetilde{LM}$ assigns to each element $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$ a U(1)-bilinear map

$$\tilde{\mu}_{\beta_1,\beta_2,\beta_3} : \widetilde{LM}_{\beta_1\cup\beta_2} \times \widetilde{LM}_{\beta_2\cup\beta_3} \to \widetilde{LM}_{\beta_1\cup\beta_3},$$

such that the following two conditions are satisfied:

(i) Associativity: for all $(\beta_1, \beta_2, \beta_3, \beta_4) \in PM^{[4]}$ and all $q_{ij} \in \widetilde{LM}_{\beta_i\cup\beta_j}$,

$$\tilde{\mu}_{\beta_1,\beta_3,\beta_4}(\tilde{\mu}_{\beta_1,\beta_2,\beta_3}(q_{12}, q_{23}), q_{34}) = \tilde{\mu}_{\beta_1,\beta_2,\beta_4}(q_{12}, \tilde{\mu}_{\beta_2,\beta_3,\beta_4}(q_{23}, q_{34})).$$

(ii) Smoothness: the map

$$\widetilde{LM}\,{}_q\!\times_{\pi_1} PM^{[3]}\,{}_{\pi_2}\!\times_q \widetilde{LM} \to \widetilde{LM}\ ,\ (q_{12}, \beta_1, \beta_2, \beta_3, q_{23}) \mapsto \tilde{\mu}_{\beta_1,\beta_2,\beta_3}(q_{12}, q_{23})$$

is a smooth map between diffeological spaces.

Now, we suppose that the smooth manifold M is actually a Lie group, G. We note that the maps $\pi_1, \pi_2, \mu : PG^{[3]} \to LG$ are then smooth group homomorphisms. Suppose that $\mathrm{U}(1) \to \widetilde{LG} \to LG$ is a Fréchet central extension.

Definition 6.1.2. A fusion product on $\widetilde{LG}$ is called *multiplicative*, if it is a group homomorphism; i.e., for all $(\beta_1, \beta_2, \beta_3), (\beta_1', \beta_2', \beta_3') \in PG^{[3]}$, $q_{ij} \in \widetilde{LG}_{\beta_i\cup\beta_j}$, and $q_{ij}' \in \widetilde{LG}_{\beta_i'\cup\beta_j'}$,

$$\tilde{\mu}_{\beta_1,\beta_2,\beta_3}(q_{12}, q_{23}) \cdot \tilde{\mu}_{\beta_1',\beta_2',\beta_3'}(q_{12}', q_{23}') = \tilde{\mu}_{\beta_1\beta_1',\beta_2\beta_2',\beta_3\beta_3'}(q_{12}q_{12}', q_{23}q_{23}').$$

Early versions of fusion products have been studied in [Bry93] and in [ST]. For a more complete treatment of this topic we refer to [Wal16b, Wal16a, Wal17]. Fusion products are a characteristic feature of the image of transgression, see Section 7.1 and [Wal16b]. The basic central extension of any compact simple Lie group can be obtained by transgression; hence, these models are automatically equipped with a multiplicative fusion product [Wal16a, Wal17]. In the present section, we will show that our operator-algebraic model constructed in Section 3.7 comes with an operator-algebraically defined multiplicative fusion product.

In order to treat connections and fusion products at the same time, we require differential forms on diffeological spaces. A differential form on a diffeological space X is a collection $\varphi = \{\varphi_c\}_c$ of differential forms $\varphi_c \in \Omega^k(U)$, one for each plot $c : U \to X$, such that $f^*\varphi_{c'} = \varphi_c$ for all smooth maps $f : U \to U'$ between the domains of plots $c : U \to X$ and $c' : U' \to X'$ with $c' \circ f = c$. Differential forms can be pulled back along smooth maps $f : X \to Y$, by simply putting $(f^*\varphi)_c := \varphi_{c\circ f}$. If a smooth manifold or Fréchet manifold is considered as a diffeological space, then diffeological and ordinary differential forms are the same thing, upon identifying $\varphi_c = c^*\varphi$.

Let $\widetilde{LG} \to LG$ be a central extension equipped with a fusion product. Additionally, we consider it as a principal U(1)-bundle $q : \widetilde{LG} \to LG$, and suppose that it is equipped with a connection ν. We consider the three smooth maps

$$\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\mu} : \widetilde{LG}\,{}_q\!\times_{\pi_1} PG^{[3]}\,{}_{\pi_2}\!\times_q \widetilde{LG} \to \widetilde{LG}, \tag{6.2}$$

where $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are the projections to the first and the third factor, respectively, and $\tilde{\mu}$ is the map of condition (ii) of Definition 6.1.1.

Definition 6.1.3. A fusion product $\tilde{\mu}$ is called *connection-preserving* with respect to a connection ν on $\widetilde{LG}$ if $\tilde{\mu}^*\nu = \tilde{\pi}_1^*\nu + \tilde{\pi}_2^*\nu$, where $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\mu}$ are the smooth maps of (6.2).

Remark 6.1.4. Fusion products are best understood using the theory of principal bundles over diffeological spaces, see [Wal12b, Wal16b]. In that terminology, a fusion product is just a smooth bundle morphism

$$\tilde{\mu} : \pi_1^*\widetilde{LG} \otimes \pi_2^*\widetilde{LG} \to \mu^*\widetilde{LG}$$

of principal $U(1)$-bundles over $PG^{[3]}$; this (plus a corresponding associativity condition) is equivalent to Definition 6.1.1. Moreover, a fusion product is connection-preserving in the sense of Definition 6.1.3 if that bundle morphism is connection-preserving.

6.2. Fusion factorizations

In this section, we will introduce a new method of defining multiplicative fusion products on central extensions of loop groups from certain minimal data, called a fusion factorization. We first define a class of central extensions that are admissible for this method. Let $1 \in PG$ denote the path constantly equal to the unit element in G.

Definition 6.2.1. A Fréchet central extension $\mathrm{U}(1) \to \widetilde{LG} \to LG$ is called *admissible* if it has the following property. For $\beta_1, \beta_3 \in PG$ with endpoints the unit of G, and $q_{12} \in \widetilde{LG}_{\beta_1 \cup 1}$ and $q_{23} \in \widetilde{LG}_{1 \cup \beta_3}$ we have $q_{12}q_{23} = q_{23}q_{12}$.

Let $\Delta : PG \to LG, \beta \mapsto \beta \cup \beta$ be the doubling map.

Definition 6.2.2. Let $\mathrm{U}(1) \to \widetilde{LG} \to LG$ be an admissible Fréchet central extension of LG. Then, a *fusion factorization* is a smooth group homomorphism $\rho : PG \to \widetilde{LG}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\rho}{\nearrow} & \widetilde{LG} \\ & & \downarrow \\ PG & \xrightarrow[\Delta]{} & LG. \end{array}$$

The main result of this section is that a fusion factorization induces a multiplicative fusion product. Indeed, let ρ be a fusion factorization for an admissible Fréchet central extension $\mathrm{U}(1) \to \widetilde{LG} \to LG$. For each triple $(\beta_1, \beta_2, \beta_3) \in PG^{[3]}$ we set

$$\tilde{\mu}^{\rho}_{\beta_1,\beta_2,\beta_3} : \widetilde{LG}_{\beta_1 \cup \beta_2} \times \widetilde{LG}_{\beta_2 \cup \beta_3} \to \widetilde{LG}_{\beta_1 \cup \beta_3} \ , \ (q_{12}, q_{23}) \mapsto q_{12}\rho(\beta_2)^{-1}q_{23}. \tag{6.3}$$

Theorem 6.2.3. *The map $\tilde{\mu}^{\rho}_{\beta_1,\beta_2,\beta_3}$ is a multiplicative fusion product.*

Proof. First of all, the codomain of $\tilde{\mu}^{\rho}_{\beta_1,\beta_2,\beta_3}$ is indeed $\widetilde{LG}_{\beta_1 \cup \beta_3}$, because

$$\beta_1 \cup \beta_3 = (\beta_1 \cup \beta_2)\Delta(\beta_2)^{-1}(\beta_2 \cup \beta_3).$$

The map is clearly U(1)-bilinear, and the associativity is straightforward. Next, we prove multiplicativity. We start by computing

$$\tilde{\mu}^{\rho}_{\beta_1,\beta_2,\beta_3}(q_{12}, q_{23})\tilde{\mu}^{\rho}_{\beta_1',\beta_2',\beta_3'}(q_{12}', q_{23}') = q_{12}\rho(\beta_2)^{-1}q_{23}q_{12}'\rho(\beta_2')^{-1}q_{23}'$$

on the one hand, and

$$\tilde{\mu}^{\rho}_{\beta_1\beta_1',\beta_2\beta_2',\beta_3\beta_3'}(q_{12}q_{12}', q_{23}q_{23}') = q_{12}q_{12}'\rho(\beta_2\beta_2')^{-1}q_{23}q_{23}' = q_{12}q_{12}'\rho(\beta_2')^{-1}\rho(\beta_2)^{-1}q_{23}q_{23}'.$$

We see that to prove multiplicativity it suffices to show that

$$\rho(\beta_2)^{-1}q_{23}q_{12}'\rho(\beta_2')^{-1} = q_{12}'\rho(\beta_2')^{-1}\rho(\beta_2)^{-1}q_{23}.$$

This equation holds by the assumption that the central extension $\widetilde{LG}$ is admissible. Finally, let us prove smoothness. The relevant map is

$$\widetilde{LG}\,{}_q\!\times_{\pi_1} PG^{[3]}\,{}_{\pi_2}\!\times_q \widetilde{LG} \to \widetilde{LG}\ ,\ (q_{12}, \beta_1, \beta_2, \beta_3, q_{23}) \mapsto q_{12}\rho(\beta_2)^{-1}q_{23}.$$

Since projections, multiplication, inversion, and ρ are smooth maps, this is a composition of smooth maps and hence smooth. □

In the remainder of this subsection we impose a condition between a fusion factorization ρ and a local section σ of the central extension and prove (Proposition 6.2.8) that this condition guarantees that the associated fusion product $\tilde{\mu}^{\rho}$ is connection-preserving for the connection ν_σ associated to σ, see Remark 3.5.11.

Definition 6.2.4. Let $\mathrm{U}(1) \to \widetilde{LG} \to LG$ be an admissible Fréchet central extension, and suppose that $\sigma : U \to \widetilde{LG}$ is a smooth local section defined in a neighbourhood U of $1 \in LG$. A fusion factorization ρ is called *compatible* with σ, if there exists an open neighbourhood $U' \subset \Delta^{-1}(U) \subset PG$ of $1 \in PG$, such that $\sigma(\Delta\beta) = \rho(\beta)$ for all $\beta \in U'$.

The following three lemmas prepare the proof of Proposition 6.2.8 below.

Lemma 6.2.5. *Suppose a fusion factorization ρ is compatible with a section σ. Then, ρ is flat with respect to the connection ν_σ, i.e., $\rho^*\nu_\sigma = 0$.*

Proof. We have to show $(\rho^*\nu_\sigma)_c = 0$ for every plot $c : U \to PG$. We first obtain from the definition of ν_σ and the definition of ρ that

$$(\rho^*\nu_\sigma)_c = (\nu_\sigma)_{\rho\circ c} = (\rho\circ c)^*\nu_\sigma = (\rho\circ c)^*\theta^{\widetilde{LG}} - \sigma_*(c^*\Delta^*\theta^{LG}).$$

Consider a smooth curve $\varphi : \mathbb{R} \to U$, with $\varphi(0) =: x \in U$ and $\dot{\varphi}(0) =: v \in T_xU$. Then, we have

$$(\rho^*\nu_\sigma)_c(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \rho(c(x))^{-1}\rho(c(\varphi(t))) - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \sigma(\Delta(c(x))^{-1}\Delta(c(\varphi(t)))).$$

The compatibility condition of Definition 6.2.4 now shows that this expression vanishes. □

The section σ induces a map $Z_\sigma : LG \times L\mathfrak{g} \to \mathbb{R}$ defined by

$$Z_\sigma(\gamma, X) := \mathrm{Ad}_\gamma^{-1}(\sigma_*(X)) - \sigma_*(\mathrm{Ad}_\gamma^{-1}(X));$$

i.e., it measures the error for the derivative σ_* being an intertwiner for the adjoint action of LG. It is related to the cocycle ω_σ by the formula

$$\omega_\sigma(X, Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Z_\sigma(e^{-tX}, Y),$$

and satisfies

$$Z_\sigma(\gamma_1\gamma_2, X) = Z_\sigma(\gamma_1, X) + Z(\gamma_2, \mathrm{Ad}_{\gamma_1}^{-1}(X)). \tag{6.4}$$

We will use the map Z_σ in order to describe a relation between the connection ν_σ and the group structure on $\widetilde{LG}$. We denote by $\tilde{m}, \mathrm{pr}_1, \mathrm{pr}_2 : \widetilde{LG} \times \widetilde{LG} \to \widetilde{LG}$ the multiplication and the two projections.

Lemma 6.2.6. *The equality*

$$\tilde{m}^*\nu_\sigma = \mathrm{pr}_1^*\,\nu_\sigma + \mathrm{pr}_2^*\,\nu_\sigma + (q\times q)^* Z_\sigma(\mathrm{pr}_2, \mathrm{pr}_1^*\,\theta^{LG})$$

of 1-forms on $\widetilde{LG}\times\widetilde{LG}$ *holds. Here,* $q:\widetilde{LG}\to LG$ *is the projection, and the expression* $Z_\sigma(\mathrm{pr}_2,\mathrm{pr}_1^*\,\theta^{LG})$ *denotes a 1-form on* $LG\times LG$*, whose value at a point* (γ_1,γ_2) *and a tangent vector* $(X_1,X_2)\in T_{\gamma_1,\gamma_2}(LG\times LG)$ *is given by* $Z_\sigma(\gamma_2,\gamma_1^{-1}X_1)$*.*

Proof. A straightforward calculation that only uses the definition of ν_σ. □

Next, we consider the set $P\mathfrak{g}$ of smooth paths in the Lie algebra $\mathfrak{g}$ with sitting instants, analogous to (6.1). We have a corresponding map $P\mathfrak{g}^{[2]}\to L\mathfrak{g}:(X_1,X_2)\mapsto X_1\cup X_2:=\bar{X}_2\star X_1$.

Lemma 6.2.7. *Suppose* $\widetilde{LG}$ *is admissible. Let* $\beta\in PG$ *with endpoints the unit of* G*, and let* $X\in P\mathfrak{g}$ *with endpoints zero. Then,* $Z_\sigma(\beta\cup 1, 0\cup X)=0$*.*

Proof. Since the adjoint action of LG on $L\mathfrak{g}$ is pointwise, we have $\mathrm{Ad}^{-1}_{\beta\cup 1}(0\cup X)=0\cup X$, so that $Z_\sigma(\beta\cup 1,0\cup X)=(\mathrm{Ad}^{-1}_{\beta\cup 1}-id)(\sigma_*(0\cup X))$. We may represent $0\cup X$ as the derivative of a smooth curve $1\cup\Gamma$ in LG, and obtain

$$Z_\sigma(\beta\cup 1,0\cup X)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \widetilde{\beta\cup 1}^{-1}\cdot\sigma(1\cup\Gamma(t))\cdot\widetilde{\beta\cup 1}-\sigma_*(0\cup X),$$

where $\widetilde{\beta\cup 1}$ is any lift of $\beta\cup 1$ to $\widetilde{LG}$. Admissibility implies now that $Z_\sigma(\beta\cup 1,0\cup X)=0$. □

Now we are in position to prove the following.

Proposition 6.2.8. *Let* $\mathrm{U}(1)\to\widetilde{LG}\to LG$ *be an admissible Fréchet central extension, equipped with a smooth section* σ *defined in a neighbourhood of the unit of* LG*, and equipped with a compatible fusion factorization* ρ*. Then, the fusion product* $\tilde{\mu}_\rho$ *is connection-preserving with respect to the connection* ν_σ *in the sense of Definition 6.1.1.*

Proof. Using the definition of $\tilde{\mu}^\rho$ and Lemma 6.2.6 we obtain, in the notation of (6.2),

$$(\tilde{\mu}^\sigma)^*\nu_\sigma=\tilde{\pi}_1^*\nu_\sigma+\tilde{\pi}_2^*\nu_\sigma+\zeta$$

where $\zeta\in\Omega^1(PG^{[3]})$ is given by

$$\zeta:=p_2^*i^*\rho^*\nu_\sigma+Z_\sigma(\pi_2,p_2^*i^*\Delta^*\theta^{LG})+Z_\sigma(1\cup p',\pi_1^*\theta^{LG}),\tag{6.5}$$

where the maps $p_2,p':PG^{[3]}\to PG$ are $p_2(\beta_1,\beta_2,\beta_3):=\beta_2$ and $p'(\beta_1,\beta_2,\beta_3):=i(\beta_2)\beta_3$, and the map $i:PG\to PG$ is the pointwise inversion. We shall prove that $\zeta=0$. By Lemma 6.2.5 the first summand in (6.5) vanishes. We write the second summand using (6.4) as

$$Z_\sigma(\pi_2,p_2^*i^*\Delta^*\theta^{LG})=Z_\sigma((\Delta\circ p_2)\cdot(1\cup p'),p_2^*i^*\Delta^*\theta^{LG})=p_2^*\Delta^*Z_\sigma(id,i^*\theta^{LG})-Z_\sigma(1\cup p',p_2^*\Delta^*\theta^{LG}).$$

It is straightforward to show that $\Delta^*Z_\sigma(id,i^*\theta^{LG})=0$, using that the composition $\sigma\circ\Delta$ is a group homomorphism (see Definitions 6.2.2 and 6.2.4). All together, we obtain

$$\zeta=-Z_\sigma(1\cup p',p_2^*\Delta^*\theta^{LG})+Z_\sigma(1\cup p',\pi_1^*\theta^{LG})=-Z_\sigma(1\cup p',p_2^*\Delta^*\theta^{LG}-\pi_1^*\theta^{LG}).$$

We claim that the values of the 1-form $p_2^*\Delta^*\theta^{LG} - \pi_1^*\theta^{LG}$ on $PG^{[3]}$ are of the form $X \cup 0 \in L\mathfrak{g}$, which proves via Lemma 6.2.7 that $\zeta = 0$. We consider a plot $c : U \to PG^{[3]}$, consisting of three plots $c_1, c_2, c_3 : U \to PG$. Let $x \in U$ and $v \in T_xU$. We compute

$$\begin{aligned}(\pi_1^*\theta^{LG})_c|_x(v)(t) &= (\pi_1 \circ c)^*\theta^{LG}|_x(v)(t)\\ &= \theta^G|_{(c_1\cup c_2)(x)(t)}(\mathrm{d}(c_1 \cup c_2)|_x(v)(t))\\ &= \begin{cases}\theta^G|_{c_1^\vee(x,2t)}(\mathrm{d}c_1^\vee|_{(x,2t)}(v, 2t)) & \text{if } 0 \leqslant t \leq \pi\\ \theta^G|_{c_2^\vee(x,2-2t)}(\mathrm{d}c_2^\vee|_{(x,2-2t)}(v, 2-2t)) & \text{if } \pi \leqslant t \leqslant 2\pi\end{cases}\end{aligned}$$

and similarly,

$$(p_2^*\Delta^*\theta^{LG})_c|_x(v)(t) = \begin{cases}\theta^G|_{c_2^\vee(x,2t)}(\mathrm{d}c_2^\vee|_{(x,2t)}(v, 2t)) & \text{if } 0 \leqslant t \leq \pi\\ \theta^G|_{c_2^\vee(x,2-2t)}(\mathrm{d}c_2^\vee|_{(x,2-2t)}(v, 2-2t)) & \text{if } \pi \leqslant t \leqslant 2\pi\end{cases}$$

This proves the claim. □

6.3. Fusion factorization for implementers

In this subsection we equip our operator-algebraic model of $\widetilde{L\,\mathrm{Spin}}(d)$ discussed in Section 3.7 with a multiplicative fusion product. For this purpose, we first prove that this central extension is admissible, and then construct a canonical fusion factorization.

Proposition 6.3.1. *The central extension* $\mathrm{U}(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$ *is admissible.*

Proof. Let $\beta_1, \beta_3 \in P\,\mathrm{Spin}(d)$ with endpoints equal to the identity of $L\,\mathrm{Spin}(d)$. We see that $M(\beta_1 \cup 1) \in \mathrm{O}_L(V)_-$ and $M(1 \cup \beta_3) \in \mathrm{O}_L(V)_+$, see Section 5.3. Now, let $q_{12} \in \widetilde{L\,\mathrm{Spin}}(d)_{\beta_1\cup 1}$ and let $q_{23} \in \widetilde{L\,\mathrm{Spin}}(d)_{1\cup\beta_3}$. Proposition 3.7.3 tells us that $M(q_{12}) \in \mathrm{Imp}_L(V)_-$ and $M(q_{23}) \in \mathrm{Imp}_L(V)_+$ are even. Then we apply Proposition 5.4.2 to conclude that $M(q_{12})$ commutes with $M(q_{23})$ and hence q_{12} commutes with q_{23}, and we are done. □

In the remainder of this section we will construct a fusion factorization for $\widetilde{L\,\mathrm{Spin}}(d)$. In fact, we will define a smooth group homomorphism $\rho : P\,\mathrm{Spin}(d) \to \mathrm{Imp}_L(V)$ such that the diagram

$$\begin{array}{ccccc} & & & & \mathrm{Imp}_L(V)\\ & & & \overset{\rho}{\nearrow} & \downarrow q\\ P\,\mathrm{Spin}(d) & \xrightarrow[\Delta]{} & L\,\mathrm{Spin}(d) & \xrightarrow[M]{} & \mathrm{O}_L(V)\end{array} \tag{6.6}$$

is commutative; this induces a fusion factorization in the obvious way. We start by considering the diffeological group

$$\mathrm{Imp}'_L(V) := (M \circ \Delta)^*\,\mathrm{Imp}_L(V) = P\,\mathrm{Spin}(d)\;{}_{\Delta\circ M}\!\times_q \mathrm{Imp}_L(V),$$

which is a central extension of $P\,\mathrm{Spin}(d)$ by $\mathrm{U}(1)$. We will first reduce it to a central extension by $\mathbb{Z}_2$. Let $(\beta, U) \in \mathrm{Imp}'_L(V)$, i.e. $q(U) = M(\Delta(\beta))$. We first observe that $\tau(M(\Delta(\beta))) = M(\Delta(\beta))$. Then, combining this observation with Proposition 5.1.9, we see that

$$q(\kappa(U)) = \tau(q(U)) = q(U).$$

Hence, $U\kappa(U)^*$ implements the identity operator, so that $U\kappa(U)^* \in \mathrm{U}(1)$. This allows us to define a map w as follows

$$w : \mathrm{Imp}'_L(V) \to \mathrm{U}(1)\ ,\ (\beta, U) \mapsto U\kappa(U)^*;$$

this map is smooth, because the projection $\mathrm{Imp}'_L(V) \to \mathrm{Imp}_L(V)$ is smooth, $\mathrm{Imp}_L(V)$ is a Lie group, and κ is smooth by Proposition 5.1.11. It is straightforward to show that w is a group homomorphism and satisfies $w(\beta, \lambda U) = \lambda^2 w(\beta, U)$ for all $\lambda \in \mathrm{U}(1)$. It is well-known that such a map determines a reduction of a central extension from $\mathrm{U}(1)$ to $\mathbb{Z}_2$; in our case, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_2 & \longrightarrow & \mathrm{U}(1) \\ \downarrow & & \downarrow \\ w^{-1}(1) & \longrightarrow & \mathrm{Imp}'_L(V) \\ \downarrow & & \downarrow \\ P\,\mathrm{Spin}(d) & = & P\,\mathrm{Spin}(d) \end{array}$$

of diffeological groups and smooth group homomorphisms, whose vertical sequences are exact sequences of groups.

Next we use the modular conjugation $J : \mathcal{F} \to \mathcal{F}$ corresponding to the triple $(\mathrm{Cl}(V_-)'', \mathcal{F}, \Omega)$, see Chapter 5. Let $(\beta, U) \in \mathrm{Imp}'_L(V)$, then using that U is even, Lemma 5.1.6 and Proposition 5.2.2 one sees that $\kappa(U) = JUJ$. Hence if $(\beta, U) \in w^{-1}(1)$, then $1 = U\kappa(U)^* = UJU^*J$, and hence $UJ = JU$.

The next step is to define a group homomorphism $r : w^{-1}(1) \to \mathbb{Z}_2$; such a group homomorphism then induces a splitting. To this end, we require the theory of standard forms of von Neumann algebras, see Sections 4.1 and 5.2. Let $(\beta, U) \in w^{-1}(1)$. We define two cones in $\mathcal{F}$ as follows

$$P_\Omega = \{aJa\Omega \in \mathcal{F} \mid a \in \mathrm{Cl}(V_-)''\}^{\mathrm{cl}},$$
$$P_{U\Omega} = \{aJaU\Omega \in \mathcal{F} \mid a \in \mathrm{Cl}(V_-)''\}^{\mathrm{cl}}.$$

We then have that the quadruples $(\mathrm{Cl}(V_-)'', \mathcal{F}, J, P_\Omega)$ and $(\mathrm{Cl}(V_-)'', \mathcal{F}, J, P_{U\Omega})$ are standard forms for the von Neumann algebra $\mathrm{Cl}(V_-)''$, see Definition 4.1.2 and Lemma 4.1.5. To see that the second quadruple is a standard form, note that the modular conjugation $J_{U\Omega}$ corresponding to the cyclic and separating vector $U\Omega$ is equal to J, since $J_{U\Omega} = UJU^* = J$. Theorem 4.1.3 then implies that there is a unique unitary $u : \mathcal{F} \to \mathcal{F}$ with the following properties

(1) For all $a \in \mathrm{Cl}(V_-)''$ we have $a = uau^*$.

(2) $u = JuJ$.

(3) $uP_\Omega = P_{U\Omega}$.

We define $r(\beta, U) := u$. In the first place, this defines a map $r : w^{-1}(1) \to \mathrm{U}(\mathcal{F})$.

It is clear that the operators $\pm\mathbb{1}$ satisfy (1) and (2). The next point of business is to show that $P_\Omega = \pm P_{U\Omega}$, from which it follows that $u = \pm\mathbb{1}$. In the sequel, we shall require the even Fock space. Recall that U is even. We define two even cones in $\mathcal{F}_0$ as

$$P_{\Omega;0} = \{aJa\Omega \in \mathcal{F}_0 \mid a \in \mathrm{Cl}(V_-)''_0\}^{\mathrm{cl}},$$
$$P_{U\Omega;0} = \{aJaU\Omega \in \mathcal{F}_0 \mid a \in \mathrm{Cl}(V_-)''_0\}^{\mathrm{cl}},$$

where the superscript cl indicates that we take the closure. The quadruples $(\mathrm{Cl}(V_-)''_0, \mathcal{F}_0, J, P_{\Omega;0})$ and $(\mathrm{Cl}(V_-)''_0, \mathcal{F}_0, J, P_{U\Omega;0})$ are standard forms for the von Neumann algebra $\mathrm{Cl}(V_-)''_0$. The following result is [Ara74, Theorem 4, parts (5) and (4)].

Lemma 6.3.2. *Let ξ be a cyclic and separating vector in $\mathcal{F}_0$. Then $\xi \in P_{\Omega;0}$ if and only if $J_\xi = J$ and*

$$\langle \xi, z\Omega \rangle \geqslant 0 \tag{6.7}$$

for all $z \in \mathrm{Cl}(V_-)_0'' \cap (\mathrm{Cl}(V_-)_0'')'$, with $z \geqslant 0$. Furthermore, if $\xi \in P_{\Omega;0}$ is a cyclic and separating vector, then $P_\xi = P_{\Omega;0}$.

Note that in our case, we have $\mathrm{Cl}(V_-)_0'' \cap (\mathrm{Cl}(V_-)_0'')' = \mathbb{C}$ (see Theorem 5.2.3), hence we may replace z in (6.7) with 1.

Lemma 6.3.3. *If $(\beta, U) \in w^{-1}(1)$, then either $P_{U\Omega} = P_\Omega$ (and then $P_{U\Omega;0} = P_{\Omega;0}$) or $P_{U\Omega} = -P_\Omega$ (and then $P_{U\Omega;0} = -P_{\Omega;0}$).*

Proof. We compute

$$\langle \Omega, U\Omega \rangle = \langle J\Omega, U\Omega \rangle = \langle JU\Omega, \Omega \rangle = \langle U\Omega, \Omega \rangle,$$

from which follows that $\langle \Omega, U\Omega \rangle$ is real. We now distinguish the following three cases.

$\langle \Omega, U\Omega \rangle > 0$: In this case, Lemma 6.3.2 tells us that $U\Omega \in P_{\Omega;0} \subset P_\Omega$ and hence $P_\Omega = P_{U\Omega}$.

$\langle \Omega, U\Omega \rangle < 0$: In this case we have $\langle \Omega, -U\Omega \rangle > 0$, and hence Lemma 6.3.2 tells us that $P_\Omega = P_{-U\Omega} = -P_{U\Omega}$.

$\langle \Omega, U\Omega \rangle = 0$: Using Lemma 6.3.2 it follows that $P_{U\Omega} = P_\Omega = -P_{U\Omega}$, a contradiction, hence this case cannot occur. □

It follows that $r(\beta, U) \in \mathbb{Z}_2$.

Lemma 6.3.4. *The map $r : w^{-1}(1) \to \mathbb{Z}_2$ is a group homomorphism.*

Proof. One sees easily that r is $\mathbb{Z}_2$-equivariant. Now, it suffices to show that for all $(\beta, U), (\beta', U') \in w^{-1}(1)$ with

$$r(\beta, U) = 1 = r(\beta', U')$$

we have $r(\beta\beta', UU') = 1$. So, let (β, U) and (β', U') have this property. It is now sufficient to show that $P_{UU'\Omega} = P_\Omega$. By assumption we have that $P_\Omega = P_{U'\Omega}$, which, by Lemma 6.3.2 implies that

$$0 \leqslant \langle U'\Omega, \Omega \rangle = \langle UU'\Omega, U\Omega \rangle,$$

from which, again using Lemma 6.3.2, it follows that $P_{UU'\Omega} = P_{U\Omega} = P_\Omega$, which concludes the proof. □

The group homomorphism r trivializes the central extension $\mathbb{Z}_2 \to w^{-1}(1) \to P\,\mathrm{Spin}(d)$; the corresponding splitting assigns to $\beta \in P\,\mathrm{Spin}(d)$ the unique element (β, U) in $w^{-1}(1)$ with $r(\beta, U) = \mathbb{1}$, i.e., the unique (β, U) with $UJ = JU$ and $P_\Omega = P_{U\Omega}$. In turn, we obtain via $w^{-1}(1) \subset \mathrm{Imp}_L'(V) \to \mathrm{Imp}_L(V)$ the claimed group homomorphism

$$\rho : P\,\mathrm{Spin}(d) \to \mathrm{Imp}_L(V), \tag{6.8}$$

making the diagram (6.6) commutative. We shall summarize the following two characterizations of this map.

Lemma 6.3.5. *Let $\beta \in P\,\mathrm{Spin}(d)$. Then,*

1. *$\rho(\beta)$ is the unique implementer of $M(\beta) \oplus \tau M(\beta)\tau^*$ such that $\rho(\beta)J = J\rho(\beta)$ and $P_\Omega = P_{\rho(\beta)\Omega}$.*

2. $\rho(\beta)$ is the unique implementer of $M(\beta)\oplus \tau M(\beta)\tau^*$ such that $\langle \Omega, \rho(\beta)\Omega\rangle > 0$.

Proof. The first characterization only repeats what we have. We now argue that the second characterization follows from the first. Applying Lemma 6.3.2 for $\xi = \rho(\beta)\Omega$ and $z = 1$, and using the fact that $P_\Omega = P_{\rho(\beta)\Omega}$ and that $J_{\rho(\beta)\Omega} = \rho(\beta)J\rho(\beta)^* = J_\Omega$ it follows that $\langle \Omega, \rho(\beta)\Omega\rangle \geqslant 0$. That the inequality is strict then follows from Lemma 6.3.3. There is only one such $\rho(\beta)$, because any two implementers of $M(\beta)\oplus \tau M(\beta)\tau^*$ are related by a unique $\lambda \in \mathrm{U}(1)$. □

Now we are in position to finalize our construction of a fusion factorization.

Proposition 6.3.6. *The map $\rho : P\,\mathrm{Spin}(d) \to \mathrm{Imp}_L(V)$ defined in (6.8) induces a fusion factorization*

$$P\,\mathrm{Spin}(d) \to \widetilde{L\,\mathrm{Spin}}(d) = M^*\,\mathrm{Imp}_L(V)$$
$$\beta \mapsto (\Delta(\beta), \rho(\beta)).$$

Proof. It remains to show that ρ is smooth, and for this, it suffices to show that the group homomorphism $r : w^{-1}(1) \to \mathbb{Z}_2$ is smooth, where $w^{-1}(1) \subset \mathrm{Imp}'_L(V)$ is equipped with the subspace diffeology. The subspace diffeology consists of those plots $c : U \to \mathrm{Imp}'_L(V)$ whose image is in $w^{-1}(1)$. In particular, if $c : U \to w^{-1}(1)$ is a plot, then the projection to $\mathrm{Imp}_L(V)$ is smooth. We have to show that $r \circ c : U \to \mathbb{Z}_2$ is smooth, i.e., it is locally constant. Consider $x \in U$, and let $(\beta_x, U_x) := c(x)$. Consider the open ball around U in $\mathrm{Imp}_L(V)$ of radius 1/2, and let $\mathcal{O} \subset U$ be its preimage under the smooth map $U \to \mathrm{Imp}_L(V)$. We will show that r is constant on $\mathcal{O}$, this proves the lemma.

Let $y \in \mathcal{O}$, and write $(\beta_y, U_y) := c(y)$. We want to show that the cone $P_{U_x\Omega;0}$ is equal to the cone $P_{U_y\Omega;0}$. Note that $(\beta_y, U_y) \in w^{-1}(1)$ and $\|U_x - U_y\| \leqslant 1/2$. We set $A := U_x - U_y$; note that $AJ = JA$. The computation

$$\langle U_x\Omega, A\Omega\rangle = \langle U_x\Omega, AJ\Omega\rangle = \langle U_x\Omega, JA\Omega\rangle = \langle A\Omega, JU_x\Omega\rangle = \langle A\Omega, U_x\Omega\rangle,$$

implies that $\langle U_x\Omega, A\Omega\rangle$ is real. We then compute

$$\langle U_x\Omega, U_y\Omega\rangle = 1 + \langle U_x\Omega, A\Omega\rangle \geqslant 1 - |\langle U_x\Omega, A\Omega\rangle| \geqslant 1 - \|A\| \geqslant 1/2.$$

Lemma 6.3.2 now proves that $P_{U_x\Omega;0} = P_{U_y\Omega;0}$. □

We recall that the central extension $\mathrm{U}(1) \to \mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$ comes with a local section $\sigma : U \to \mathrm{Imp}_L(V)$ defined in an open $\mathbb{1}$-neighbourhood U, see Section 3.5. Hence, its pullback $\widetilde{L\,\mathrm{Spin}}(d) = M^*\,\mathrm{Imp}_L(V)$ is equipped with a local section $\tilde{\sigma} := M^*\sigma$ defined on $\tilde{U} := M^{-1}(U)$.

Lemma 6.3.7. *The fusion factorization of Proposition 6.3.6 constructed above is compatible with the local section $\tilde{\sigma}$ in the sense of Definition 6.2.4.*

Proof. Let $U' \subseteq U \subset \mathrm{O}_L(V)$ be an open neighbourhood such that $|\langle \Omega, \sigma(g)\Omega\rangle - 1| < 1$ for $g \in U'$, we put $\tilde{U}' := \Delta^{-1}M^{-1}(U') \subset \Delta^{-1}(\tilde{U}) \subset P\,\mathrm{Spin}(d)$ and shall prove that $\tilde{\sigma}(\Delta(\beta)) = \rho(\beta)$ for all $\beta \in \tilde{U}'$. We recall from Lemma 5.1.10 that $\kappa \circ \sigma = \sigma \circ \tau$ hence $\kappa(\tilde{\sigma}(\Delta(\beta))) = \tilde{\sigma}(\Delta(\beta))$. This implies that $\tilde{\sigma}(\Delta(\beta))$ commutes with the modular conjugation J, and hence

$$\langle \Omega, \tilde{\sigma}(\Delta(\beta))\Omega\rangle = \langle \Omega, \tilde{\sigma}(\Delta(\beta))J\Omega\rangle = \langle \Omega, J\tilde{\sigma}(\Delta(\beta))\Omega\rangle = \langle \tilde{\sigma}(\Delta(\beta))\Omega, J\Omega\rangle = \overline{\langle \Omega, \tilde{\sigma}(\Delta(\beta))\Omega\rangle},$$

whence $\langle \Omega, \tilde{\sigma}(\Delta(\beta))\Omega\rangle \in \mathbb{R}$. Now, because $M(\Delta(\beta)) \in U'$ we have $\langle \Omega, \tilde{\sigma}(\Delta(\beta))\Omega\rangle > 0$, and Lemma 6.3.5 shows $\tilde{\sigma}(\Delta(\beta)) = \rho(\beta)$. □

Remark 6.3.8. From [Nee10b, Proof of Theorem 10.2] it follows that $\langle \Omega, \sigma(g)\Omega \rangle > 0$ for all $g \in U$; hence, the reduction to $U' \subseteq U$ is in fact not strictly necessary.

As a consequence of Lemma 6.3.7 and Proposition 6.2.8 we obtain:

Proposition 6.3.9. *The fusion product $\tilde{\mu}^\rho$ induced by the fusion factorization of Proposition 6.3.6 is connection-preserving with respect to the connection ν_σ of Remark 3.5.11.*

In the remainder of this section, we relate the fusion product $\tilde{\mu}^\rho$ induced by our fusion factorization to the (Connes) fusion of implementers, $\hat{\mu}$, defined in Section 5.4, and show that the two notions of fusion coincide. First, we require the following lemma.

Lemma 6.3.10. *Let $\beta = (\beta_1, \beta_2, \beta_3) \in P^{[3]} \operatorname{Spin}(d)$, let $(U_1, U_2) \in \widetilde{L\operatorname{Spin}}(d)_{\pi_1(\beta)} \times \widetilde{L\operatorname{Spin}}(d)_{\pi_2(\beta)}$, and let $K := \rho(\beta_2)^{-1} \in \operatorname{Imp}_L(V)$. Then, $P_\Omega = P_{K\Omega} = P_{K^{-1}\Omega}$, and $U_1 K \in \operatorname{Cl}(V_-)_0''$, and $KU_2 \in \operatorname{Cl}(V_+)_0''$.*

Proof. We have $P_\Omega = P_{K\Omega}$ because $K \in r^{-1}(1)$. To prove that $P_{K\Omega} = P_{K^{-1}\Omega}$ we compute

$$0 \leqslant \langle \Omega, K\Omega \rangle = \langle K^{-1}\Omega, \Omega \rangle = \langle \Omega, K^{-1}\Omega \rangle,$$

and apply Lemma 6.3.2. Finally, one may verify that $U_1 K$ implements an element of $\mathrm{O}_L(V)_-$ and hence $U_1 K \in (\operatorname{Cl}(V_+)_0'')' = \operatorname{Cl}(V_-)_0''$, and similarly for KU_2. □

The following result now tells us that the fusion product $\tilde{\mu}^\rho$ on the basic central extension of $\widetilde{L\operatorname{Spin}}(d)$ is given by the Connes fusion of operators on $\mathcal{F}$.

Theorem 6.3.11. *Let $\beta = (\beta_1, \beta_2, \beta_3) \in P^{[3]} \operatorname{Spin}(d)$, and $U_1, U_2 \in \operatorname{Imp}_L(V)$ such that $(U_1, U_2) \in \widetilde{L\operatorname{Spin}}(d)_{\pi_1(\beta)} \times \widetilde{L\operatorname{Spin}}(d)_{\pi_2(\beta)}$. Then*

$$\tilde{\mu}^\rho_\beta(U_1 \otimes U_2) = \hat{\mu}(U_1, U_2).$$

Proof. Let $K \in \operatorname{Imp}_L(V)$ be the element from Lemma 6.3.10, with the property $\tilde{\mu}^\rho_\beta(U_1 \otimes U_2) = U_1 K U_2$. We compute, using the multiplicativity of $\hat{\mu}$ (Lemma 5.4.6),

$$\hat{\mu}(U_1, U_2) = \hat{\mu}(U_1 K K^{-1}, U_2 K K^{-1}) = \hat{\mu}(U_1 K, U_2 K)\hat{\mu}(K^{-1}, K^{-1}).$$

Hence, it suffices to show that

$$\hat{\mu}(U_1 K, U_2 K) = U_1 K U_2 K,$$

and

$$\hat{\mu}(K^{-1}, K^{-1}) = K^{-1}.$$

We see that because $U_1 K \in (\operatorname{Cl}(V_+)_0'')'$ (Lemma 6.3.10) we have that the unitary $u : \mathcal{F} \to \mathcal{F}$ used in the construction of $\mu_0(U_1 K, U_2 K)$ (see Lemma 4.2.5) is simply the identity. Let $v \in \mathcal{F}$, then to determine $\hat{\mu}(U_1 K, U_2 K)$ we compute

$$\begin{aligned} \mu\mu_0(U_1 K, U_2 K)\mu^{-1}(v) &= \mu(U_1 K f^{-1} \otimes U_2 K v) \\ &= \mu(U_1 K f^{-1} \otimes v \triangleleft J K^* U_2^* J) \\ &= U_1 K v \triangleleft J K^* U_2^* J \\ &= U_1 K U_2 K v. \end{aligned}$$

Because of the fact that K^{-1} has the property that $JK^{-1} = K^{-1}J$ and $P_{K^{-1}\Omega} = P_\Omega$, we see that the corresponding $u : L^2_\phi(\mathcal{A}_-) \to L^2_\phi(\mathcal{A}_-)$ is $fK^{-1}f^{-1}$. We compute:

$$\begin{aligned} \mu_0(K^{-1}, K^{-1})(f^{-1} \otimes v) &= K^{-1} f^{-1} f K f^{-1} \otimes K^{-1} v \\ &= f^{-1} \otimes K^{-1} v. \end{aligned}$$

□

7. Implementers and string geometry

In this chapter we show that our operator-algebraic model of the central extension $\widetilde{L\,\mathrm{Spin}}(d)$ (Section 3.7), equipped with the connection defined in Remark 3.5.11, and with the connection-preserving, multiplicative fusion product defined in Section 6.3, yields a so-called *fusion extension with connection*. We then establish that our model is canonically isomorphic, as fusion extension with connection, to the usual model obtained by transgression, so that both models can be used interchangeably in string geometry.

7.1. Multiplicative gerbes and transgression

Let G be a Lie group; important for us is $G = \mathrm{Spin}(d)$. We consider multiplicative bundle gerbes $\mathcal{G}$ over G. A quick introduction to bundle gerbes is given in Appendix D. We shall recall some minimal facts on multiplicative bundles gerbes. A multiplicative bundle gerbe over G is a bundle gerbe $\mathcal{G}$ over G equipped with an isomorphism

$$\mathcal{M} : \mathrm{pr}_1^*\,\mathcal{G} \otimes \mathrm{pr}_2^*\,\mathcal{G} \to m^*\mathcal{G}$$

over $G \times G$, where pr_i denote the projections, and $m : G \times G \to G$ is the multiplication of G. Moreover, this isomorphism has to be coherently associative [CJM+05]. Multiplicative bundle gerbes have characteristic classes in $H^4(BG, \mathbb{Z})$; forgetting the multiplicative structure realizes the usual homomorphism $H^4(BG, \mathbb{Z}) \to H^3(G, \mathbb{Z})$, see [CJM+05, Wal10]. We now assume that G is compact, cimple, simply-connected and connected, in which case both $H^4(BG, \mathbb{Z})$ and $H^3(G, \mathbb{Z})$ are isomorphic to $\mathbb{Z}$, and the above homomorphism is the identity. A bundle gerbe $\mathcal{G}$ over G that represents a generator in $H^3(G, \mathbb{Z})$ is called a *basic bundle gerbe*; thus, a basic bundle gerbe admits a (up to isomorphism) unique multiplicative structure. Concrete constructions of a basic bundle gerbe are described in [GR02, Mei02], while concrete constructions of a corresponding multiplicative structure have not yet been carried out (one proposal is described on the last page of [Wal12a]).

The Lie group $\mathrm{Spin}(d)$ fulfils above conditions, hence it has an (up to isomorphism) unique basic bundle gerbe $\mathcal{G}_{bas}$. The characteristic class of $\mathcal{G}_{bas}$ is $\frac{1}{2}p_1 \in H^4(B\,\mathrm{Spin}(d), \mathbb{Z})$ [McL92, Wal13]. String geometry is based on the geometry of the basic bundle gerbe $\mathcal{G}_{bas}$. The geometry consists of a connection on $\mathcal{G}_{bas}$ which is compatible with the multiplicative structure. The curvature of this connection is the Cartan 3-form $H = \frac{1}{24\pi}\langle \theta \wedge [\theta \wedge \theta]\rangle$, where $\langle -, - \rangle$ denotes the basic inner product on $\mathfrak{spin}(d)$, as in Section 3.7, and θ denotes the left-invariant Maurer-Cartan form on G. The 3-form H satisfies the equation

$$\mathrm{pr}_1^*\,H + \mathrm{pr}_2^*\,H = m^*H + \mathrm{d}\rho, \tag{7.1}$$

where $\rho \in \Omega^2(G \times G)$ is the 2-form $\rho := \frac{1}{4\pi}\langle \mathrm{pr}_1^*\,\theta \wedge \mathrm{pr}_2^*\,\bar{\theta}\rangle$; here, $\bar{\theta}$ is the right-invariant Maurer-Cartan form.

In general, the curvature of a multiplicative bundle gerbe is a pair (H, ρ) consisting of a 3-form $H \in \Omega^3(G)$ and a 2-form $\rho \in \Omega^2(G \times G)$ satisfying (7.1) and an additional "simplicial" condition over $G \times G \times G$, see [Wal10, Section 2.3]. Indeed, such a pair defines a degree four cocycle in the simplicial de Rham cohomology of G, which computes $H^4(BG, \mathbb{R})$ [BSS76]. Similar as in Chern-Weil theory, the class of this cocycle coincides with the image of the characteristic class of the multiplicative bundle gerbe in real cohomology [Wal10, Prop. 2.1].

Transgression (to loop groups) is a homomorphism in cohomology, defined as

$$H^3(G, \mathbb{Z}) \to H^2(LG, \mathbb{Z}) \ , \ \xi \mapsto \int_{S^1} ev^*\xi,$$

where $ev : S^1 \times LG \to G$ is the evaluation map. There is an analogous homomorphism in de Rham cohomology. Transgression can also be defined on a geometrical level, taking bundle gerbes with connection over G to principal U(1)-bundles with connection over LG, see [Bry93, GR02]. A multiplicative structure on a bundle gerbe $\mathcal{G}$ transgresses to a group structure on the corresponding U(1)-bundle, turning it into a central extension which we will denote by $\mathscr{T}_{\mathcal{G}}$ [Wal10]. The basic gerbe $\mathcal{G}_{bas}$ over a compact, simple, connected, simply-connected Lie group G transgresses to the basic central extension of LG, i.e. $\mathscr{T}_{\mathcal{G}_{bas}} \cong \widetilde{LG}$, as we will recall below. This establishes the relation between string geometry and the geometry of the basic central extension of $L\,\mathrm{Spin}(d)$.

In general, central extensions $\mathscr{T}_{\mathcal{G}}$ of a loop group LG in the image of the transgression functor come equipped with the following additional structure [Wal17, Section 5.2]:

(a) a multiplicative fusion product $\tilde{\mu}$ as in Definition 6.1.1.

(b) a connection ν that is preserved by $\tilde{\mu}$ in the sense of Definition 6.1.3, and additionally has the property of being superficial and of symmetrizing $\tilde{\mu}$.

(c) a multiplicative, contractible path splitting κ of the error 1-form of the connection ν.

The notions of superficial and symmetrizing connections have been defined in [Wal16b]; these will only play a minor role here. Likewise, the notion of a path splitting defined in [Wal17] is only listed for completeness. Central extensions of LG equipped with the structure (a) to (c) are called *fusion extensions with connection*; they form a category $\mathcal{F}us\mathcal{E}xt^{\nabla}(LG)$, whose morphisms are fusion-preserving, connection-preserving isomorphisms of central extensions. Transgression is a functor

$$\mathscr{T} : \mathrm{h}_1\mathcal{M}ult\mathcal{G}rb^{\nabla}(G) \to \mathcal{F}us\mathcal{E}xt^{\nabla}(LG) \tag{7.2}$$

defined on the 1-truncation of the bicategory $\mathcal{M}ult\mathcal{G}rb^{\nabla}(G)$ of multiplicative bundle gerbes with connection. All details of these structures can be found in [Wal17]. We remark that the list (a) to (c) of additional structures is complete, i.e. the transgression functor (7.2) is an equivalence of categories, whenever G is compact and connected [Wal17, Theorem 5.3.1].

Proposition 7.1.1. *Let* $\mathrm{U}(1) \to \mathcal{L} \to LG$ *be a fusion extension with connection, and with fusion product* $\tilde{\mu}$*. Then,* $\mathcal{L}$ *is admissible in the sense of Definition 6.2.1. Moreover, there is a unique flat fusion factorization* $\chi : PG \to \mathcal{L}$ *such that* $\tilde{\mu} = \tilde{\mu}^{\chi}$*.*

Proof. Admissibility is weaker than being disjoint-commutative, which is a property of any fusion extension with connection, see [Wal17, Theorem 3.3.1]. The uniqueness of the fusion factorization can be seen easily from definition (6.3) of the associated fusion product. We infer from [Wal16b, Lemma 2.1.2] the existence of a flat section χ of $\Delta^*\mathcal{L}$, and from [Wal17, Prop. 3.1.1] that this section is a group homomorphism and neutral with respect to fusion. Using this neutrality together with the multiplicativity of $\tilde{\mu}$ we check that

$$\tilde{\mu}(q_{12}, q_{23}) = \tilde{\mu}(q_{12}, \chi(\beta_2))\tilde{\mu}(\chi(\beta_2)^{-1}, \chi(\beta_2)^{-1})\tilde{\mu}(\chi(\beta_2), q_{23}) = q_{12}\chi(\beta_2)^{-1}q_{23} = \tilde{\mu}^{\chi}(q_{12}, q_{23}). \quad \square$$

We remark that the connection ν of (b) of a fusion extension induces a splitting σ_ν on the level of Lie algebras; namely, the one whose image is the horizontal subspace at the unit element. The splitting gives rise to a 2-cocycle $\omega_{\sigma_\nu} : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ defined from σ_ν just as in (3.3). The section σ_ν, in turn, induces another connection $\nu' = \nu_{\sigma_\nu}$, analogously as described in Remark 3.5.11. The new connection ν' does in general *not* coincide with the original connection ν, and it will be important to distinguish both. For example, the connection ν' is in general not superficial as required in (b). In a quite general context, it is possible to determine the 2-cocycle ω_{σ_ν} as well as the difference between the two connections, see [Wal15, Lemmas 2.2.2 and 2.2.3].

Lemma 7.1.2. *Let $\mathcal{G}$ be a multiplicative bundle gerbe over a Lie group G, whose curvature (H, ρ) is of the form $H = \frac{1}{24\pi}\langle \theta \wedge [\theta \wedge \theta]\rangle$ and $\rho = \frac{1}{4\pi}\left\langle \mathrm{pr}_1^* \theta \wedge \mathrm{pr}_2^* \bar{\theta}\right\rangle$, for some invariant bilinear form $\langle -, - \rangle$ on the Lie algebra $\mathfrak{g}$. Let $\mathscr{T}_\mathcal{G}$ be the transgressed central extension, and let ν be the connection on $\mathscr{T}_\mathcal{G}$ that appears under (b). Then, the following holds:*

(a) The 2-cocycle determined by the section σ_ν is

$$\omega_{\sigma_\nu}(X, Y) = \frac{1}{2\pi i}\int_0^{2\pi} \langle X(t), Y'(t)\rangle \mathrm{d}t$$

for $X, Y \in L\mathfrak{g}$.

(b) The connection ν' determined by the section σ_ν differs from the connection ν by a canonical 1-form $\beta \in \Omega^1(LG)$; more precisely, we have $\nu' = \nu + q^*\beta$ with

$$\beta_\tau(X) = \frac{1}{4\pi i}\int_0^{2\pi} \left\langle \tau(t)^{-1}\partial_t \tau(t), \tau(t)^{-1}X(t)\right\rangle \mathrm{d}t$$

for $\tau \in LG$ and $X \in T_\tau LG$.

In the next subsection, we will apply Lemma 7.1.2 to the case where $G = \mathrm{Spin}(d)$ and $\langle -, - \rangle$ is the basic inner product. Then we have $\mathcal{G} = \mathcal{G}_{bas}$, and Lemma 7.1.2 (a) implies (see Theorem 3.7.6) that $\mathscr{T}_{\mathcal{G}_{bas}}$ is the basic central extension.

7.2. Transgression and implementers

One of our goals is to provide operator-algebraic constructions of the loop group perspective to string geometry. In Section 3.7 we have constructed an operator-algebraic model for the basic central extension $\widetilde{L\,\mathrm{Spin}}(d)$ of the loop group $L\,\mathrm{Spin}(d)$, together with a local section σ, inducing a connection ν_σ. In Section 6.3 we have defined a connection-preserving, multiplicative fusion product $\tilde{\mu}^\rho$ on $\widetilde{L\,\mathrm{Spin}}(d)$. In the following we compare that structure with the central extension $\mathscr{T}_{\mathcal{G}_{bas}}$ obtained by transgression from the basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(d)$, as described in Section 7.1. We recall that $\mathscr{T}_{\mathcal{G}_{bas}}$ comes equipped with a fusion product $\tilde{\mu}$ and a connection ν, see (a) and (b) above.

Because both central extensions are the basic one (Theorem 3.7.6 and Lemma 7.1.2), there exists an isomorphism $\widetilde{L\,\mathrm{Spin}}(d) \cong \mathscr{T}_{\mathcal{G}_{bas}}$ of central extensions of $L\,\mathrm{Spin}(d)$. Each central extension comes equipped with a section of the associated Lie algebra extension: the section σ_* of $\widetilde{L\,\mathrm{Spin}}(d)$ is induced by the local section σ of Section 3.5, and the section σ_ν of $\mathscr{T}_{\mathcal{G}_{bas}}$ is induced by the connection ν.

Lemma 7.2.1. *There exists a unique isomorphism $\varphi : \widetilde{L\,\mathrm{Spin}}(d) \to \mathscr{T}_{\mathcal{G}_{bas}}$ of central extensions that exchanges the two Lie algebra sections, i.e. $\sigma_\nu = \varphi_* \circ \sigma_*$. Moreover, φ is connection-preserving* *for the induced connections ν' on $\mathscr{T}_{\mathcal{G}_{bas}}$ and ν_σ on $\widetilde{L\,\mathrm{Spin}}(d)$, and it takes the fusion product $\tilde{\mu}$ on $\mathscr{T}_{\mathcal{G}_{bas}}$ to the fusion product $\tilde{\mu}^\rho$ on $\widetilde{L\,\mathrm{Spin}}(d)$.*

Proof. Uniqueness is clear. For existence, we choose any isomorphism φ, and observe that $\sigma_\nu = (\varphi_* \circ \sigma_*) + f$, for a bounded linear map $f : L\mathfrak{spin}(d) \to \mathbb{R}$. We infer that the 2-cocycles associated to both sections, σ_* and σ_ν, coincide: they both give the basic 2-cocycle, see Theorem 3.7.6 and Lemma 7.1.2. Thus, using the formula (3.3) for the 2-cocycle, we see that f vanishes on all commutators, in other words, it is a Lie algebra homomorphism. We would like to integrate it to a Lie group homomorphism $F : L\,\mathrm{Spin}(n) \to \mathrm{U}(1)$. To this end, we note that $L\,\mathrm{Spin}(n)$ is 1-connected and $\mathrm{U}(1)$ is regular, and that both are Lie groups modelled on a locally convex topological vector space. The integration is hence possible due to a theorem of Milnor [Mil83, Theorem 8.1], also see [Nee06, Theorem III.1.5]. Now, the isomorphism $\varphi' := \varphi \cdot F$ will have the claimed property.

Indeed, since φ exchanges the sections σ_ν and σ_*, it follows immediately that it is connection-preserving for the induced connections $\nu' = \nu_{\sigma_\nu}$ and ν_σ, respectively. The fusion products on both sides can be characterized by fusion factorizations that are flat with respect to the connections ν' and ν_σ (see Lemma 6.2.5 and Proposition 7.1.1). Using the fact that φ is connection-preserving, $\varphi \circ \rho$ is another flat fusion factorization of $\mathscr{T}_{\mathcal{G}_{bas}}$. Two flat sections differ by a locally constant smooth map $PG \to \mathrm{U}(1)$, and since PG is connected and both sections map the constant path 1 to $1 \in \mathscr{T}_{\mathcal{G}_{bas}}$, this map is constant and equal to $1 \in \mathrm{U}(1)$. Thus, φ preserves the fusion factorizations, and hence the corresponding fusion products. □

We may now shift the connection ν_σ on $\widetilde{L\,\mathrm{Spin}}(d)$ by the 1-form β of Lemma 7.1.2 (b), and obtain a new connection $\tilde{\nu} := \nu_\sigma - q^*\beta$. The isomorphism φ is then connection-preserving for the connections $\tilde{\nu}$ and ν on $\mathscr{T}_{\mathcal{G}_{bas}}$. In particular, this implies that $\tilde{\nu}$ is superficial and symmetrizing, and that we may use the path splitting κ in (c) of $\mathscr{T}_{\mathcal{G}_{bas}}$ for the connection $\tilde{\nu}$. Now, we have equipped our operator-algebraic construction of $\widetilde{L\,\mathrm{Spin}}(d)$ with all of the structure (a) to (c). Summarizing, we have the following result.

Theorem 7.2.2. *Our operator-algebraic model $\widetilde{L\,\mathrm{Spin}}(d)$ of the basic central extension of $L\,\mathrm{Spin}(d)$ equipped with the fusion product $\tilde{\mu}^\rho$, the connection $\tilde{\nu} = \nu_\sigma + q^*\beta$, and the path splitting κ is a fusion extension with connection, and it is canonically isomorphic to the fusion extension $\mathscr{T}_{\mathcal{G}_{bas}}$ obtained by transgression of the basic gerbe over $\mathrm{Spin}(d)$, as objects of the category $\mathcal{F}us\mathcal{E}xt^\nabla(L\,\mathrm{Spin}(d))$.*

By [Wal17, Theorem 5.3.1] every fusion extension of $L\,\mathrm{Spin}(d)$ with connection corresponds to a diffeological multiplicative bundle gerbe with connection over $\mathrm{Spin}(d)$, via a procedure called *regression*. The underlying regressed bundle gerbe is described in [Wal16b, Section 5.1]. It has the subduction (the diffeological analog of a surjective submersion) $ev_1 : P_1\,\mathrm{Spin}(d) \to \mathrm{Spin}(d)$, where $ev_t : P\,\mathrm{Spin}(d) \to \mathrm{Spin}(d)$ is the evaluation at t, and $P_1\,\mathrm{Spin}(d) := ev_0^{-1}(1)$ is the subspace consisting of paths starting at the identity. On the 2-fold fibre product we have a smooth map $P_1\,\mathrm{Spin}(d)^{[2]} \subset P\,\mathrm{Spin}(d)^{[2]} \to L\,\mathrm{Spin}(d)$, along which we pull back the central extension $\widetilde{L\,\mathrm{Spin}}(d)$, considered as a principal $\mathrm{U}(1)$-bundle. Under the pullback, the fusion product $\tilde{\mu}^\rho$ becomes precisely a bundle gerbe product. The connection $\tilde{\nu}$ gives one part of the connection on the regressed bundle gerbe. The construction of a corresponding curving is more involved; it uses that $\tilde{\nu}$ is superficial, see [Wal16b, Section 5.2].

The regressed multiplicative structure is strict; it is composed of the fact that $P_1\,\mathrm{Spin}(d)$ and $\widetilde{L\,\mathrm{Spin}}(d)$ are diffeological groups, and that the fusion product $\tilde{\mu}^\rho$ is multiplicative. This was mentioned in [Wal12a, Section 5] and is explained in more detail in [Wal17, Section 5.3]. By

Theorem 7.2.2 and the fact that regression is inverse to transgression ([Wal17, Theorem 5.3.1]), the above construction results in a diffeological, operator-algebraical construction of the basic gerbe over $\mathrm{Spin}(d)$, with the correct connection and multiplicative structure.

7.3. String structures and fusive spin structures

In this section we recall some results on the spin geometry on the loop space of a string manifold. We start by recalling the notion of a spin structure on the loop space LM. Next, we recall when such a spin structure is *fusive*. After this, we recall the definition of a string structure on a spin manifold, and indicate how a string structure on a spin manifold M induces a fusive spin structure on LM. In the remainder of this work, it is only the notion of (fusive) spin structure on LM that is used, and not the richer notion of a string structure on M. However, the notion of string structure is interesting in its own right, and moreover it is expected to play an important role in the further development of the theory started here. This is why we give a cursory review of the notion here.

Let M be a spin manifold, and let $\mathrm{Spin}(M)$ be its spin frame bundle. The Fréchet manifold $L\,\mathrm{Spin}(M)$ is then a Fréchet principal $L\,\mathrm{Spin}(d)$-bundle over LM, (see [SW07, Proposition 1.8]). We now have the following definition due to Killingback, [Kil87].

Definition 7.3.1. A *spin structure on loop space* is a principal $\widetilde{L\,\mathrm{Spin}}(d)$-bundle lifting $L\,\mathrm{Spin}(M)$.

A fundamental result due to Killingback is that if the first fractional Pontryagin class $p_1(M)/2 \in H^4(M,\mathbb{Z})$ vanishes, then LM admits a spin structure. In Section 8.2 we shall give a construction of a spinor bundle on loop space that takes a spin structure on loop space as input.

Let $\widetilde{L\,\mathrm{Spin}}(M)$ be a spin structure on LM, then we can view $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ as a principal $\mathrm{U}(1)$-bundle. We recall the notion of a fusive spin structure on LM from [Wal16a, Definition 3.6], which fits in the general pattern of our treatment of principal $\mathrm{U}(1)$-bundles over loop spaces. We shall build on Definition 6.1.1.

Definition 7.3.2. A spin structure $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ on LM is called *fusive* if it comes equipped with a fusion product λ, which is equivariant with respect to the fusion product $\tilde{\mu}^{\rho}$ on $\widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$. Explicitly

$$\lambda(\Phi_{12} \cdot g_{12} \otimes \Phi_{23} \cdot g_{23}) = \lambda(\Phi_{12} \otimes \Phi_{23}) \cdot \tilde{\mu}^{\rho}(g_{12} \otimes g_{23}), \tag{7.3}$$

for all $\Phi_{12} \otimes \Phi_{23} \in \pi_1^*\widetilde{L\,\mathrm{Spin}}(M) \otimes \pi_2^*\widetilde{L\,\mathrm{Spin}}(M)$ and all $g_{12} \otimes g_{23} \in \pi_1^*\widetilde{L\,\mathrm{Spin}}(d) \otimes \pi_2^*\widetilde{L\,\mathrm{Spin}}(d)$.

As mentioned above, the notion of fusive spin structure on LM will be instrumental in our construction of a fusion product on the spinor bundle on loop space in Section 8.3.

In the remainder of this section we recall the notion of string structure on M, and recall how a string structure on M induces a fusive spin structure on LM. None of what follows is required for the remainder of this work, however, it is an important piece of context, and will undoubtedly be important for the further development of the theory.

The following definition of a string structure, taken from [Wal13, Definition 1.1.5] involves some notions that will be explained later in this section.

Definition 7.3.3. A string structure on M is a trivialization of the Chern-Simons 2-gerbe $\mathbb{CS}(M)$.

The Chern-Simons 2-gerbe was defined in [CJM+05], we shall have more to say about it momentarily. Let $\underline{\mathrm{String}}(d)$ be any topological 2-group model for the string group. The following theorem, which collects some well-known results from the literature on string structures, then gives alternative ways to think about string structures.

Theorem 7.3.4. *There is a commutative diagram*

$$\begin{array}{ccc} \left\{\begin{array}{c}\textit{Equivalence classes of}\\ \textit{string structures on } M\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\textit{Isomorphism classes of lifts of}\\ \mathrm{Spin}(M) \textit{ to a } \underline{\mathrm{String}}(d)\textit{-2-bundle}\end{array}\right\} \\ & \searrow \qquad \swarrow & \\ & \left\{\begin{array}{c}\textit{Homotopy classes of lifts of}\\ M \to B\,\mathrm{Spin}(d) \textit{ to } M \to B\,\mathrm{String}(d)\end{array}\right\} & \end{array}$$

where all the arrows are isomorphisms. Moreover, these sets are non-empty if and only if the first fractional Pontryagin class $p_1(M)/2 \in H^4(M,\mathbb{Z})$ is zero.

The top arrow in the diagram is constructed in [NW13b], see in particular Theorem 6.9. The arrow on the right hand side of the diagram is constructed in [NW13a], see in particular Theorem 4.6. The fact that the first fractional Pontryagin class is the obstruction to the existence of string structures is Theorem 1.1.3 in [Wal13].

As mentioned above, the vanishing of the first fractional Pontryagin class implies that the loop space admits a spin structure. In particular this means that if a manifold admits a string structure, then its loop space admits a spin structure. However, true to our slogan "A geometric structure on a space M induces a geometric structure of one degree lower on its loop space LM, which comes equipped with a fusion product", this is not all. In fact, a string structure on a manifold can be transgressed to a *fusive spin structure* on its loop space. In a series of papers, culminating in [Wal15], Waldorf described how a string structure on a manifold induces a fusive spin structure on its loop space. In this construction, one must use the model $\mathscr{T}_{\mathcal{G}_{bas}}$ described in Section 7.1 for the fusion extension of $\mathrm{Spin}(d)$. Theorem 7.2.2 allows us to use our operator algebraic model $\widetilde{L\,\mathrm{Spin}}(d)$ equipped with the fusion product $\tilde{\mu}^\rho$ described in Section 6.3 instead.

We finish this section with a sketch of the map that takes a string structure on M to a fusive spin structure on LM. Recall that a bundle gerbe over M (see Appendix D) consists of a triple $(\mathcal{L}, Y, m)$, where Y is equipped with a surjective submersion $\pi : Y \to M$, and $\mathcal{L} \to Y^{[2]}$ is a principal $\mathrm{U}(1)$-bundle, equipped with a multiplication m. Now, bundle gerbes are supposed to be higher versions of $\mathrm{U}(1)$-bundles, which suggests that one may try to replace the line bundle $\mathcal{L}$ with a bundle gerbe, say $\mathcal{P}$. We then also replace the multiplication m by an isomorphism

$$\mathcal{M} : \pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P} \to \pi_{13}^*\mathcal{P}.$$

Unlike in Definition D.1 we demand that this isomorphim is associative only up to a some fixed morphism, called the *associator*, which is required to satisfy a certain pentagon identity. The object that we end up with is called a bundle 2-gerbe, see [Wal13, Definition 2.1.1] for details. A trivialization of this bundle 2-gerbe is then a bundle gerbe $\mathcal{S}$ over Y, equipped with an isomorphism $\mathcal{P} \otimes \pi_2^*\mathcal{S} \to \pi_1^*\mathcal{S}$ subject to some compatibility conditions, see [Wal13, Definition 2.2.1]. The Chern-Simons 2-gerbe is then schematically given by the following diagram

$$\begin{array}{ccc} & & \Delta^*\mathcal{G}_{bas} \\ & & \downarrow \\ \mathrm{Spin}(M) & \Longleftarrow & \mathrm{Spin}(M)^{[2]} \\ \downarrow & & \\ M & & \end{array}$$

where $\Delta^*\mathcal{G}_{bas}$ is the pullback of the basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(d)$ along the map $\Delta : \mathrm{Spin}(M)^{[2]} \to \mathrm{Spin}(d)$, (c.f. Definition D.3). The multiplication $\mathcal{M}$ and its associator are defined using the multiplicative structure of the bundle gerbe $\mathcal{G}_{bas}$. It is proven in [Wal16a, Section 5.2] that the Chern-Simons 2-gerbe transgresses to the lifting gerbe

$$\begin{array}{ccc} & & \Delta^*\mathscr{T}_{\mathcal{G}_{bas}} \\ & & \downarrow \\ L\,\mathrm{Spin}(M) & \leftleftarrows & L\,\mathrm{Spin}(M)^{[2]} \\ \downarrow & & \\ LM & & \end{array}$$

in a way compatible with the fusion structures. Now let $\mathcal{S}$ be a string structure, i.e. a bundle gerbe over $\mathrm{Spin}(M)$ which trivializes the Chern-Simons 2-gerbe, is given. It is proven in [Wal15] that $\mathcal{S}$ transgresses to a fusion trivialization of the above lifting gerbe, and hence yields a fusive spin structure on LM.

8. The spinor bundle on loop space and its fusion product

In this chapter we tie everything together, and complete the main construction of this work. That is, we construct the spinor bundle on loop space, equip it with the action of a Clifford algebra bundle on loop space, and finally construct a family of fusion isomorphisms over compatible loops.

The spinor bundle on loop space will be a vector bundle of infinite rank, this means that it does not fit in the usual framework for smooth finite rank vector bundles. In Section 8.1 we define the notions of rigged Hilbert space bundles and of rigged C*-algebra bundles, which form the appropriate framework for the spinor bundle on loop space.

In Section 8.2 we define the Clifford algebra bundle and the spinor bundle on loop space as associated bundles, and we equip the spinor bundle on loop space with a smooth action of the Clifford bundle on loop space.

In Section 8.3 we construct the family of fusion isomorphisms, completing a conjecture put forth by Stolz and Teichner in [ST, Theorem 1].

8.1. Bundles of rigged Hilbert spaces and bundles of C*-algebras

We first define the notion of rigged Hilbert space and rigged C*-algebra. The use of rigged Hilbert spaces originates from quantum mechanics, where they were introduced to provide Dirac's bra-ket formalism with a rigorous mathematical basis, see [Rob66, Ant69].

The reason that rigged Hilbert spaces appear here is as follows. The spinor bundle on loop space should be a *smooth* vector bundle of some sort. Our intent is to define the spinor bundle on loop space as a bundle associated to $\widetilde{L\operatorname{Spin}}(M)$. Because the map $\widetilde{L\operatorname{Spin}}(d) \times \mathcal{F}_L \to \mathcal{F}_L$ is not smooth, the space $\mathcal{F}_L$ is not eligible to this method of defining smooth vector bundles. However, in Section 3.5 we constructed a Fréchet space $\mathcal{F}_L^{\mathrm{s}}$, which is a dense subspace of $\mathcal{F}_L$ with the property that the map $\widetilde{L\operatorname{Spin}}(d) \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_L^{\mathrm{s}}$ is smooth. This means that $\mathcal{F}_L^{\mathrm{s}}$ is eligible to this method of defining smooth vector bundles, which is what we shall do. It turns out to be important to keep track of the relation between $\mathcal{F}_L^{\mathrm{s}}$ and $\mathcal{F}_L$. This is exactly what the notion of a rigged Hilbert space does for us.

Definition 8.1.1. A *rigged Hilbert space* is a Fréchet space equipped with a continuous inner product.

Remark 8.1.2. This definition is equivalent to saying that a rigged Hilbert space consists of a pair (F, H), where H is a Hilbert space, and F is a Fréchet space equipped with a continuous linear injection $\iota : F \to H$ such that $\iota(F)$ is dense in H. We use Definition 8.1.1 because it is better suited to our definition of rigged Hilbert space bundle later on.

Example 8.1.3. The Schwartz space of the real line, $\mathcal{S}(\mathbb{R})$, is a Fréchet space, which is dense in

$L^2(\mathbb{R})$. The restriction of the L^2-inner product is continuous on $\mathcal{S}(\mathbb{R})$, and hence turns $\mathcal{S}(\mathbb{R})$ into a rigged Hilbert space.

Example 8.1.4. Because the inclusion $\mathcal{F}_L^{\mathrm{s}} \to \mathcal{F}_L$ is continuous, the restriction of the inner product on $\mathcal{F}_L$ to $\mathcal{F}_L^{\mathrm{s}}$ is continuous, hence $\mathcal{F}_L^{\mathrm{s}}$ is a rigged Hilbert space.

Definition 8.1.5. A *rigged C*-algebra* is a Fréchet algebra A, equipped with a continuous norm $\|\cdot\| : A \to \mathbb{R}_{\geqslant 0}$ and a continuous involution $* : A \to A$ such that the completion with respect to the norm is a C*-algebra.

Example 8.1.6. We equip $C^\infty(S^1, \mathbb{C})$ with its usual Fréchet structure. Then, equipped with the supremum norm and with pointwise complex conjugation, $C^\infty(S^1, \mathbb{C})$ becomes a rigged C*-algebra.

Example 8.1.7. The Fréchet algebra $\mathrm{Cl}(V)^{\mathrm{s}}$ equipped with the norm from $\mathrm{Cl}(V)$ is a rigged C*-algebra.

Let $\mathcal{M}$ be a Fréchet manifold. We shall consider several different notions of bundles over $\mathcal{M}$. We start with the definition of Fréchet principal bundle from [Ham82, Definition 4.6.5]. Let $\mathcal{G}$ be a Fréchet Lie group.

Definition 8.1.8. A *Fréchet principal $\mathcal{G}$-bundle* over $\mathcal{M}$ is a Fréchet manifold P, equipped with the following structure:

- A smooth surjection $\pi : P \to \mathcal{M}$.
- A fibre-preserving right action of $\mathcal{G}$ on P.

Such that for each $x \in \mathcal{M}$ there exists an open neighbourhood U of x, and a fibre-preserving diffeomorphism $\Phi : P|_U \to U \times \mathcal{G}$, which intertwines the right action of $\mathcal{G}$ on P with right multiplication.

Let E be a Fréchet space. We define the notion of Fréchet vector bundle, following [Ham82, Definition 4.3.1].

Definition 8.1.9. A *Fréchet vector bundle* over $\mathcal{M}$ with fibre E is a Fréchet manifold $\mathcal{E}$ equipped with the following structure:

- A smooth surjection $\pi : \mathcal{E} \to \mathcal{M}$.
- For each $x \in \mathcal{M}$, the fibre $\mathcal{E}_x := \pi^{-1}(x)$ has the structure of a vector space.

Such that for each $x \in \mathcal{M}$ there exists an open neighbourhood $U \subset \mathcal{M}$ containing x, and a diffeomorphism $\Phi_U : \mathcal{E}|_U := \pi^{-1}(U) \to U \times E$ which is fibre-preserving and fibrewise linear.

As usual, the maps Φ_U are called *local trivializations* of $\mathcal{E}$.

Remark 8.1.10. Using the local trivializations Φ_U the fibres $\mathcal{E}_x$ can be equipped with a unique structure of Fréchet space such that $\Phi_U|_x : \mathcal{E}_x \to E$ is a homeomorphism.

The tangent space of a Fréchet manifold is a Fréchet vector bundle over that manifold, see [Ham82, Example 4.3.2].

Next, we prove that the operation of taking associated bundles behaves well with respect to Fréchet manifolds. Let $\pi : P \to \mathcal{M}$ be a Fréchet principal $\mathcal{G}$-bundle, and suppose that $\mathcal{G}$ acts on a Fréchet space E, and that the map $\mathcal{G} \times E \to E$ is smooth. We consider the space $(P \times E)/\mathcal{G}$, equipped with the projection map to $\mathcal{M}$ inherited from P. For each $x \in \mathcal{M}$ we equip $((P \times E)/\mathcal{G})_x$ with the structure of vector space as follows. The map

$$\begin{aligned} ((P \times E)/\mathcal{G})_x \times ((P \times E)/\mathcal{G})_x &\to ((P \times E)/\mathcal{G})_x, \\ ([p, v], [p, w]) &\mapsto [p, v + w], \end{aligned}$$

defines an addition. Scalar multiplication is defined as $(\lambda, [p, v]) \mapsto [p, \lambda v]$ for all $\lambda \in \mathbb{C}$.

Lemma 8.1.11. *The space $(P \times E)/\mathcal{G}$ has a unique structure of Fréchet manifold, such that the map $P \times E \to (P \times E)/\mathcal{G}$ is a surjective submersion. When the fibres are equipped with the vector space structure as above, the Fréchet manifold $(P \times E)/\mathcal{G}$ is a Fréchet vector bundle over $\mathcal{M}$.*

Proof. There is at most one structure of Fréchet manifold on $(P \times E)/\mathcal{G}$ such that the projection $P \times E \to (P \times E)/\mathcal{G}$ is a surjective submersion. Hence, all we need to do is show that there exists one such structure. Let us write F for the Fréchet space on which $\mathcal{M}$ is modelled. Let $\{U_\alpha, \psi_\alpha\}_{\alpha \in I}$ be an atlas of $\mathcal{M}$ that trivializes P. Let $\alpha \in I$ be arbitrary, let $\varphi_\alpha : P|_{U_\alpha} \to U_\alpha \times \mathcal{G}$ be the corresponding trivialization. Let us write $g_\alpha : P|_{U_\alpha} \to \mathcal{G}$ for φ_α followed by projection onto $\mathcal{G}$. We have

$$((P \times E)/\mathcal{G}))\,|_{U_\alpha} = \{[p, v] \mid p \in P|_{U_\alpha}, v \in E\}.$$

Define the chart

$$\Psi_\alpha : ((P \times E)/\mathcal{G}))\,|_{U_\alpha} \to F \times E, \qquad [p, v] \mapsto (\psi_\alpha(\pi(p)),\, g_\alpha(p)v).$$

Using the right $\mathcal{G}$ equivariance of the map g_α, it is easy to check that this map is well-defined and linear. Its inverse is given by

$$(f, v) \mapsto [\varphi_\alpha^{-1}(\psi_\alpha^{-1}(f), e), v].$$

Now suppose that $\beta \in I$. Let us check that

$$\Psi_\beta \Psi_\alpha^{-1} : F \times E \to F \times E$$

is smooth. We compute

$$\Psi_\beta \Psi_\alpha^{-1}(f, v) = (\psi_\beta(\pi(p)), g_\beta(p)v),$$

where

$$p = \varphi_\alpha^{-1}(\psi_\alpha^{-1}(f), e).$$

Using the fact that $\varphi_\beta \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathcal{G} \to (U_\alpha \cap U_\beta) \times \mathcal{G}$ acts as the identity on the first component we see that

$$\psi_\beta(\pi(\varphi_\alpha^{-1}(\psi_\alpha^{-1}(f), e))) = \psi_\beta \psi_\alpha^{-1}(f).$$

Next, we consider

$$g_\beta \varphi_\alpha^{-1}(\psi_\alpha^{-1}(f), e)v.$$

The map $F \to \mathcal{G}, f \mapsto g_\beta \varphi_\alpha^{-1}(\psi_\alpha^{-1}(f), e)$ is smooth because P is a principal $\mathcal{G}$ bundle. The assumption that the map $\mathcal{G} \times E \to E$ is smooth implies that the map $(f, v) \mapsto g_\beta \varphi_\alpha^{-1}(\psi_\alpha^{-1}(f), e)v$ is smooth. Hence, the map $\Psi_\beta \Psi_\alpha^{-1} : F \times E \to F \times E$ is the Cartesian product of two smooth maps, and hence smooth. It follows that the system $\{((P \times E)/\mathcal{G}))\,|_{U_\alpha}, \Psi_\alpha\}_{\alpha \in I}$ is a smooth atlas for $(P \times E)/\mathcal{G}$. It is obvious that the projection map $(P \times E)/\mathcal{G} \to \mathcal{M}$ is smooth.

Moreover, the maps $((P \times E)/\mathcal{G})|_{U_\alpha} \to U_\alpha \times E, [p, v] \mapsto (\pi(p), g_\alpha(p)v)$ are local trivializations, which are fibrewise linear.

Let us show that the projection map $P \times E \to (P \times E)/\mathcal{G}$ is a surjective submersion. Surjectivity is clear. It is of course sufficient to check submersivity locally. One computes that the composite map

$$F \times \mathcal{G} \times E \xrightarrow{(\varphi_\alpha^{-1} \circ (\psi_\alpha^{-1} \times \mathbb{1})) \times \mathbb{1}} P|_{U_\alpha} \times E \to ((P \times E)/\mathcal{G})|_{U_\alpha} \xrightarrow{\Psi_\alpha} F \times E$$

is simply projection onto the first and third component, and hence a submersion. ☐

Now suppose that E is a rigged Hilbert space.

Definition 8.1.12. A *rigged Hilbert space bundle* $\mathcal{E}$ over $\mathcal{M}$ with fibre E is a Fréchet vector bundle $\mathcal{E} \to M$ equipped with a map $g : \mathcal{E} \times_\pi \mathcal{E} \to \mathbb{C}$, such that the following conditions hold for each $x \in \mathcal{M}$:

- The map $g_x : \mathcal{E}_x \times \mathcal{E}_x \to \mathbb{C}$ is an inner product.
- There exists an open neighbourhood $U \subset \mathcal{M}$ of x and a diffeomorphism $\Phi_U : \mathcal{E}|_U \to U \times E$, which is fibre-preserving, fibrewise linear, and unitary.

The map g is called the *Hermitian metric* of $\mathcal{E}$.

Remark 8.1.13. Let $\mathcal{E} \to \mathcal{M}$ be a rigged Hilbert space bundle over $\mathcal{M}$ with Hermitian metric g. Let $\Phi_U : \mathcal{E}|_U \to U \times E$ be a fibrewise linear and unitary local trivialization. The diagram

$$\begin{array}{ccc} \mathcal{E}|_U \times_\pi \mathcal{E}|_U & \longrightarrow & \mathbb{C} \\ {\scriptstyle \Phi_U \times_\pi \Phi_U}\downarrow & \nearrow & \\ U \times E \times E & & \end{array}$$

then commutes, which proves that g is smooth, and in particular continuous. This means that each fibre $\mathcal{E}|_x$ is a rigged Hilbert space, and each local trivialization is fibrewise an isomorphism of rigged Hilbert spaces.

If the fibre E is finite-dimensional, then a rigged Hilbert space bundle $\mathcal{E} \to \mathcal{M}$ with fibre E is a Hermitian vector bundle.

If the base manifold $\mathcal{M}$ is a point, then a rigged Hilbert space bundle over $\mathcal{M}$ is a rigged Hilbert space.

Let $P \to \mathcal{M}$ be a Fréchet principal $\mathcal{G}$-bundle, and suppose that $\mathcal{G}$ acts on a Fréchet space E, and that the map $\mathcal{G} \times E \to E$ is smooth. Moreover, suppose that E is a rigged Hilbert space and that $\mathcal{G}$ acts by unitary transformations. We define a Hermitian metric g on $(P \times E)/\mathcal{G}$ as follows. Let

$$(x, y) \in (P \times E)/\mathcal{G} \times_\pi (P \times E)/\mathcal{G}.$$

Then there exist a $p \in P$ and $v, w \in E$ such that $x = [p, v]$ and $y = [p, w]$. We set $g(x, y) = \langle v, w \rangle_E$. Using the fact that $\mathcal{G}$ acts by unitary transformations it becomes easy to check that the map g is well-defined and that the local trivializations Ψ_α are fibrewise unitary. Hence, we conclude that the following lemma holds.

Lemma 8.1.14. *Equipped with the Hermitian metric g, the Fréchet vector bundle $(P \times E)/\mathcal{G}$ is a rigged Hilbert space bundle.*

Example 8.1.15. Let $\mathcal{Q}^* \to \mathrm{Lag}_L(V)$ be the principal U(1)-bundle defined in Section 3.6. Then, $E^{\mathrm{s}} := \mathcal{Q}^* \times_{\mathrm{U}(1)} \mathcal{F}_L^{\mathrm{s}}$ is a rigged Hilbert space bundle over $\mathrm{Lag}_L(V)$.

Let A be a rigged C*-algebra.

Definition 8.1.16. A *rigged C*-algebra bundle* over $\mathcal{M}$ with fibre A is a Fréchet vector bundle $\pi : \mathcal{E} \to \mathcal{M}$, equipped with the following structure:

- A map $\| \cdot \| : \mathcal{E} \to \mathbb{R}_{\geqslant 0}$.
- A fibre-preserving map $m : \mathcal{E} \times_\pi \mathcal{E} \to \mathcal{E}$.
- A fibre-preserving map $* : \mathcal{E} \to \mathcal{E}$.

such that the following conditions hold for each $x \in \mathcal{M}$

- The map $\|\cdot\|_x : \mathcal{E}_x \to \mathbb{R}_{\geqslant 0}$ is a norm.
- The map $m_x : \mathcal{E}_x \times \mathcal{E}_x \to \mathcal{E}_x$ turns the vector space $\mathcal{E}_x$ into an algebra.
- The map $*_x : \mathcal{E}_x \to \mathcal{E}_x$ is an isometry with respect to $\|\cdot\|_x$, and turns the algebra $\mathcal{E}_x$ into a $*$-algebra.
- The norm $\|\cdot\|_x$ is submultiplicative.
- The norm $\|\cdot\|_x$ satisfies the C*-identity.
- There exists an open neighbourhood $U \subset \mathcal{M}$ of x and a diffeomorphism $\Phi_U : \mathcal{E}|_U \to U \times A$ which is fibrewise linear, isometric, and a homomorphism of $*$-algebras.

Remark 8.1.17. Let $\mathcal{E} \to \mathcal{M}$ be a rigged C*-algebra bundle with fibre A. A standard argument proves that the maps $*$ and m are smooth and that $\|\cdot\|$ is smooth away from the image of the zero section, c.f. Remark 8.1.13. A routine verification shows that, equipped with the maps $\|\cdot\|_x$, m_x and $*_x$, the vector space $\mathcal{E}_x$ is a rigged C*-algebra isomorphic to A.

Let $P \to \mathcal{M}$ be a Fréchet principal $\mathcal{G}$-bundle, and suppose that $\mathcal{G}$ acts on a rigged C*-algebra A by isometric $*$-homomorphisms, and that the map $\mathcal{G} \times A \to A$ is smooth. We equip $\mathcal{E} := (P \times A)/\mathcal{G}$ with the structure of Fréchet vector bundle as in Lemma 8.1.11. The map $\|\cdot\| : P \times A \to \mathbb{R}_{\geqslant 0}, (p, v) \mapsto \|v\|$ then descends to the quotient. The map $m : \mathcal{E} \times_\pi \mathcal{E} \to \mathcal{E}$ is defined by the equation $([p, v_1], [p, v_2]) \mapsto [p, v_1 v_2]$. Finally, the map $* : P \times A \to P \times A, (p, v) \mapsto (p, v^*)$ descends to a map $* : \mathcal{E} \to \mathcal{E}$. It is a routine exercise to check that these maps satisfy the properties laid out in Definition 8.1.16. Moreover, it follows from a routine verification that the trivializations Ψ_α are fibrewise linear, isometric $*$-homomorphisms. We conclude that the following result holds.

Lemma 8.1.18. *The Fréchet vector bundle $(P \times A)/\mathcal{G}$ equipped with the maps $\|\cdot\|$, m, and $*$ is a rigged C*-algebra bundle over $\mathcal{M}$ with fibre A.*

8.2. The spinor bundle on loop space and its Clifford action

Let M be a spin manifold equipped with a spin structure $\widetilde{L\,\mathrm{Spin}}(M)$ on its loop space. We now have all the ingredients required to define the spinor bundle on loop space. In particular, we have a rigged Hilbert space $\mathcal{F}^{\mathrm{s}}$, equipped with a unitary action by $\widetilde{L\,\mathrm{Spin}}(d)$, with the property that the map $\widetilde{L\,\mathrm{Spin}}(d) \times \mathcal{F}^{\mathrm{s}} \to \mathcal{F}^{\mathrm{s}}$ is smooth, see Section 3.5.

Definition 8.2.1. The *spinor bundle on loop* space is the rigged Hilbert space bundle

$$\mathcal{F}^{\mathrm{s}}(LM) := \left(\widetilde{L\,\mathrm{Spin}}(M) \times \mathcal{F}^{\mathrm{s}}\right) / \widetilde{L\,\mathrm{Spin}}(d).$$

It then follows from Lemma 8.1.14 that the spinor bundle on loop space is indeed a rigged Hilbert space bundle over LM. This means in particular that each fibre $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ is equipped with an inner product. Let us write $\mathcal{F}(M)_\gamma$ for the completion of $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ with respect to this inner product.

Similarly, using the representation of $L\,\mathrm{Spin}(d)$ on $\mathrm{Cl}(V)^{\mathrm{s}}$, see Section 3.2, we define a bundle of Clifford algebras over the loop space of M:

Definition 8.2.2. The *Clifford bundle on loop space* is the rigged C*-algebra bundle

$$\mathrm{Cl}^{\mathrm{s}}(LM) := (L\,\mathrm{Spin}(M) \times \mathrm{Cl}(V)^{\mathrm{s}})/L\,\mathrm{Spin}(d).$$

That this is indeed a rigged C*-algebra bundle follows Lemma 8.1.18.

Remark 8.2.3. Instead of the spin frame bundle we could have used the oriented orthonormal frame bundle $\mathrm{SO}(M)$ to define the Clifford algebra bundle, i.e. we could equally well have defined $\mathrm{Cl}^{\mathrm{s}}(LM)$ to be $(L\,\mathrm{SO}(M) \times \mathrm{Cl}(V)^{\mathrm{s}})/L\,\mathrm{SO}(d)$. The advantage of this description is that it is clear that bundle of Clifford algebras does not require the manifold M to be spin. However, we prefer the definition we have given, because it makes the following sections lighter on notation.

We now define a bundle map $\mathrm{Cl}^{\mathrm{s}}(LM) \times_\pi \mathcal{F}^{\mathrm{s}}(LM) \to \mathcal{F}^{\mathrm{s}}(LM)$. Let $q : \widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ be the projection map.

Lemma 8.2.4. *Let $\gamma \in LM$ be arbitrary but fixed. Then the map*

$$\begin{aligned}\rho_\gamma : \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \times \mathcal{F}^{\mathrm{s}}(LM)_\gamma &\to \mathcal{F}^{\mathrm{s}}(LM)_\gamma, \\ ([q(\Phi), a], [\Phi, v]) &\mapsto [\Phi, a \triangleright v],\end{aligned}$$

is well-defined. Moreover, it defines a smooth bundle map $\rho : \mathrm{Cl}^{\mathrm{s}}(LM) \times_\pi \mathcal{F}^{\mathrm{s}}(LM) \to \mathcal{F}^{\mathrm{s}}(LM)$, with the property that the diagram

$$\begin{array}{ccc}\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \times \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \times \mathcal{F}^{\mathrm{s}}(LM)_\gamma & \xrightarrow{m_\gamma \times \mathbb{1}} & \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \times \mathcal{F}^{\mathrm{s}}(LM)_\gamma \\ \Big\downarrow {\scriptstyle \mathbb{1} \times \rho_\gamma} & & \Big\downarrow {\scriptstyle \rho_\gamma} \\ \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \times \mathcal{F}^{\mathrm{s}}(LM)_\gamma & \xrightarrow{\rho_\gamma} & \mathcal{F}^{\mathrm{s}}(LM)_\gamma\end{array} \tag{8.1}$$

commutes for all $\gamma \in LM$.

Proof. First, we check well-definedness. Let $(\varphi, a) \in L\,\mathrm{Spin}(d) \times \mathrm{Cl}(V)^{\mathrm{s}}$ represent an element of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ and let $(\Phi, \psi) \in \widetilde{L\,\mathrm{Spin}}(M) \times \mathcal{F}^{\mathrm{s}}$ represent an element of $\mathcal{F}^{\mathrm{s}}(LM)$. There is a unique element $g \in L\,\mathrm{Spin}(d)$ such that $q(\Phi) = \varphi \cdot g$, it follows that $[\varphi, a] = [q(\Phi), \theta_{g^{-1}}(a)]$. The map ρ_γ is then defined as follows

$$\begin{aligned}\rho_\gamma : \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \times \mathcal{F}^{\mathrm{s}}(LM)_\gamma &\to \mathcal{F}^{\mathrm{s}}(LM)_\gamma, \\ ([\varphi, a], [\Phi, \psi]) &\mapsto [\Phi, \theta_{g^{-1}}(a)\psi].\end{aligned}$$

To prove well-definedness, we suppose that $h \in L\,\mathrm{Spin}(d)$ and $U \in \widetilde{L\,\mathrm{Spin}}(d)$ are arbitrary. We must then show that

$$([\varphi \cdot h^{-1}, ha], [\Phi U^{-1}, U\psi]) \mapsto [\Phi, \theta_{g^{-1}}(a)\psi].$$

Suppose that U implements θ_T for $T \in \mathrm{O}_L(V)$, then we have

$$p(\Phi U^{-1}) = p(\Phi) \cdot T^{-1} = \varphi \cdot (gT^{-1}) = \varphi \cdot h^{-1} \cdot (hgT^{-1}).$$

It follows that

$$(\varphi \cdot h^{-1}, \theta_h(a)) \cdot (\Phi U^{-1}, U\psi) = (\Phi U^{-1}, \theta_{(Tg^{-1}h^{-1})}\theta_h(a)U\psi).$$

Finally, using the fact that θ is a group homomorphism, and using the fact that U implements θ_T, we obtain

$$(\varphi \cdot h^{-1}, \theta_h(a)) \cdot (\Phi U^{-1}, U\psi) = (\Phi U^{-1}, U\theta_{g^{-1}}(a)\psi),$$

which completes the proof. The fact that the resulting map is smooth follows from a routine but tedious computation in local charts and the fact that the map $\mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}^{\mathrm{s}} \to \mathcal{F}^{\mathrm{s}}$ is smooth, Lemma 3.5.7. The fact that Diagram (8.1) commutes follows from a straightforward computation. □

Lemma 8.2.5. *Let $\gamma \in LM$ be arbitrary. Then, the action of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ on $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ extends to an action of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ on the Hilbert completion, $\mathcal{F}(M)_\gamma$, of $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$. Moreover, the operator norm on $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ with respect to this representation is identical to the norm of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ that is part of the rigged C^*-algebra bundle structure of $\mathrm{Cl}^{\mathrm{s}}(LM)$.*

Proof. This follows from the fact that $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ is isomorphic to $\mathcal{F}^{\mathrm{s}}$ and that $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ is isomorphic to $\mathrm{Cl}(V)^{\mathrm{s}}$ and that these isomorphisms can be chosen to intertwine the action of $\mathrm{Cl}(V)^{\mathrm{s}}$ on $\mathcal{F}^{\mathrm{s}}$ with the action of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ on $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ and to intertwine the inner product on $\mathcal{F}^{\mathrm{s}}$ with the inner product on $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$. □

Next, we wish to equip each fibre of the bundle of Fock spaces $\mathcal{F}^{\mathrm{s}}(LM)$ with the structure of bimodule over appropriate algebras. Let $\gamma \in LM$ be arbitrary, and let $\varphi \in L\,\mathrm{Spin}(d)$. Then we define A_γ to be the subset of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$ given by

$$A_\gamma := \{[\varphi, a_- \otimes \mathbb{1}] \in \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \mid a_- \in \mathrm{Cl}(V_-)^{\mathrm{s}}\}.$$

A straightforward check shows that A_γ does not depend on the choice of φ and that A_γ is a $*$-subalgebra of $\mathrm{Cl}^{\mathrm{s}}(LM)_\gamma$.

According to Lemma 8.2.5 the $*$-algebra A_γ then acts on $\mathcal{F}^{\mathrm{s}}(LM)_\gamma$ as well. Let us write $\mathcal{A}_\gamma$ for the von Neumann completion with respect to this representation. We can represent elements of $\mathcal{A}_\gamma$ as equivalence classes $[\varphi, a \otimes \mathbb{1}]$, where $\varphi \in L\,\mathrm{Spin}(M)_\gamma$ and $a \in \mathrm{Cl}(V_-)''$.

Lemma 8.2.6. *For any $\varphi \in L\,\mathrm{Spin}(M)$ the map $A_\gamma \ni [\varphi, a_- \otimes \mathbb{1}] \mapsto a_- \in \mathrm{Cl}(V_-)^{\mathrm{s}}$ extends to an isomorphism of von Neumann algebras $\mathcal{A}_\gamma \to \mathrm{Cl}(V_-)''$.*

Proof. Let $\varphi \in L\,\mathrm{Spin}(M)_\gamma$ and let $\Phi \in \widetilde{L\,\mathrm{Spin}}(M)_\gamma$ lie over φ. Let us write $u : A_\gamma \to \mathrm{Cl}(V)^{\mathrm{s}}$ for the map $u([\varphi, a_- \otimes \mathbb{1}]) = a_-$. Then the map $U : \mathcal{F}^{\mathrm{s}} \to \mathcal{F}^{\mathrm{s}}(LM)_\gamma, v \mapsto [\Phi, v]$ extends to a unitary map $\nu : \mathcal{F}^{\mathrm{s}} \to \mathcal{F}^{\mathrm{s}}(LM)_\gamma$. Moreover, for all $x \in A_\gamma$ we have that $u(x) = \nu^* x \nu$, which completes the proof. □

Lemma 8.2.7. *Let γ' be a smooth loop with the property that $\gamma(t) = \gamma'(t)$ for $t \in [0, \pi]$. Then, there is a natural isomorphism $A_\gamma \to A_{\gamma'}$ of $*$-algebras, which extends to an isomorphism $\mathcal{A}_\gamma \to \mathcal{A}_{\gamma'}$ of von Neumann algebras.*

Proof. Choose elements $\varphi \in L\,\mathrm{Spin}(M)_\gamma$ and $\varphi' \in L\,\mathrm{Spin}(M)_{\gamma'}$ such that $\varphi(t) = \varphi'(t)$ for $t \in [0, \pi]$. Then, the map $A_\gamma \ni [\varphi, a_- \otimes \mathbb{1}] \mapsto [\varphi', a_- \otimes \mathbb{1}] \in A_{\gamma'}$ is an isomorphism of $*$-algebras that does not depend on the choice of φ and φ', which can be shown by a straightforward computation. The fact that this map extends to an isomorphism of von Neumann algebras follows from the fact that it factors as $A_\gamma \to \mathrm{Cl}(V_-) \to A_{\gamma'}$, where each of these arrows extends to an isomorphism of von Neumann algebras, see Lemma 8.2.6. □

Remark 8.2.8. Lemma 8.2.7 tells us that in fact the von Neumann algebra $\mathcal{A}_\gamma$ only depends on the restriction of γ to $[0, \pi]$. This allows us to introduce the following notation. If $\beta \in PM$, then we denote by $\mathcal{A}_\beta$ the von Neumann algebra $\mathcal{A}_\gamma$, where $\gamma \in LM$ is any smooth loop which restricts to β.

In a similar way we define $*$-algebras $B_\gamma = \{[\varphi, \mathbb{1} \otimes a_+] \in \mathrm{Cl}^{\mathrm{s}}(LM)_\gamma \mid a_+ \in \mathrm{Cl}(V_+)^{\mathrm{s}}\}$ and von Neumann algebras $\mathcal{B}_\gamma$, which act on $\mathcal{F}(M)_\gamma$. Lemma 8.2.7 then has the following analogue.

Lemma 8.2.9. *Let γ' be a loop with the property that $\gamma(t) = \gamma'(t)$ for $t \in [\pi, 2\pi]$. Then there is a natural isomorphism $B_\gamma \to B_{\gamma'}$ of $*$-algebras, which extends to an isomorphism $\mathcal{B}_\gamma \to \mathcal{B}_{\gamma'}$.*

An immediate consequence of Proposition 5.2.3 is that the graded commutant of $\mathcal{A}_\gamma$ is $\mathcal{B}_\gamma$. For each loop γ, we equip the Hilbert space $\mathcal{F}(M)_\gamma$ with the structure of $\mathcal{A}_\gamma$-$\mathcal{B}_\gamma^{\mathrm{op}}$ bimodule by defining the right action for homogeneous elements to be

$$\begin{aligned}\mathcal{F}(M)_\gamma \times \mathcal{B}_\gamma^{\mathrm{op}} &\to \mathcal{F}(M)_\gamma,\\ (\xi, b) &\mapsto \mathrm{k}^{-1}\, b\, \mathrm{k} \triangleright \xi.\end{aligned}$$

(See Section 5.2.) Now, let $\gamma' \in LM$ be a loop with the property that $\gamma'(t) = \gamma(2\pi - t)$ for $t \in [0, \pi]$.

Lemma 8.2.10. *Let $\varphi \in L\,\mathrm{Spin}(M)_\gamma$ and $\varphi' \in L\,\mathrm{Spin}(M)_{\gamma'}$ with the property that $\varphi'(t) = \varphi(2\pi - t)$ for $t \in [0, \pi]$. Then, the map*

$$\begin{aligned}\tilde{\tau} : \mathcal{A}_{\gamma'} &\to \mathcal{B}_\gamma^{\mathrm{op}}\\ [\varphi', a \otimes \mathbb{1}] &\mapsto [\varphi, \mathbb{1} \otimes \Lambda_{\alpha\tau} a^* \Lambda_{\alpha\tau}]\end{aligned}$$

does not depend on the choice of φ and φ', and is an isomorphism of von Neumann algebras.

Proof. The fact that $\tilde{\tau}$ is an isomorphism follows from the fact that it factors through isomorphisms:

$$\mathcal{A}_{\gamma'} \to \mathrm{Cl}(V_-)'' \to (\mathrm{Cl}(V_+)'')^{\mathrm{op}} \to \mathcal{B}_\gamma^{\mathrm{op}}.$$

Let $\varphi, \psi \in L\,\mathrm{Spin}(M)_\gamma$ and let $\varphi', \psi' \in L\,\mathrm{Spin}(M)_{\gamma'}$ with the property that $\varphi'(t) = \varphi(2\pi - t)$ and $\psi'(t) = \psi(2\pi - t)$ for $t \in [0, 2\pi]$. Then, there exist unique elements $g, g' \in L\,\mathrm{Spin}(d)$ with the property that $\psi = \varphi \cdot g$ and $\psi' = \varphi' \cdot g'$. Moreover $g'(t) = g(2\pi - t)$ for $t \in [0, \pi]$. Let $a \in \mathrm{Cl}(V_-)''$. Then we compute using Lemma 5.1.8,

$$\theta_g(\mathbb{1} \otimes \Lambda_{\alpha\tau} a^* \Lambda_{\alpha\tau}) = \mathbb{1} \otimes \Lambda_{\alpha\tau} \theta_{\tau(g)}(a)^* \Lambda_{\alpha\tau} = \mathbb{1} \otimes \Lambda_{\alpha\tau} \theta_{g'}(a)^* \Lambda_{\alpha\tau}.$$

Now, we compute

$$\begin{aligned}\tilde{\tau}([\psi', a \otimes \mathbb{1}]) &= [\psi, \mathbb{1} \otimes \Lambda_{\alpha\tau} a^* \Lambda_{\alpha\tau}]\\ &= [\varphi \cdot g, \mathbb{1} \otimes \Lambda_{\alpha\tau} a^* \Lambda_{\alpha\tau}]\\ &= [\varphi, \theta_g(\mathbb{1} \otimes \Lambda_{\alpha\tau} a^* \Lambda_{\alpha\tau})]\\ &= [\varphi, \mathbb{1} \otimes \Lambda_{\alpha\tau} \theta_{g'}(a)^* \Lambda_{\alpha\tau}]\\ &= \tilde{\tau}[\varphi', \theta_{g'}(a) \otimes \mathbb{1}].\end{aligned}$$

□

Using the isomorphism $\tilde{\tau}$ we may view the Hilbert space $\mathcal{F}(M)_\gamma$ as an $\mathcal{A}_\gamma$-$\mathcal{A}_{\gamma'}$ bimodule. Now let us denote by $\beta_1 \in PM$ the path $\beta_1(t) = \gamma(t)$ and by $\beta_2 \in PM$ the path $\beta_2(t) = \gamma'(t) = \gamma(2\pi - t)$, for $t \in [0, \pi]$. Then, using the notation introduced in Remark 8.2.8, we have that $\mathcal{F}(M)_\gamma$ is an $\mathcal{A}_{\beta_1}$-$\mathcal{A}_{\beta_2}$ bimodule.

For future reference we give an explicit formula for the right $\mathcal{A}_{\beta_2}$ action on $\mathcal{F}(M)_\gamma$, assuming that we are given $\Phi \in \widetilde{L\,\mathrm{Spin}}(M)_\gamma$ with basepoint $\varphi \in L\,\mathrm{Spin}(M)_\gamma$, and $\varphi' \in L\,\mathrm{Spin}(M)_{\gamma'}$, with $\varphi'(t) = \varphi(2\pi - t)$ for $t \in [0, \pi]$. We then obtain, for $v \in \mathcal{F}$ and $a \in \mathrm{Cl}(V_-)''$,

$$[\Phi, v] \triangleleft [\varphi', a \otimes \mathbb{1}] = [\Phi, v] \triangleleft [\varphi, \mathbb{1} \otimes \Lambda_{\alpha\tau} a^* \Lambda_{\alpha\tau}] = [\Phi, J a^* J \triangleright v].$$

8.3. Fusion

In this section we construct the family of fusion isomorphisms, whose existence was conjectured in [ST].

Let M be a spin manifold equipped with a fusive spin structure $\widetilde{L\,\mathrm{Spin}}(M)$ on its loop space. We denote the fusion product on $\widetilde{L\,\mathrm{Spin}}(M)$ by λ. If $(\beta_i, \beta_j) \in PM^{[2]}$, we write $\gamma_{ij} = \beta_i \cup \beta_j$. We then set $\mathcal{A}_i := \mathcal{A}_{\beta_i}$, and $\mathcal{F}_{ij} := \mathcal{F}(LM)_{\gamma_{ij}}$, and $\mathcal{F}^{\mathrm{s}}_{ij} := \mathcal{F}^{\mathrm{s}}(LM)_{\gamma_{ij}}$. With this notation we see that $\mathcal{F}_{ij}$ is an $\mathcal{A}_i$-$\mathcal{A}_j$ bimodule. Our goal in this section is to prove the following result.

Theorem 8.3.1. *For every triple $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$ there exists a natural isomorphism of $\mathcal{A}_1$-$\mathcal{A}_3$ bimodules from $\mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23}$ into $\mathcal{F}_{13}$. These isomorphisms are associative, in the sense that if $(\beta_1, \beta_2, \beta_3, \beta_4) \in PM^{[4]}$, then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23} \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34} & \longrightarrow & \mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{24} \\ \downarrow & & \downarrow \\ \mathcal{F}_{13} \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34} & \longrightarrow & \mathcal{F}_{14} \end{array} \tag{8.2}$$

Remark 8.3.2. Lemmas 4.2.7 and 4.2.8 allow us to write $\mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23} \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34}$ instead of $(\mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23}) \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34}$ or $\mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} (\mathcal{F}_{23} \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34})$.

The construction of the family of isomorphisms promised in Theorem 8.3.1 will take the remainder of this section. Before getting into the thick of things, we summarize the main idea as follows. Essentially, we pick trivializations $\nu_{ij} : \mathcal{F}_{ij} \to \mathcal{F}$ in a clever way, and then use the fusion isomorphism $\mu : \mathcal{F} \boxtimes \mathcal{F} \to \mathcal{F}$ on the free fermions defined in Section 5.2, as in Diagram (8.3) to define the fusion isomorphism promised in Theorem 8.3.1.

Let $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$ be arbitrary. Choose $\Phi_{12} \in \widetilde{L\,\mathrm{Spin}}(M)_{\gamma_{12}}$ and $\Phi_{23} \in \widetilde{L\,\mathrm{Spin}}(M)_{\gamma_{23}}$ such that the corresponding basepoints $\varphi_{12} \in L\,\mathrm{Spin}(M)_{\gamma_{12}}$ and $\varphi_{23} \in L\,\mathrm{Spin}(M)_{\gamma_{23}}$ have sitting instants at 0 and π and, moreover, satisfy $\varphi_{23}(t) = \varphi_{12}(2\pi - t)$ for $t \in [0, \pi]$. Set $\Phi_{13} = \lambda(\Phi_{12} \otimes \Phi_{23})$. Note that its basepoint $\varphi_{13} \in L\,\mathrm{Spin}(M)_{\gamma_{13}}$ satisfies $\varphi_{13}(t) = \varphi_{12}(t)$ for $t \in [0, \pi]$ and $\varphi_{13}(t) = \varphi_{23}(t)$ for $t \in [\pi, 2\pi]$. Let $\nu_{ij} : \mathcal{F} \to \mathcal{F}_{ij}$ be the unitary map given by the extension of $\mathcal{F}^{\mathrm{s}} \ni v \mapsto [\Phi_{ij}, v] \in \mathcal{F}^{\mathrm{s}}_{ij}$, for $ij = 12, 23, 13$. By Lemma 8.2.6 the map $u_1 : \mathcal{A}_1 \to \mathrm{Cl}(V_-)'', a \mapsto \nu_{12}^* a \nu_{12}$ is then an isomorphism of von Neumann algebras, which has the property that $\nu_{12}(u_1(a) \triangleright v) = a \triangleright \nu_{12}(v)$ for all $a \in \mathcal{A}_1$ and all $v \in \mathcal{F}$. Similarly, the map $u_2 : \mathcal{A}_2 \to \mathrm{Cl}(V_-)'', a \mapsto \nu_{23}^* a \nu_{23}$ is an isomorphism of von Neumann algebras, with the property that $\nu_{23}(u_2(a) \triangleright v) = a \triangleright \nu_{23}(v)$ for all $a \in \mathcal{A}_2$ and all $v \in \mathcal{F}$.

We compute, for $a \in \mathrm{Cl}(V_-)''$ and $v \in \mathcal{F}$,

$$\nu_{12}(v) \triangleleft u_2^{-1}(a) = [\Phi_{12}, v] \triangleleft [\varphi_{23}, a \otimes \mathbb{1}] = [\Phi_{12}, J a^* J \triangleright v] = [\Phi_{12}, v \triangleleft a] = \nu_{12}(v \triangleleft a).$$

We have now in fact proven the following result.

Lemma 8.3.3. *The triple of maps*

$$u_1 : \mathcal{A}_1 \to \mathrm{Cl}(V_-)'', \qquad \nu_{12}^* : \mathcal{F}_{12} \to \mathcal{F}, \qquad u_2 : \mathcal{A}_2 \to \mathrm{Cl}(V_-)''$$

constitutes an isomorphism of bimodules between $\mathcal{F}_{12}$ and $\mathcal{F}$.

Of course, the triple of maps

$$u_2 : \mathcal{A}_2 \to \mathrm{Cl}(V_-)'', \qquad \nu_{23}^* : \mathcal{F}_{23} \to \mathcal{F}, \qquad u_3 : \mathcal{A}_3 \to \mathrm{Cl}(V_-)''$$

then constitutes an isomorphism of bimodules between $\mathcal{F}_{23}$ and $\mathcal{F}$, and similarly, we have that the triple

$$u_1 : \mathcal{A}_1 \to \mathrm{Cl}(V_-)'', \qquad \nu_{13}^* : \mathcal{F}_{13} \to \mathcal{F}, \qquad u_3 : \mathcal{A}_3 \to \mathrm{Cl}(V_-)'',$$

is an isomorphism of bimodules. Using Lemma 4.2.5 we obtain an isomorphism $\nu_{12}^* \boxtimes \nu_{23}^* : \mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23} \to \mathcal{F} \boxtimes_{\mathrm{Cl}(V_-)''} \mathcal{F}$. It is clear that the triple of maps $(u_1, \nu_{12}^* \boxtimes \nu_{23}^*, u_3)$ then constitutes an isomorphism of bimodules. We define the map $\mu_{123} : \mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23} \to \mathcal{F}_{13}$ by the following diagram

$$\mu_{123} : \mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23} \xrightarrow{\nu_{12}^* \boxtimes \nu_{23}^*} \mathcal{F} \boxtimes_{\mathrm{Cl}(V_-)''} \mathcal{F} \xrightarrow{\mu} \mathcal{F} \xrightarrow{\nu_{13}} \mathcal{F}_{13} \tag{8.3}$$

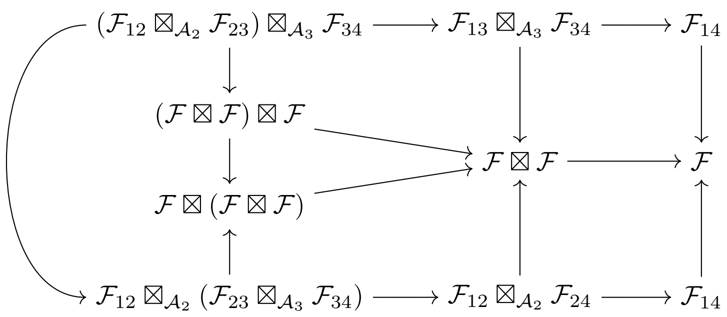
It is clear by construction that μ_{123} is an isomorphism of $\mathcal{A}_1$-$\mathcal{A}_3$ bimodules. The above construction required us to choose certain Φ_{12} and Φ_{23}. We will now show that the resulting map μ_{123} does not depend on these choices. Let $\Phi'_{12} \in \widetilde{L\,\mathrm{Spin}}(M)_{\gamma_{12}}$ and let $\Phi'_{23} \in \widetilde{L\,\mathrm{Spin}}(M)_{\gamma_{23}}$, and suppose that the corresponding elements $\varphi'_{12} \in L\,\mathrm{Spin}(M)_{\gamma_{12}}$ and $\varphi'_{23} \in L\,\mathrm{Spin}(M)_{\gamma_{23}}$ have the property that $\varphi'_{23}(t) = \varphi'_{12}(2\pi - t)$. Then, there are unique elements $U_{12}, U_{23} \in \widetilde{L\,\mathrm{Spin}}(d)$ such that $\Phi'_{12} = \Phi_{12} \cdot U_{12}$ and $\Phi'_{23} = \Phi_{23} \cdot U_{23}$. We set $\Phi'_{13} = \lambda(\Phi'_{12} \otimes \Phi'_{23})$. Equation (7.3) then implies that $\Phi'_{13} = \Phi_{13} \cdot \tilde{\mu}^{\rho}(U_{12} \otimes U_{23})$. We write $\nu'_{ij} : \mathcal{F} \to \mathcal{F}_{ij}, v \mapsto [\Phi'_{ij}, v]$. We then have $\nu'_{12} = \nu_{12} \circ U_{12}^*$ and $\nu'_{23} = \nu_{23} \circ U_{23}^*$, and, using Theorem 6.3.11, $\nu'_{13} = \nu_{13} \circ \hat{\mu}(U_{12}^*, U_{23}^*)$. We now compute, using Lemma 4.2.5 and Diagram (5.1)

$$\begin{aligned} \nu'_{13} \circ \mu \circ ((\nu'_{12})^* \boxtimes (\nu'_{23})^*) &= \nu_{13} \circ \hat{\mu}(U_{12}^*, U_{23}^*) \circ \mu \circ (U_{12} \boxtimes U_{23}) \circ (\nu_{12}^* \boxtimes \nu_{23}^*) \\ &= \nu_{13} \circ \mu \circ (U_{12}^* \boxtimes U_{23}^*) \circ (U_{12} \boxtimes U_{23}) \circ (\nu_{12}^* \boxtimes \nu_{23}^*) \\ &= \nu_{13} \circ \mu \circ (\nu_{12}^* \boxtimes \nu_{23}^*). \end{aligned}$$

We complete this section by proving that Diagram (8.2) commutes. Hence, let $(\beta_1, \beta_2, \beta_3, \beta_4) \in PM^{[4]}$. Choose $\Phi_{ij} \in \widetilde{L\,\mathrm{Spin}}(M)_{\gamma_{ij}}$, for $ij \in \{12, 23, 34\}$ such that the corresponding basepoints $\varphi_{ij} \in L\,\mathrm{Spin}(M)_{\gamma_{ij}}$ satisfy $\varphi_{34}(t) = \varphi_{23}(2\pi - t)$ and $\varphi_{23}(t) = \varphi_{12}(2\pi - t)$ for $t \in [0, \pi]$. We set $\Phi_{13} = \lambda(\Phi_{12} \otimes \Phi_{23})$ and $\Phi_{24} = \lambda(\Phi_{23} \otimes \Phi_{34})$, and $\Phi_{14} = \lambda(\Phi_{13} \otimes \Phi_{34})$. Notice that by the associativity of λ, (see Definition 6.1.1), we have that $\Phi_{14} = \lambda(\Phi_{12} \otimes \Phi_{24})$. First of all, one verifies, using Lemma 4.2.5 that the following diagram commutes

$$\begin{array}{ccccc} (\mathcal{F}_{12} \boxtimes_{\mathcal{A}_2} \mathcal{F}_{23}) \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34} & \xrightarrow[\mu_{123} \boxtimes \mathbb{1}]{} & \mathcal{F}_{13} \boxtimes_{\mathcal{A}_3} \mathcal{F}_{34} & \xrightarrow[\mu_{134}]{} & \mathcal{F}_{14} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{F} \boxtimes \mathcal{F}) \boxtimes \mathcal{F} & \longrightarrow & \mathcal{F} \boxtimes \mathcal{F} & \longrightarrow & \mathcal{F} \end{array}$$

Using Lemmas 4.2.8 and 5.2.1 the commutativity of the above diagram implies that the following diagram commutes as well



The diagram above is in fact nothing but Diagram (8.2). We have thus proven Theorem 8.3.1.

9. Geometric aspects of the spinor bundle on loop space

In Section 8.2 we constructed the spinor bundle on loop space as an associated bundle. This point of view was very fruitful in Section 8.3, where we used it to define the fusion isomorphisms, see Theorem 8.3.1.

In Section 9.1 we consider an alternative construction of the spinor bundle on loop space. Informally speaking the construction goes as follows. For each loop $\gamma \in LM$ we consider the Hilbert space $\mathcal{H}_\gamma := L^2(S^1, \mathbb{S} \otimes \gamma^* TM_{\mathbb{C}})$, which should be thought of as a twisted version of the tangent space of LM at γ. Because every Hilbert space is a Fréchet space, every Hilbert space is trivially a rigged Hilbert space. The rigged Hilbert spaces $\mathcal{H}_\gamma$ fit together into a rigged Hilbert space bundle, $\mathcal{H} \to LM$. Taking fibrewise Clifford algebras yields the bundle of Clifford algebras discussed in Section 8.2. The orthonormal frame bundle of $\mathcal{H}$ has a natural reduction of structure group to $L\,\mathrm{Spin}(M)$, which induces a choice of equivalence class of Lagrangians in $\mathcal{H}_\gamma$ for each $\gamma \in LM$. For each loop we then pick a Lagrangian in this equivalence class, and consider its Fock space. We then construct a bundle gerbe over LM, which encodes the obstruction to patching these Fock spaces together into a bundle of Fock spaces on which the bundle of Clifford algebras acts irreducibly. A spin structure on the loop space LM induces a trivialization of this bundle gerbe, and hence lifts the obstruction to patching together the aforementioned Fock spaces into a bundle on LM. We prove that the bundle of Fock spaces induced by a spin structure on loop space in this way is the spinor bundle on loop space defined in Section 8.2.

In [Amb12] Ambler considers a bundle $H \to LM$ of Hilbert spaces whose fibre over a loop γ is $L^2(S^1, \gamma^* E_{\mathbb{C}})$, where $E \to M$ is a real oriented vector bundle of even rank. Using essentially the same construction as we just outlined, Ambler then constructs a bundle gerbe on LM, and gives a construction that takes a trivialization of this bundle gerbe as input and produces a bundle of irreducible modules for the Clifford algebra bundle $\mathrm{Cl}(H)$. We shall point out the connection between our work and Ambler's in Section 9.1.

In the preceding discussion we argued that we can use the bundle $L\,\mathrm{Spin}(M)$ to choose an equivalence class of Lagrangians in $\mathcal{H}_\gamma$ for each loop $\gamma \in LM$. In Section 9.2 we give an interesting characterization of the equivalence class induced in this way. We expect this characterization to be useful when constructing a conformal connection on the spinor bundle on loop space.

9.1. A bundle gerbe construction of the spinor bundle on loop space

Let M be a spin manifold. For each loop $\gamma \in LM$ we set $\mathcal{H}_\gamma := L^2(S^1, \mathbb{S} \otimes \gamma^* TM_{\mathbb{C}})$. We then define a vector bundle $\mathcal{H} \to LM$ by setting $\mathcal{H} := \coprod_{\gamma \in LM} \mathcal{H}_\gamma$, and we define a principal $\mathrm{O}(V)$-bundle $\mathrm{Fr}_{\mathrm{O}}(\mathcal{H}) := \coprod_{\gamma \in LM} \mathrm{O}(V, \mathcal{H}_\gamma)$. At the moment, neither of these bundles is equipped with a topology. However, it is clear that $\mathcal{H} = \mathrm{Fr}_{\mathrm{O}}(\mathcal{H}) \times_{\mathrm{O}(V)} V$. Hence, to equip $\mathcal{H}$ with the structure of rigged Hilbert space bundle it is sufficient, by Lemma 8.1.14, to equip $\mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$ with the structure of Fréchet principal $\mathrm{O}(V)$-bundle. We now provide such a structure.

First, we construct an equivariant bundle map $\Theta : L\,\mathrm{Spin}(M) \to \mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$. Let $\gamma \in LM$ be arbitrary. Let $\varphi \in L\,\mathrm{Spin}(M)_\gamma$, we write $\pi(\varphi) \in L\,\mathrm{SO}(M)$ for its basepoint. Then, for each $t \in S^1$, we have an orthogonal map $\pi(\varphi)(t) : \mathbb{R}^d \to T_{\gamma(t)}M$. Let $v \in V$, then we define $\Theta(\varphi)(v) \in \mathcal{H}_\gamma$ by the expression

$$\Theta(\varphi)(v)(t) = (\mathbb{1} \otimes \pi(\varphi)(t))v(t).$$

A routine verification shows that this indeed defines an equivariant fibre-preserving map $\Theta : L\,\mathrm{Spin}(M) \to \mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$. The map $L\,\mathrm{Spin}(M) \times \mathrm{O}(V) \to \mathrm{Fr}_{\mathrm{O}}(\mathcal{H}), (\varphi, g) \mapsto \Theta(\varphi) \cdot g$ then descends to an equivariant fibrewise bijection $L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{O}(V) \to \mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$. Using this bijection we transfer the smooth structure from $L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{O}(V)$ to $\mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$. In fact, because Θ factors through $L\,\mathrm{SO}(M)$ we have

$$L\,\mathrm{SO}(M) \times_{L\,\mathrm{SO}(d)} \mathrm{O}(V) = L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{O}(V) = \mathrm{Fr}_{\mathrm{O}}(\mathcal{H}).$$

We recall that the bundle of Clifford algebras $\mathrm{Cl}^{\mathrm{s}}(LM)$ is defined to be $L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{Cl}(V)^{\mathrm{s}}$. For each loop $\gamma \in LM$ the map $L\,\mathrm{Spin}(M) \times \mathrm{Cl}(V)^{\mathrm{s}} \to \mathrm{Cl}(\mathcal{H}_\gamma)^{\mathrm{s}}, (\nu, a) \mapsto \mathrm{Cl}(\nu)(a)$ descends to an isomorphism of rigged C*-algebras $\mathrm{Cl}^{\mathrm{s}}(LM) \to \mathrm{Cl}(\mathcal{H}_\gamma)^{\mathrm{s}}$, see Lemma 3.2.6. In other words, the fibre of $\mathrm{Cl}^{\mathrm{s}}(LM)$ over γ can be canonically identified with $\mathrm{Cl}(\mathcal{H}_\gamma)^{\mathrm{s}}$.

We obtain a reduction of $\mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$ to $\mathrm{O}_L(V)$, denoted $\mathrm{Fr}_L(\mathcal{H})$, by setting

$$\mathrm{Fr}_L(\mathcal{H}) \coloneqq L\,\mathrm{SO}(M) \times_{L\,\mathrm{SO}(d)} \mathrm{O}_L(V) = L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{O}_L(V)$$

Because $\mathrm{Fr}_L(\mathcal{H})$ is a subset of $\mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$ we may view elements of $\mathrm{Fr}_L(\mathcal{H})$ as frames of $\mathcal{H}$. That is, if $\nu \in \mathrm{Fr}_L(\mathcal{H})_\gamma$ for some $\gamma \in LM$, then ν can be viewed as an orthogonal map $\nu : V \to \mathcal{H}_\gamma$. Thus, if $\nu \in \mathrm{Fr}_L(\mathcal{H})_\gamma$, then $\nu(L) \subset \mathcal{H}_\gamma$ is a Lagrangian.

Lemma 9.1.1. *Let $\gamma \in LM$ and $\nu_1, \nu_2 \in \mathrm{Fr}_L(\mathcal{H})_\gamma$ be arbitrary. Then, the Lagrangians $\nu_1(L)$ and $\nu_2(L)$ are equivalent.*

Proof. Let $P_i : \mathcal{H}_\gamma \to \nu_i(L)$, for $i = 1, 2$, be the orthogonal projections. Then, according to Lemma 3.6.1 it suffices to show that $P_1 - P_2$ is Hilbert-Schmidt. To that end, we compute

$$P_1 - P_2 = \nu_1 P_L \nu_1^{-1} - \nu_2 P_L \nu_2^{-1} = \nu_1(P_L - \nu_1^{-1}\nu_2 P_L \nu_2^{-1}\nu_1)\nu_1^{-1}.$$

We know that $\nu_1^{-1}\nu_2 \in \mathrm{O}_L(V)$, hence it follows from Lemma 3.6.1 that $P_L - \nu_1^{-1}\nu_2 P_L \nu_2^{-1}\nu_1$ is Hilbert-Schmidt. □

Recall from Section 3.6 that the space of Lagrangians equivalent to L is equal to $\mathrm{Lag}_L(V) \coloneqq \mathrm{O}_L(V)/\,\mathrm{U}(L)$. We define $\mathrm{Lag}_L(\mathcal{H})$ to be the topological space $\mathrm{Fr}_L(\mathcal{H})/\,\mathrm{U}(L)$. Lemma 9.1.1 then implies that the fibre $\mathrm{Lag}_L(\mathcal{H})_\gamma$ really consists of Lagrangians in $\mathcal{H}_\gamma$ all of which are pairwise equivalent. For reasons that will become clear in Section 9.2 we call $\mathrm{Lag}_L(\mathcal{H})$ the *bundle of Dirac Lagrangians* in $\mathcal{H}$. Let us write $\tilde{\pi}$ for the projection $\mathrm{Fr}_L(\mathcal{H}) \to \mathrm{Lag}_L(\mathcal{H})$, and p_2 for the projection $\mathrm{Lag}_L(\mathcal{H}) \to LM$.

We now equip the topological space $\mathrm{Lag}_L(\mathcal{H})$ with a structure of Fréchet manifold, such that the following two conditions are satisfied:

1. The map $\tilde{\pi} : \mathrm{Fr}_L(\mathcal{H}) \to \mathrm{Lag}_L(\mathcal{H})$ is a surjective submersion.

2. For every $\gamma \in LM$ there exists an open neigbourhood U of γ such that there exists a diffeomorphism $\varphi : \mathrm{Lag}_L(\mathcal{H})|_U \to U \times \mathrm{Lag}_L(V)$ such that for all $x \in \mathrm{Lag}_L(\mathcal{H})|_U$ we have $p_2(x) = \mathrm{pr}_1(\varphi(x))$.

Let $\{U_\alpha\}_{\alpha \in I}$ be a trivializing cover for $\mathrm{Fr}_L(\mathcal{H})$. The collection $\{\mathrm{Lag}_L(\mathcal{H})|_{U_\alpha}\}_{\alpha \in I}$ is then an open cover of $\mathrm{Lag}_L(\mathcal{H})$. We have

$$\mathrm{Lag}_L(\mathcal{H})|_{U_\alpha} = \mathrm{Fr}_L(\mathcal{H})|_{U_\alpha} / \mathrm{U}(L) \simeq (U_\alpha \times \mathrm{O}_L(V)) / \mathrm{U}(L) = U_\alpha \times \mathrm{Lag}_L(V),$$

where the first equality is the definition of $\mathrm{Lag}_L(\mathcal{H})$ and the second identification follows from the assumption that U_α is a trivializing cover for $\mathrm{Fr}_L(\mathcal{H})$. The condition that this identification is a diffeomorphism then defines a smooth structure on $\mathrm{Lag}_L(\mathcal{H})$, which satisfies condition 2. The projection $\tilde{\pi}$ can locally be described as

$$\mathrm{Fr}_L(\mathcal{H})|_{U_\alpha} \simeq U_\alpha \times \mathrm{O}_L(V) \to U_\alpha \times \mathrm{Lag}_L(V) \simeq \mathrm{Lag}_L(\mathcal{H})|_{U_\alpha},$$

from which follows that it is a submersion.

Now, the bundle $\mathrm{Fr}_L(\mathcal{H})$ is a principal $\mathrm{O}_L(V)$-bundle. Hence, one may wonder if there is a lift of the structure group to $\mathrm{Imp}_L(V)$; if the loop space LM is equipped with a spin structure $\widetilde{L\,\mathrm{Spin}}(M)$, then a lift is given by the associated bundle $\widetilde{L\,\mathrm{Spin}}(M) \times_{\widetilde{L\,\mathrm{Spin}(d)}} \mathrm{Imp}_L(V)$. In general, one may consider the lifting gerbe $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$ corresponding to this lifting problem, see [Mur96, Section 4] and Appendix D. This bundle gerbe can be described as follows. Because $\mathrm{Fr}_L(\mathcal{H})$ is a principal $\mathrm{O}_L(V)$-bundle we have a map

$$\Delta : \mathrm{Fr}_L(\mathcal{H})^{[2]} \to \mathrm{O}_L(V),\ (\nu_1, \nu_2) \mapsto \nu_2^{-1}\nu_1.$$

The line bundle $\mathcal{L}$ is now defined to be $\Delta^* \mathrm{Imp}_L(V)$. The bundle gerbe multiplication is then simply multiplication in $\mathrm{Imp}_L(V)$. The following result tells us that the fibre of $\mathcal{L}$ over a pair $(\nu_1, \nu_2) \in \mathrm{Fr}_L(\mathcal{H})^{[2]}$ is the space of unitary equivalences from $\mathcal{F}_{\nu_2(L)}$ into $\mathcal{F}_{\nu_1(L)}$.

Lemma 9.1.2. *For every $(\nu_2, \nu_1) \in \mathrm{Fr}_L(\mathcal{H})^{[2]}$, the map*

$$\begin{aligned} \mathcal{L}_{\nu_2,\nu_1} &\to \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{\nu_2(L)}, \mathcal{F}_{\nu_1(L)}), \\ U &\mapsto \Lambda_{\nu_2} U \Lambda_{\nu_1}^{-1}, \end{aligned}$$

is an isomorphism of $\mathrm{U}(1)$-torsors, with the property that for all $(\nu_3, \nu_2, \nu_1) \in \mathrm{Fr}_L(\mathcal{H})^{[3]}$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_{\nu_3,\nu_2} \otimes \mathcal{L}_{\nu_2,\nu_1} & \longrightarrow & \mathcal{L}_{\nu_3,\nu_1} \\ \downarrow & & \downarrow \\ \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{\nu_3(L)}, \mathcal{F}_{\nu_2(L)}) \otimes \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{\nu_2(L)}, \mathcal{F}_{\nu_1(L)}) & \longrightarrow & \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{\nu_3(L)}, \mathcal{F}_{\nu_1(L)}) \end{array}$$

where the top arrow is the bundle gerbe multiplication, and the bottom arrow is composition of operators.

Proof. Let $\nu_2, \nu_1 \in \mathrm{Fr}_L(\mathcal{H})_\gamma$ be arbitrary. Let $U \in \mathcal{L}_{\nu_2,\nu_1}$ be arbitrary. This means that U implements $\nu_2^{-1}\nu_1$. Now, let $a \in \mathrm{Cl}(\mathcal{H})_\gamma$ be arbitrary, then we compute

$$\Lambda_{\nu_2} U \Lambda_{\nu_1}^{-1} a \Lambda_{\nu_1} U^* \Lambda_{\nu_2}^{-1} = \theta_{\nu_2}(U \theta_{\nu_1}^{-1}(a) U^*)) = a.$$

The fact that the diagram from the lemma commutes follows from a straightforward computation. □

Lemma 9.1.2 suggests that the fibre of $\mathcal{L}$ over a pair $(\nu_1, \nu_2) \in \mathrm{Fr}_L(\mathcal{H})^{[2]}$ really only depends on the Lagrangians $\nu_1(L)$ and $\nu_2(L)$. What's more, the bundle gerbe $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$ is a refinement of a bundle gerbe on $\mathrm{Lag}_L(\mathcal{H})$, as recorded in the following proposition.

Proposition 9.1.3. *There is a bundle gerbe $(\mathcal{U}, \mathrm{Lag}_L(\mathcal{H}))$ with the following properties:*

- *The $\mathrm{U}(1)$-bundle $\mathcal{L}$ is the pullback of $\mathcal{U}$ along $\tilde{\pi} \times \tilde{\pi} : \mathrm{O}_L(V)^{[2]} \to \mathrm{Lag}_L(V)^{[2]}$.*
- *The multiplication on $\mathcal{U}$ pulls back to the multiplication on $\mathcal{L}$, i.e. the following diagram commutes*

$$\begin{array}{ccc} \pi_3^*\mathcal{L} \otimes \pi_1^*\mathcal{L} & \longrightarrow & \pi_2^*\mathcal{L} \\ \downarrow & & \downarrow \\ \pi_3^*\mathcal{U} \otimes \pi_1^*\mathcal{U} & \longrightarrow & \pi_2^*\mathcal{U} \end{array}$$

Moreover, for each point $(L_1, L_2) \in \mathrm{Lag}_L(\mathcal{H})^{[2]}$ there is an isomorphism of $\mathrm{U}(1)$-torsors from $\mathcal{U}(L_1, L_2)$ to $\mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{L_1}, \mathcal{F}_{L_2})$ which takes the bundle gerbe multiplication to the composition of operators.

Proof. First, we prove that the $\mathrm{U}(1)$-bundle $\mathcal{L} \to \mathrm{O}_L(V)^{[2]}$ descends to a $\mathrm{U}(1)$-bundle on $\mathrm{Lag}_L(V)^{[2]}$. To that end, we shall give a descent isomorphism, see [Bry93, Proposition 5.1.3]. Let us introduce some notation. We set $X = \mathrm{O}_L(\mathcal{H})$ and $Y = \mathrm{Lag}_L(\mathcal{H})$. We then write $\tilde{\pi}^{[2]} = \tilde{\pi} \times \tilde{\pi} : X^{[2]} \to Y^{[2]}$. We write q_1 and q_2 for the two projection maps $X^{[2]} \times_{Y^{[2]}} X^{[2]} \rightrightarrows X^{[2]}$. A descent isomorphism is now an isomorphism $\phi : q_1^*\mathcal{L} \to q_2^*\mathcal{L}$, which satisfies the cocyle condition. Let

$$(l, (x_1, x_2), (x_1', x_2')) \in q_1^*\mathcal{L}$$

be arbitrary. Unravelling the definitions, this means that $x_1, x_2, x_1', x_2' \in \mathrm{Fr}_L(\mathcal{H})_\gamma$ for some $\gamma \in LM$ and that l is an implementer for $x_2^{-1}x_1 \in \mathrm{O}_L(V)$. Moreover, we have that $x_1(L) = x_1'(L)$ and $x_2(L) = x_2'(L)$, which implies that $x_1^{-1}x_1' \in \mathrm{U}(L)$ and $(x_2')^{-1}x_2 \in \mathrm{U}(L)$. It then follows that $\Lambda_{(x_2')^{-1}x_2} l \Lambda_{x_1^{-1}x_1'}$ implements $(x_2')^{-1}x_1'$. We thus define ϕ to be the map

$$\phi : (l, (x_1, x_2), (x_1', x_2')) \mapsto (\Lambda_{(x_2')^{-1}x_2} l \Lambda_{x_1^{-1}x_1'}, (x_1, x_2), (x_1', x_2')).$$

This map satisfies the cocycle condition because the map $\mathrm{U}(L) \to \mathrm{Imp}_L(V), g \mapsto \Lambda_g$ is a group homomorphism. That ϕ is smooth follows from the fact that it is the composition of smooth maps. According to [Bry93], we then obtain a $\mathrm{U}(1)$-bundle $\mathcal{U} \to Y^{[2]}$, which pulls back to $\mathcal{L}$, by setting $\mathcal{U} = \mathcal{L}/\sim$, where $\sim$ is the following equivalence relation: The element $(l, (x_1, x_2)) \in \mathcal{L} = \Delta^* \mathrm{Imp}_L(V)$ is equivalent to $(l', (x_1', x_2')) \in \mathcal{L}$ if $(l, (x_1, x_2), (x_1', x_2')) \in q_1^*\mathcal{L}$ and if, moreover, $\phi(l, (x_1, x_2), (x_1', x_2')) = (l', (x_1, x_2), (x_1', x_2')) \in q_2^*\mathcal{L}$.

Next, we define, for each element $(L_1, L_2) \in Y^{[2]}$, an isomorphism $\mathcal{U}_{(L_1, L_2)} \to \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{L_1}, \mathcal{F}_{L_2})$. Let $u \in \mathcal{U}$, and pick a representative $(l, (x_1, x_2)) \in \mathcal{L}$. The map $\Lambda_{x_2} l \Lambda_{x_1^{-1}}$ is then a $\mathrm{Cl}(\mathcal{H}_\gamma)$-equivariant isomorphism from $\mathcal{F}_{L_1}$ into $\mathcal{F}_{L_2}$. Let us show that this does not depend on the representative chosen. Indeed, suppose that we are given a second representative (l', x_1', x_2') of u. The assumption that $(l, (x_1, x_2)) \sim (l', (x_1', x_2'))$ implies that $l' = \Lambda_{(x_2')^{-1}x_2} l \Lambda_{x_1^{-1}x_1'}$. We then compute

$$\Lambda_{x_2'} l' \Lambda_{(x_1')^{-1}} = \Lambda_{x_2'} \Lambda_{(x_2')^{-1}x_2} l \Lambda_{x_1^{-1}x_1'} \Lambda_{(x_1')^{-1}} = \Lambda_{x_2} l \Lambda_{x_1^{-1}},$$

as desired. It is clear that the assignment $u \mapsto \Lambda_{x_2} l \Lambda_{x_1^{-1}}$ is an isomorphism of $\mathrm{U}(1)$-torsors.

To turn the pair $(\mathcal{U}, Y) = (\mathcal{U}, \mathrm{Lag}_L(\mathcal{H}))$ into a bundle gerbe, we still need to define its multiplication. We claim that the map $\pi_3^*\mathcal{L} \otimes \pi_1^*\mathcal{L} \to \pi_2^*\mathcal{L} \to \pi_2^*(\mathcal{L}/\sim)$ descends to the quotient $\pi_3^*(\mathcal{L}/\sim) \otimes \pi_1^*(\mathcal{L}/\sim)$. Let $(l_{12} \otimes l_{23}, (x_1, x_2, x_3)) \in \pi_3^*\mathcal{L} \otimes \pi_1^*\mathcal{L}$ be arbitrary. Its image in $\pi_2^*(\mathcal{L}/\sim)$

is $[l_{23}l_{12}, (x_1, x_2, x_3)]$. Now, suppose we are given a second element $(l'_{12} \otimes l'_{23}, (x'_1, x'_2, x'_3)) \in \pi_3^*\mathcal{L} \otimes \pi_1^*(\mathcal{L})$ with the property that

$$(l_{12}, (x_1, x_2)) \sim (l'_{12}, (x'_1, x'_2)) \qquad \text{and} \qquad (l_{23}, (x_2, x_3)) \sim (l'_{23}, (x'_2, x'_3)).$$

This means that we have $l'_{12} = \Lambda_{(x'_2)^{-1}x_2} l_{12} \Lambda_{x_1^{-1}x'_1}$ and $l'_{23} = \Lambda_{(x'_3)^{-1}x_3} l_{23} \Lambda_{x_2^{-1}x'_2}$, which implies that $l'_{23}l'_{12} = \Lambda_{(x'_3)^{-1}x_3} l_{23} l_{12} \Lambda_{x_1^{-1}x'_1}$, and hence that $(l_{23}l_{12}, (x_1, x_3)) \sim (l'_{23}l'_{12}, (x'_1, x'_3))$. We thus have a commutative square

$$\begin{array}{ccc} \pi_3^*\mathcal{L} \otimes \pi_1^*\mathcal{L} & \longrightarrow & \pi_2^*\mathcal{L} \\ \downarrow & & \downarrow \\ \pi_3^*(\mathcal{L}/\sim) \otimes \pi_1^*(\mathcal{L}/\sim) & \longrightarrow & \pi_2^*(\mathcal{L}/\sim) \end{array}$$

The bottom arrow in this diagram is the multiplication that turns $(Y, \mathcal{L}/\sim) = (\mathrm{Lag}_L(\mathcal{H}), \mathcal{U})$ into a bundle gerbe. A routine computation shows that, for each triple $(L_1, L_2, L_3) \in Y^{[3]}$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{U}_{(L_1,L_2)} \otimes \mathcal{U}_{(L_2,L_3)} & \longrightarrow & \mathcal{U}_{(L_1,L_3)} \\ \downarrow & & \downarrow \\ \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{L_1}, \mathcal{F}_{L_2}) \otimes \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{L_2}, \mathcal{F}_{L_3}) & \longrightarrow & \mathrm{U}_{\mathrm{Cl}(\mathcal{H}_\gamma)}(\mathcal{F}_{L_1}, \mathcal{F}_{L_3}) \end{array}$$

where the bottom arrow is simply composition of maps, and where $L_i \subset \mathcal{H}_\gamma$. □

Remark 9.1.4. The bundle gerbe $(\mathcal{U}, \mathrm{Lag}_L(\mathcal{H}))$ is analogous to the bundle gerbe (P, Y) of [Amb12, Proposition 11.8]. The difference is that where we have used the Hilbert space $L^2(S^1, \mathbb{S} \otimes \gamma^* TM_{\mathbb{C}})$, Ambler used the Hilbert space $L^2(S^1, \gamma^* E)$, where $E \to M$ is a real oriented vector bundle of even rank. Moreover, Ambler works in a topological setting, where we use Fréchet manifolds.

We now return to the bundle gerbe $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$, in particular, we shall construct a bundle gerbe module for it. We start by considering the trivial rigged Hilbert space bundle $\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}_L^{\mathrm{s}} \to \mathrm{Fr}_L(\mathcal{H})$. For every $\nu \in \mathrm{Fr}_L(\mathcal{H})$, the map

$$\begin{aligned} (\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}_L^{\mathrm{s}})_\nu &\to \mathcal{F}^{\mathrm{s}}_{\nu(L)}, \\ (\nu, v) &\mapsto \Lambda_\nu(v), \end{aligned}$$

is an isomorphism (see Lemma 3.5.8). In other words, we obtain an identification

$$\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}_L^{\mathrm{s}} \xrightarrow{\sim} \coprod_{\nu \in \mathrm{Fr}_L(\mathcal{H})} \mathcal{F}^{\mathrm{s}}_{\nu(L)},$$

through which we view the right-hand side as a rigged Hilbert space bundle over $\mathrm{Fr}_L(\mathcal{H})$. Now let $\gamma \in LM$ be arbitrary. If $\nu_1, \nu_2 \in \mathrm{Fr}_L(\mathcal{H})_\gamma$, then $\mathcal{F}^{\mathrm{s}}_{\nu_1(L)}$ and $\mathcal{F}^{\mathrm{s}}_{\nu_2(L)}$ are unitarily equivalent representations of $\mathrm{Cl}(\mathcal{H}_\gamma)$. If we were given a coherent family of unitary equivalences of these $\mathrm{Cl}(\mathcal{H}_\gamma)$-modules, for all $\gamma \in LM$, then we would have descent data that would allow us to obtain a bundle on loop space. Lemma 9.1.2 tells us that the bundle gerbe $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$ encodes these unitary equivalences. Motivated by this we will show that a trivialization of $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$ thus allows us to construct a spinor bundle on loop space. We proceed by using the theory of bundle gerbe modules, (see [BCM+02] and Appendix D), as follows.

We shall equip the rigged Hilbert space bundle $\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}_L^{\mathrm{s}} \to \mathrm{Fr}_L(\mathcal{H})$ with the structure of bundle gerbe module for $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$. Because $\mathcal{L}$ is defined to be $\Delta^* \mathrm{Imp}_L(V)$ it comes equipped with a map $s : \mathcal{L} \to \mathrm{Imp}_L(V)$. We now define the action of the bundle gerbe $\mathcal{L}$ on $\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}^{\mathrm{s}}_L$ to be

$$\begin{aligned} \mathcal{L} \otimes (\mathrm{Fr}_L(\mathcal{H})^{[2]} \times \mathcal{F}^{\mathrm{s}}_L) &\to \mathrm{Fr}_L(\mathcal{H})^{[2]} \times \mathcal{F}^{\mathrm{s}}_L, \\ l \otimes (\nu_1, \nu_2, v) &\mapsto (\nu_1, \nu_2, s(l)v). \end{aligned}$$

Let $p : \mathrm{Fr}_L(\mathcal{H}) \to LM$ denote the projection map. We now suppose that M is a string manifold, then the bundle gerbe $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$ has a trivialization $\mathrm{Imp}_L(\mathcal{H}) \to \mathrm{Fr}_L(\mathcal{H})$, which can be used as descent data for the bundle $(\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}^{\mathrm{s}}_L) \otimes \mathrm{Imp}_L(\mathcal{H}) \to \mathrm{Fr}_L(\mathcal{H})$, [BCM+02, Proposition 4.2]. I.e. it can be used to construct a vector bundle $F^{\mathrm{s}} \to LM$ with the property that $p^* F^{\mathrm{s}} \otimes \mathrm{Imp}_L(\mathcal{H})^*$ is isomorphic, as a bundle gerbe module, to $\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}^{\mathrm{s}}_L$. (See Appendix D or [BCM+02, Proposition 4.2] for the bundle gerbe module structure of $p^* F^{\mathrm{s}} \otimes \mathrm{Imp}_L(\mathcal{H})^*$.)

Proposition 9.1.5. *The bundle $F^{\mathrm{s}} \to LM$ is isomorphic to the spinor bundle on loop space $\mathcal{F}^{\mathrm{s}}(LM) \to LM$.*

Proof. We prove the proposition by giving an isomorphism from $p^* \mathcal{F}^{\mathrm{s}}(LM) \otimes \mathrm{Imp}_L(\mathcal{H})^*$ into $\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}^{\mathrm{s}}_L$. Let us write $\Delta : \mathrm{Imp}_L(\mathcal{H}) \times_{LM} \mathrm{Imp}_L(\mathcal{H}) \to \mathrm{Imp}_L(V)$ for the map characterized by the equation $\Phi \Delta(\Phi, \Phi') = \Phi'$. A straightforward calculation shows that Δ satisfies the equations

$$\Delta(\Phi_\nu, \Phi U) = \Delta(\Phi_\nu, \Phi) U \qquad\qquad \Delta(\Phi_\nu U, \Phi) = U^{-1} \Delta(\Phi_\nu, \Phi),$$

for all $U \in \mathrm{Imp}_L(V)$. Using this, a routine calculation shows that for each $\nu \in \mathrm{Fr}_L(\mathcal{H})_\gamma$ the following map is well-defined

$$\begin{aligned} (p^* \mathcal{F}^{\mathrm{s}}(LM) \otimes \mathrm{Imp}_L(\mathcal{H})^*)_\nu &\to \mathrm{Fr}_L(\mathcal{H})_\nu \times \mathcal{F}^{\mathrm{s}}_L \\ (\nu, [\Phi, v]) \otimes \Phi_\nu &\mapsto (\nu, \Delta(\Phi_\nu, \Phi) v). \end{aligned}$$

Its inverse is given by the map $(\nu, v) \mapsto (\nu, [\Phi, v]) \otimes \Phi$, where Φ is any element $\Phi \in \mathrm{Imp}_L(\mathcal{H})_\nu$. The statement that this map is an isomorphism of bundle gerbe modules is then equivalent to the statement that the following diagram commutes, for each $(\nu_1, \nu_2) \in \mathrm{Fr}_L(\mathcal{H})^{[2]}$,

$$\begin{array}{ccc} \mathcal{L}_{\nu_1,\nu_2} \otimes (\mathrm{Fr}_L(\mathcal{H})_{\nu_2} \times \mathcal{F}^{\mathrm{s}}_L) & \longrightarrow & (\mathrm{Fr}_L(\mathcal{H})_{\nu_1} \times \mathcal{F}^{\mathrm{s}}_L) \\ \downarrow & & \downarrow \\ \mathcal{L}_{\nu_1,\nu_2} \otimes (p^* \mathcal{F}^{\mathrm{s}}(LM) \otimes \mathrm{Imp}_L(\mathcal{H})^*)_{\nu_2} & \longrightarrow & (p^* \mathcal{F}^{\mathrm{s}}(LM) \otimes \mathrm{Imp}_L(\mathcal{H})^*)_{\nu_1} \end{array}$$

Indeed, we have for arbitrary $(\nu_1, \nu_2) \in \mathrm{Fr}_L(\mathcal{H})^{[2]}$ and arbitrary $\Phi_1 \otimes \Phi_2 \otimes (\nu_2, v) \in \mathcal{L}_{\nu_1,\nu_2} \otimes (\mathrm{Fr}_L(\mathcal{H})_{\nu_2} \times \mathcal{F}^{\mathrm{s}}_L)$

$$\begin{array}{ccc} \Phi_1 \otimes \Phi_2 \otimes (\nu_2, v) & \longmapsto & (\nu_1, \Delta(\Phi_2, \Phi_1) v) \\ \downarrow & & \downarrow \\ \Phi_1 \otimes \Phi_2 \otimes (\nu_2, [\Phi_2, v]) \otimes \Phi_2 & \longmapsto & (\nu_1, [\Phi_2, v]) \otimes \Phi_1 \end{array}$$

Which completes the proof. □

Let us again make contact with [Amb12]. First, we remark that the caveats from Remark 9.1.4 still apply.

We define yet another vector bundle over $\mathrm{Fr}_L(\mathcal{H})$. Recall that we have an $\mathrm{O}_L(V)$-equivariant vector bundle $D^{\mathrm{s}} := \mathrm{Imp}_L(V)^* \times_{\mathrm{U}(1)} \mathcal{F}_L^{\mathrm{s}}$, see Section 3.6. We write $p_D : D^{\mathrm{s}} \to \mathrm{O}_L(V)$ for the projection map. We then set

$$\mathcal{D}^{\mathrm{s}} := \mathrm{Fr}_L(\mathcal{H}) \times D^{\mathrm{s}} / \sim$$

where $(\nu g, v) \sim (\nu, gv)$ for all $g \in \mathrm{O}_L(V)$, $\nu \in \mathrm{Fr}_L(\mathcal{H})$, and all $v \in D^{\mathrm{s}}$. The map $(\nu, v) \mapsto \nu p_D(v)$ then descends to a map $\mathcal{D}^{\mathrm{s}} \to \mathrm{Fr}_L(\mathcal{H})$, which turns $\mathcal{D}^{\mathrm{s}}$ into a vector bundle over $\mathrm{Fr}_L(\mathcal{H})$ with fibre $\mathcal{F}_L^{\mathrm{s}}$. In fact, one may check that the map $\mathrm{Fr}_L(\mathcal{H}) \times \mathcal{F}_L^{\mathrm{s}} \to \mathcal{D}^{\mathrm{s}}, (\nu, v) \mapsto [\nu, [\mathbb{1}, v]]$ is a trivialization of $\mathcal{D}^{\mathrm{s}}$, which thus allows us to equip $\mathcal{D}^{\mathrm{s}}$ with the structure of rigged Hilbert space bundle on $\mathrm{Fr}_L(\mathcal{H})$, and moreover with the structure of bundle gerbe module for $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$. In [Amb12] Ambler constructs a vector bundle, denoted FY, over $\mathrm{Lag}_L(\mathcal{H})$, which is a bundle gerbe module for the bundle gerbe $(\mathcal{U}, \mathrm{Lag}_L(\mathcal{H}))$. The bundle gerbe module FY for $(\mathcal{U}, \mathrm{Lag}_L(\mathcal{H}))$ pulls back to the bundle gerbe module $\mathcal{D}^{\mathrm{s}}$ for $(\mathcal{L}, \mathrm{Fr}_L(\mathcal{H}))$, (recall that $\mathcal{L}$ is a refinement of $\mathcal{U}$). Using the bundle gerbe module FY, Ambler constructs from a trivialization of the bundle gerbe $\mathcal{U}$, a vector bundle on loop space $S \to LM$, which we thus conclude is completely analogous to the spinor bundle on loop space $\mathcal{F}^{\mathrm{s}}(LM)$ that we constructed.

9.2. A geometric view on the reduction $\mathrm{Fr}_L(\mathcal{H}) \to \mathrm{Fr}(\mathcal{H})$.

We still assume that M is a smooth spin manifold of dimension d, (technically, we only require M to be oriented in this section). The reduction $\mathrm{Fr}_L(\mathcal{H}) \to \mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$ is equivalent to a smooth choice of equivalence class of Lagrangians for each loop $\gamma \in LM$. In this section we give a more geometric characterization of the choice that leads to the reduction used in Section 9.1.

For each loop $\gamma \in LM$ we define the twisted Dirac operator $D\!\!\!/_\gamma : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ by

$$D\!\!\!/_\gamma = i \frac{\mathrm{d}}{\mathrm{d}t} \otimes \nabla,$$

where ∇ is the pullback of the covariant derivative corresponding to the Levi-Civita connection on TM.

There is a basis of $\mathcal{H}_\gamma$ that diagonalizes $D\!\!\!/_\gamma$. Write $\mathrm{pt}_\gamma(t) \in \mathrm{SO}(T_{\gamma(0)}M, T_{\gamma(t)}M)$ for parallel transport along γ for time t. Let $\{v_j\}_{j=1,\dots,d}$ be a collection of linearly independent vectors in $T_{\gamma(0)}M_{\mathbb{C}}$ that diagonalizes $\mathrm{pt}_\gamma(2\pi) \in \mathrm{SO}(T_{\gamma(0)}M_{\mathbb{C}})$, with eigenvalues $e^{i\varphi_j} \in S^1$. We choose φ_j to be in $[-\pi, \pi)$.

Lemma 9.2.1. *The family of vectors $\eta_{n,j} \in \mathcal{H}_\gamma$ defined by*

$$\eta_{n,j}(t) = e^{-i(n+\frac{1}{2})t - i\frac{\varphi_j}{2\pi}t} \otimes \mathrm{pt}_\gamma(t) v_j,$$

diagonalizes $D\!\!\!/_\gamma$ with eigenvalues

$$\lambda_{n,j} = n + \frac{1}{2} + \frac{\varphi_j}{2\pi}.$$

Proof. First, we show that the sections $\eta_{n,j} \in \mathcal{H}_\gamma$ are in fact smooth by evaluating

$$\eta_{n,j}(2\pi) = e^{-i\pi - i\varphi_j} \otimes \mathrm{pt}_\gamma(2\pi) v_j = -1 \otimes v_j = -\eta_{n,j}(0).$$

Next, we compute the action of the twisted Dirac operator on these sections

$$\begin{aligned}
\not{D}_\gamma \eta_{n,j}(t) &= \left(i\frac{\mathrm{d}}{\mathrm{d}t} e^{-i(n+\frac{1}{2})t - i\frac{\varphi_j}{2\pi}t} \right) \otimes v_j + e^{-i(n+\frac{1}{2})t - i\frac{\varphi_j}{2\pi}t} \otimes \nabla \operatorname{pt}_\gamma(t) v_j \\
&= \left(i\frac{\mathrm{d}}{\mathrm{d}t} e^{-i(n+\frac{1}{2})t - i\frac{\varphi_j}{2\pi}t} \right) \otimes v_j \\
&= \left(n + \frac{1}{2} + \frac{\varphi_j}{2\pi} \right) \eta_{n,j}(t).
\end{aligned}$$

Here we have used that $\nabla \operatorname{pt}_\gamma(t) v_j = 0$. □

Lemma 9.2.2. *The subspace $Eig_{>0}(\not{D}_\gamma) \subset \mathcal{H}_\gamma$ is isotropic and can be embedded in a Lagrangian subspace $L_\gamma \subset \mathcal{H}_\gamma$. The Lagrangian L_γ is not unique if $Eig_{=0}(\not{D}_\gamma)$ is non-trivial. For any choice of such a Lagrangian, the codimension of $Eig_{>0}(\not{D}_\gamma) \subseteq L_\gamma$ is finite.*

Proof. A computation shows that $\alpha(\operatorname{Eig}_{>0}(\not{D}_\gamma)) = \operatorname{Eig}_{<0}(\not{D}_\gamma)$. The fact that $\operatorname{Eig}_{>0}(\not{D}_\gamma)$ is isotropic follows from this and the fact that the basis $\eta_{n,j}$ is orthonormal. We generally have a decomposition

$$\mathcal{H}_\gamma = \operatorname{Eig}_{>0}(\not{D}_\gamma) \oplus \alpha(\operatorname{Eig}_{>0}(\not{D}_\gamma)) \oplus \operatorname{Eig}_{=0}(\not{D}_\gamma),$$

If the zero eigenspace $\operatorname{Eig}_{\gamma=0}(\not{D}_\gamma)$ is non-trivial, then $\operatorname{Eig}_{>0}(\not{D}_\gamma)$ is not Lagrangian. However if $K \subset \operatorname{Eig}_{=0}(\not{D}_\gamma)$ is Lagrangian, then $L_\gamma := \operatorname{Eig}_{>0}(\not{D}_\gamma) \oplus K \subset \mathcal{H}_\gamma$ is Lagrangian. Such a choice of K always exists, because $\operatorname{Eig}_{=0}(\not{D}_\gamma)$ is even-dimensional, ($\lambda_{n,j} = 0$ corresponds to $n = 0$ and $\varphi_j = -\pi$ and $\operatorname{Eig}_{-1}(\operatorname{pt}_\gamma(2\pi))$ is even dimensional), which also immediately shows that the codimension of $\operatorname{Eig}_{>0}(\not{D}_\gamma) \subset L_\gamma$ is finite. However, such a choice is never unique, because $\alpha(K) \subset \operatorname{Eig}_{=0}(\not{D}_\gamma)$ is also Lagrangian. □

Remark 9.2.3. The construction above leads to the reduction $\operatorname{Fr}_L(\mathcal{H}) \to \operatorname{Fr}_{\mathrm{O}}(\mathcal{H})$ as follows. We consider the bundle $\operatorname{Fr}(\mathcal{H})/\operatorname{O}_L(\mathcal{H})$. For each $\gamma \in LM$ we have that the fibre $\operatorname{Fr}_L(\mathcal{H})_\gamma/\operatorname{O}_L(V)$ is the space of Lagrangians in $\mathcal{H}_\gamma$ modulo equivalence. The assignment $\gamma \mapsto [L_\gamma]$, where L_γ is a Lagrangian completion of $\operatorname{Eig}_{>0}(\not{D}_\gamma)$ as in Lemma 9.2.2, is then a section of the bundle $\operatorname{Fr}(\mathcal{H})/\operatorname{O}_L(\mathcal{H})$, which induces the reduction. To make this argument completely rigorous, one would need to prove that the section $\gamma \mapsto [L_\gamma]$ is smooth. An approach to a similar lifting problem in this spirit is carried out in [CMM00, Section 5].

The following result implies that the reduction $\operatorname{Fr}_L(\mathcal{H}) \to \operatorname{Fr}_{\mathrm{O}}(\mathcal{H})$ outlined in Remark 9.2.3 agrees with the one defined in Section 9.1.

Proposition 9.2.4. *Let $\gamma \in LM$ and $\nu \in \operatorname{Fr}_L(\mathcal{H})$ be arbitrary. Let $L_\gamma \subset \mathcal{H}_\gamma$ be any Lagrangian that contains $Eig_{>0}(\not{D}_\gamma)$. Then, the Fock representations $\mathcal{F}_{\nu(L)}$ and $\mathcal{F}_{L_\gamma}$ of $\operatorname{Cl}(\mathcal{H}_\gamma)$ are unitarily equivalent.*

A proof strategy could be as follows. Find an element $\psi \in \operatorname{Fr}_L^U(\mathcal{H})_\gamma$, such that $\psi(L) = L_\gamma$, then a variation on Lemma 9.1.1 tells us that the Lagrangian L_γ is equivalent to any Lagrangian $\nu(L)$. Our actual proof is slightly more complicated, but in the same spirit. We first require the notion of sublagrangian.

Definition 9.2.5. A subspace $L' \subset V$ is a *sublagrangian* if it is isotropic and if $L' \oplus \alpha(L') \subseteq V$ is of finite and even codimension.

Lemma 9.2.6. *Let $L' \subset V$ be a sublagrangian. Then there exists at least one Lagrangian subspace $L \subset V$ such that $L' \subseteq L$. If L_0 and L_1 are two Lagrangians which both contain the sublagrangian L', then L_0 is equivalent to L_1. Finally, let L' and L'' be sublagrangians such that there exists an* *element $g \in \mathrm{U}_L(V)$ such that $g(L') = L''$. Then, if L_0 is a Lagrangian containing L' and L_1 is a Lagrangian containing L'', then L_0 is equivalent to L_1.*

Proof. Let L' be a sublagrangian. Then, by definition, the orthogonal complement of $L' \oplus \alpha(L')$ is of finite and even dimension. Moreover, it is clearly preserved by α, hence it contains a Lagrangian subspace $K \subset V$. It follows that $L' \oplus K \subset V$ is Lagrangian.

We note that the second part of the lemma follows from the last, if we set $L' = L''$ and $g = \mathbb{1}$.

So, let L', L'', L_0, L_1 and g be as in the lemma. A direct computation shows that $P_{L_0}^{\perp} P_{L_1}$ is Hilbert-Schmidt which suffices to prove the claim, according to Lemma 3.6.1. □

Proof of Proposition 9.2.4. Let $\mathrm{U}(M) \to M$ be the bundle of unitary frames of the complexified tangent bundle and let $L\,\mathrm{U}(M) \to LM$ be its looping. From Lemma 3.7.1 we know that $L\,\mathrm{U}(d) \to \mathrm{U}_L(V)$. This allows us to define

$$\mathrm{Fr}_L^{\mathrm{U}}(\mathcal{H}) := L\,\mathrm{U}(M) \times_{L\,\mathrm{U}(d)} \mathrm{U}_L(V).$$

Just like we may view $\mathrm{Fr}_L(\mathcal{H})_\gamma$ as a subset of $\mathrm{O}(V, \mathcal{H}_\gamma)$ we may view $\mathrm{Fr}_L^{\mathrm{U}}(\mathcal{H})_\gamma$ as a subset of $\mathrm{U}(V, \mathcal{H})_\gamma$. We view $\mathrm{Fr}_L(\mathcal{H})$ as a subset of $\mathrm{Fr}_L^{\mathrm{U}}(\mathcal{H})$ using the natural inclusion.

Now, let $\gamma \in LM$ be arbitrary. Let φ_j and v_j be as in Lemma 9.2.1. For each $t \in S^1$ define $\varphi_t \in U(M)_{\gamma(t)}$

$$\varphi_t(e_j) = e^{-i\frac{\varphi_j}{2\pi}t}\,\mathrm{pt}_\gamma(t) v_j.$$

This defines an element $(\gamma, \varphi) \in L\,\mathrm{U}(M)$, and thus an element $\psi \in \mathrm{Fr}_L^U(\mathcal{H})_\gamma \subset \mathrm{U}(V, \mathcal{H}_\gamma)$. Let $\eta_{n,j}$ be as in Lemma 9.2.1. It then follows that $\psi(\xi_{n,j}) = \eta_{n,j}$.

Now, we consider the sublagrangian $L'_\gamma = \mathrm{Eig}_{>0}(\not{D}_\gamma)$ from Lemma 9.2.2. From the characterization

$$\mathrm{Eig}_{>0}(\not{D}_\gamma) = \mathrm{span}\{\eta_{n,j} \mid n \geqslant 1, j = 1, ..., d\} \oplus \mathrm{span}\{\eta_{n,j} \mid n = 0, j : \varphi_j \neq -1\},$$

it follows readily that $L' := \psi^{-1}(L'_\gamma) \subset V$ is a sublagrangian which is contained in our standard Lagrangian L. The map $\nu\psi^{-1}$ clearly maps the sublagrangian L'_γ to the sublagrangian $\nu(L')$. Now, according to Lemma 3.6.1 and Lemma 9.2.6 it suffices to prove that $[\nu\psi^{-1}, P_{\nu(L)}]$ is Hilbert-Schmidt, which is equivalent to the statement that $\nu^{-1}[\nu\psi^{-1}, P_{\nu(L)}]\nu$ is Hilbert-Schmidt, because ν is unitary. We compute

$$\begin{aligned}\nu^{-1}[\nu\psi^{-1}, P_{\nu(L)}]\nu &= \nu^{-1}[\nu\psi^{-1}, \nu P_L \nu^{-1}]\nu \\ &= \psi^{-1}\nu P_L - P_L \nu\psi^{-1} \\ &= [\psi^{-1}\nu, P_L],\end{aligned}$$

this is Hilbert-Schmidt, because $\psi, \nu \in \mathrm{Fr}_L^U(\mathcal{H})$, and hence $\psi^{-1}\nu \in \mathrm{U}_L(V)$. □

A. Central extensions of Banach Lie groups

In this section we provide the following well-known result used in the proof of Theorem 3.5.2. See [Nee02, Proposition 4.2] for a similar statement.

Proposition A.1. *Let G be a Banach Lie group with finitely many connected components, let Z be an abelian Banach Lie group, and let*

$$1 \to Z \to \widehat{G} \xrightarrow{q} G \to 1 \tag{A.1}$$

be a central extension of groups. Let $U \subset G$ be an open 1-neighbourhood supporting a section σ, i.e. a map $\sigma : U \to \hat{G}$ such that $q \circ \sigma = id_G$. Suppose there exists an open 1-neighbourhood $V \subset U$ with $V^2 \subset U$, such that the associated 2-cocycle $f_\sigma : V \times V \to Z$ defined by

$$\sigma(g_1)\sigma(g_2) = f_\sigma(g_1, g_2)\sigma(g_1 g_2)$$

is smooth in an open $(1,1)$-neighbourhood. Then, $\widehat{G}$ carries a unique Banach Lie group structure such that σ is smooth in an open 1-neighbourhood. Moreover, when equipped with this Banach Lie group structure, (A.1) is a central extension of Banach Lie groups.

We will use the following lemma, which appears as [Nee02, Lemma 4.1] or as [Tit13, p.14] in the finite-dimensional case, which goes through without changes.

Lemma A.2. *Let G be a group and $K \subset G$ be a subset with $1 \in K$ and $K = K^{-1}$. We assume that K is a Banach manifold such that the inversion is smooth on K and there exists an open 1-neighbourhood $V \subseteq K$ with $V^2 \subseteq K$, such that the multiplication $m : V \times V \to K$ is smooth. Further, we assume that for any $g \in G$ the conjugation map $C_g : G \to G : x \mapsto gxg^{-1}$ is smooth in an open 1-neighbourhood. Then, there exists a unique Banach Lie group structure on G such that the inclusion map $K \hookrightarrow G$ is a local diffeomorphism at 1.*

Now we give the proof of Proposition A.1, first assuming that G is connected. Without loss of generality we assume that U satisfies $U^{-1} = U$. We set $K \coloneqq U \times Z$. We equip K with the product Banach manifold structure, and identify it with a subset of $\hat{G}$ along the injective map $(u, z) \mapsto \sigma(u)z$. We consider the open subset

$$W \coloneqq \{((x_1, z_1), (x_2, z_2)) \mid x_1 x_2 \in U\} \subset K \times K,$$

and choose an open 1-neighbourhood V in K such that $V \times V \subset W$. The definition of f_σ implies that the restriction of the group structure of $\hat{G}$ to $V \times V$ is given by

$$((x_1, z_1), (x_2, z_2)) \mapsto (x_1 x_2, z_1 z_2 f(x_1, x_2)),$$

which is smooth. Likewise, the inversion map on K is

$$(x, z) \mapsto (x^{-1}, z^{-1} f(x, x^{-1})^{-1}),$$

and hence smooth, too.

Next, we claim that for any $\widehat{g} \in \widehat{G}$ the map $C_{\widehat{g}} : \hat{G} \to \hat{G} : x \mapsto \widehat{g}x\widehat{g}^{-1}$ is smooth in an open 1-neighbourhood. To this end, let $X \subseteq V$ be an again smaller open 1-neighbourhood such that $X^3 \subseteq V$ and $X^{-1} = X$. It then follows that for $\widehat{g} \in X$ we have that $C_{\widehat{g}} : X \to K$ is smooth. Using the assumption that G is connected, it follows that X generates $\widehat{G}$, and hence that any $\widehat{g} \in \widehat{G}$ can be decomposed as $\widehat{g} = \widehat{g}_1 ... \widehat{g}_n$ with $n \in \mathbb{N}$ and $\widehat{g}_1, ..., \widehat{g}_n \in X$. It follows that $C_{\widehat{g}} = C_{\widehat{g}_1} ... C_{\widehat{g}_n}$ is smooth. We see that now all the conditions of Lemma A.2 are met, and it follows that there is a unique Banach Lie group structure on $\widehat{G}$ such that the inclusion $K \hookrightarrow \widehat{G}$ is a local diffeomorphism at $\mathbb{1}$.

Finally, we consider the case that G is not connected, but has finitely many connected components. We apply the previous result to the connected component G^0 of the identity. If $X \subset G$ is another connected component, then any element $\hat{g} \in q^{-1}(X)$ determines a bijection between $q^{-1}(G^0)$ and $q^{-1}(X)$, hence equipping $q^{-1}(X)$ with a Banach manifold structure, which is readily verified to be independent of the choice of $\hat{g}$. It is also easy to see that multiplication and inversion are smooth.

To complete the proof we now need to prove that $\widehat{G} \to G$ is a smooth principal Z-bundle, which boils down to prove that $q : \widehat{G} \to G$ is a smooth surjective submersion. This is true in the open $\mathbb{1}$-neighbourhood K, and hence everywhere since it is a group homomorphism.

B. Regularity of the action of the Clifford algebra on Fock space

Here, we complete the proof of Lemma 3.5.7. For the notions of derivatives and differentiability used here, we refer to [Mil83, Section 3] or [Ham82, Sections I.3.1 & I.3.6].

We start by proving the following lemma.

Lemma B.1. *Let X, Y, Z be Fréchet spaces, and let $\mu : X \times Y \to Z$ be a bilinear map. If μ is continuous, then it is smooth.*

Proof. Let $x_0, x_1 \in X$ and $y_0, y_1 \in Y$. Then we compute the derivative of μ at (x_0, y_0) in the direction of (x_1, y_1) as follows

$$\begin{aligned} D_{(x_0,y_0)}\mu(x_1, y_1) &:= \lim_{t\to 0} \frac{1}{t} \left(\mu(x_0 + tx_1, y_0 + ty_1) - \mu(x_0, y_0)\right) \\ &= \lim_{t\to 0} \left(\mu(x_0, y_1) + \mu(x_1, y_0) + t\mu(x_1, y_1)\right) \\ &= \mu(x_0, y_1) + \mu(x_1, y_0). \end{aligned}$$

Clearly the map $X^2 \times Y^2 \ni (x_0, x_1, y_0, y_1) \mapsto D_{(x_0,y_0)}\mu(x_1, y_1)$ is continuous, hence μ is continuously differentiable. Now let $x_2 \in X$ and $y_2 \in Y$. We consider the second derivative

$$\begin{aligned} D^2_{(x_0,y_0)}((x_1, y_1), (x_2, y_2)) &:= \lim_{t\to 0} \frac{1}{t} \left(D_{(x_0+tx_2,y_0+ty_2)}\mu(x_1, y_1) - D_{(x_0,y_0)}\mu(x_1, y_1)\right) \\ &= \lim_{t\to 0} \frac{1}{t} \left(\mu(x_0 + tx_2, y_1) + \mu(x_1, y_0 + ty_2) - \mu(x_0, y_1) + \mu(x_1, y_0)\right) \\ &= \mu(x_2, y_1) + \mu(x_1, y_2), \end{aligned}$$

which is obviously a continuous function on $X^3 \times Y^3$. Finally, because the second derivative does not depend on the basepoint (x_0, y_0) all higher derivatives are identically zero, and we are done. □

Let us write $\rho : \mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}^{\mathrm{s}} \to \mathcal{F}^{\mathrm{s}}$ for the action map. In the remainder of this section we shall prove that ρ is continuous, from which, using Lemma B.1, it follows that ρ is smooth.

We shall consider the family of semi-norms on $\mathcal{F}^{\mathrm{s}}$ and on $\mathrm{Cl}(V)^{\mathrm{s}}$ defined in [Nee10a, Section 4], that define the topology. Let $X \in \mathfrak{imp}(V)$, and let $v \in \mathcal{F}^{\mathrm{s}}$, then we define

$$Xv := \mathrm{d}\pi(X)(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)(v).$$

Similarly, for $X = (Y, \lambda) \in \mathfrak{imp}(V) = \mathfrak{o}_L(V) \oplus \mathbb{R}$ and $a \in \mathrm{Cl}(V)^{\mathrm{s}}$ we define

$$Xa := \mathrm{d}\theta(Y)(a) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta_{\exp(tY)}(a),$$

We then have

$$X(a \triangleright v) = Xa \triangleright v + a \triangleright Xv. \tag{B.1}$$

Let $\mathcal{P}(\mathcal{F})$ be the set of continuous semi-norms on $\mathcal{F}$. The topology on $\mathcal{F}^{\mathrm{s}}$ is then defined by the family of semi-norms, [Nee10a, Section 4],

$$p_n(v) = \sup\{p(X_1 \dots X_n v) \mid X_i \in \mathfrak{imp}(V), \|X_i\| \leqslant 1\}, \quad p \in \mathcal{P}(\mathcal{F}), n \in \mathbb{N}_0.$$

Similarly, let $\mathcal{R}(\mathrm{Cl}(V))$ be the set of continuous semi-norms on $\mathrm{Cl}(V)$. The topology on $\mathrm{Cl}(V)^{\mathrm{s}}$ is then defined by the family of semi-norms

$$r_n(a) = \sup\{r(X_1 \dots X_n a) \mid X_i \in \mathfrak{imp}(V), \|X_i\| \leqslant 1\}, \quad r \in \mathcal{R}(\mathrm{Cl}(V)), n \in \mathbb{N}_0.$$

For convenience of notation, we write s for the norm on both $\mathcal{F}$ and $\mathrm{Cl}(V)$, so that we have, for $v \in \mathcal{F}^{\mathrm{s}}$ and $a \in \mathrm{Cl}(V)^{\mathrm{s}}$

$$\begin{aligned} s_n(v) &= \sup\{\|X_1 \dots X_n v\| \mid X_i \in \mathfrak{imp}(V), \|X_i\| \leqslant 1\}, \quad n \in \mathbb{N}_0, \\ s_n(a) &= \sup\{\|X_1 \dots X_n a\| \mid X_i \in \mathfrak{imp}(V), \|X_i\| \leqslant 1\}, \quad n \in \mathbb{N}_0. \end{aligned}$$

Now, let $U = \{v \in \mathcal{F}^{\mathrm{s}} \mid p_n(v) < \varepsilon\}$ be a subbasis open around 0. It suffices to prove that for each point $(a, v) \in \rho^{-1}(U)$ there exists an open neighbourhood W of (a, v) which is contained in $\rho^{-1}(U)$.

Equation (B.1) implies that for $X_1, ..., X_n \in \mathfrak{imp}(V)$ we have the following non-commutative binomial expansion

$$X_1 ... X_n(a \triangleright v) = \sum_{k=0}^{n} \sum_{\sigma \in \mathfrak{S}_k^n} X_{\sigma(1)} ... X_{\sigma(k)} a \triangleright X_{\sigma(k+1)} ... X_{\sigma(n)} v,$$

where $\mathfrak{S}_k^n$ is the set of bijections $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ such that the restrictions $\sigma : \{1, ..., k\} \to \{1, ..., n\}$ and $\sigma : \{k+1, ..., n\} \to \{1, ..., n\}$ are order preserving. We now suppose that $\|X_i\| \leqslant 1$ and compute for arbitrary $\sigma \in \mathfrak{S}_k^n$

$$\begin{aligned} \|X_{\sigma(1)} ... X_{\sigma(k)} a \triangleright X_{\sigma(k+1)} ... X_{\sigma(n)} v\| &\leqslant \|X_{\sigma(1)} ... X_{\sigma(k)} a\| \|X_{\sigma(k+1)} ... X_{\sigma(n)} v\| \\ &\leqslant s_k(a) s_{n-k}(v). \end{aligned}$$

From which we obtain the estimate

$$\begin{aligned} \|X_1 \dots X_n(a \triangleright v)\| &\leqslant \sum_{k=0}^{n} \sum_{\sigma \in \mathfrak{S}_k^n} s_k(a) s_{n-k}(v) \\ &= \sum_{k=0}^{n} \binom{n}{k} s_k(a) s_{n-k}(v). \end{aligned}$$

Lemma B.2. *The action map ρ is continuous.*

Proof. Because ρ is bilinear it suffices to prove that it is continuous in $(0, 0) \in \mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}^{\mathrm{s}}$. A subbasis of open neighbourhoods of zero in $\mathcal{F}$ is given by the open neighbourhoods

$$U(p, n, \varepsilon) = \{v \in \mathcal{F}^{\mathrm{s}} \mid p_n(v) < \varepsilon\}, \quad p \in \mathcal{P},\ n \in \mathbb{N}_0,\ \varepsilon > 0.$$

We assume, without loss of generality, that $p \leqslant \| \cdot \|$. Hence, let $W := U(p, n, \varepsilon)$, with p, n, ε as above, be arbitrary. We shall prove that $\rho^{-1}(W)$ is open by finding for each $(a, v) \in \rho^{-1}(W)$ an open neighbourhood of (a, v) contained in $\rho^{-1}(W)$. Indeed, for each $k = 0, ..., n$ we consider the open neighbourhoods $Z_k = \{b \in \mathrm{Cl}(V)^{\mathrm{s}} \mid s_k(b - a) < \varepsilon \min\{2^{-n}, 2^{-n}/s_{n-k}(v)\}/3\}$ of a, and $Z'_k = \{w \in \mathcal{F}^{\mathrm{s}} \mid s_{n-k}(w - v) < \varepsilon \min\{2^{-n}, 2^{-n}/s_k(a)\}/3\}$ of v. The intersection $Z := \cap_k Z_k \times Z'_k$ is then an open neighbourhood of (a, v). Let $(b, w) \in \mathrm{Cl}(V)^{\mathrm{s}} \times \mathcal{F}^{\mathrm{s}}$ be arbitrary, then we compute

$$\rho(b, w) - \rho(a, v) = (b - a) \triangleright (w - v) + a \triangleright (w - v) + (b - a) \triangleright v.$$

We thus obtain the estimate, for $(b,w) \in Z$ and $X_1, \dots X_n \in \mathfrak{imp}(V)$ with $\|X_i\| \leqslant 1$,

$$\begin{aligned}
p(X_1 ... X_n(\rho(b,w) - \rho(a,v))) &\leqslant \|X_1 \dots X_n(\rho(b,w) - \rho(a,v))\| \\
&\leqslant \|X_1 \dots X_n((b-a) \triangleright (w-v))\| + \|X_1 \dots X_n(a \triangleright (w-v))\| \\
&\qquad + \|X_1 \dots X_n((b-a) \triangleright v)\| \\
&\leqslant \sum_{k=0}^{n} \binom{n}{k} \left(\begin{array}{l} s_k(b-a)s_{n-k}(w-v) \\ + s_k(a)s_{n-k}(w-v) \\ + s_k(b-a)s_{n-k}(v) \end{array} \right) \\
&< \frac{\varepsilon}{3} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2^{-2n}}{3} + 2^{-n} + 2^{-n} \right) \\
&= \frac{\varepsilon}{3} \left(2 + \frac{2^{-n}}{3} \right) \\
&< \varepsilon
\end{aligned}$$

Which implies that

$$p_n(\rho(b,w) - \rho(a,v)) = \sup\{p(X_1 ... X_n(\rho(b,w) - \rho(a,v))) \mid X_i \in \mathfrak{imp}(V), \|X_i\| \leqslant 1\} < \varepsilon.$$

Hence $Z \subseteq \rho^{-1}(W)$, and we are done. $\square$

C. Modular conjugation in the free fermions

In this section we give a proof of Proposition 5.2.2, i.e. we compute the modular conjugation J_Ω for the triple $(\mathrm{Cl}(V_-)'', \mathcal{F}, \Omega)$, see Section 5.2 for the notation. The result and possible computations are probably well-known, and have appeared in slight variations of the setting in [Was98, Section 15], [Hen14], and [Jan13]. In the latter reference, Janssens outlines how to transfer his computations into our setting, and in the following we have done this step by step, closely following [Jan13].

The strategy will be to find a unitary operator $s : L \to L$ such that the Tomita operator S is $S = \mathrm{k}^{-1}\,\Lambda_{i\alpha s}$, and then to find a polar decomposition for s. For simplicity, we set $d = 1$, and thus only work with $V = L^2(\mathbb{S})$. All statements carry over to $L^2(\mathbb{S}) \otimes \mathbb{C}^d$ in a straightforward fashion. Further, it will be convenient to identify V with the L^2-closure of $C^\infty(S^1, \mathbb{C})$, see Section 2.2.

Let us write $P_L : V \to L$, $P_L^\perp : V \to \alpha(L) = L^\perp$ and $P_\pm : V \to V_\pm$ for the orthogonal projections. We then define the operators

$$T_\pm := (P_L - P_\pm)^2,$$

on V.

Lemma C.1. *The operators $T_\pm$ have unbounded inverses $T_\pm^{-1}$.*

We will later diagonalize the operators $T_\pm$; the fact that these operators have unbounded inverses will be evident from their diagonal form. For now, we assume that this lemma holds.

We recall that the Tomita operator is $S : a \triangleright \Omega \mapsto a^* \triangleright \Omega$. Suppose that $s : V \to V$ is an operator with the properties that $s(L) = \alpha(L)$, and that $v + s(v) \in V_-$ for $v \in L$. Now, let $v \in L \subset \mathcal{F}$. We compute $v = v \triangleright \Omega = (v + s(v)) \triangleright \Omega$, and hence $S(v) = S((v + s(v)) \triangleright \Omega) = (\alpha s(v) + \alpha(v)) \triangleright \Omega = \alpha s(v)$. This means that $S|_L = \mathrm{k}^{-1}\,\Lambda_{i\alpha s}|_L$, in Lemma C.4 we shall see that $S = \mathrm{k}^{-1}\,\Lambda_{i\alpha s}$. It turns out that the densely defined operator s on V defined by

$$s := P_L P_- T_-^{-1} P_L^\perp + P_L^\perp P_- T_+^{-1} P_L,$$

does the trick, as we show below.

Lemma C.2. *The operator s commutes with α.*

Proof. Direct computation using the fact that $\alpha P_L = P_L^\perp \alpha$, and $\alpha P_\pm = P_\pm \alpha$, which implies that $\alpha T_\pm = T_\mp \alpha$. □

Lemma C.3. *The operator s maps $v \in L$ to the unique $w \in \alpha(L)$ such that $v + w \in V_-$, if such a w exists. Similarly, it maps $w \in \alpha(L)$ to the unique $v \in L$ such that $v + w \in V_-$, again if such a v exists.*

Proof. Let $v \in L$ be arbitrary. First let us prove the uniqueness claim. Suppose that there exist $w, w' \in \alpha(L)$ such that $v + w \in V_-$ and $v + w' \in V_-$. Then it follows that $(v + w) - (v + w') = w - w' \in V_- \cap \alpha(L) = \{0\}$ by Lemma 5.1.2. By direct computation one might verify that

$$v + P_L^\perp P_- T_+^{-1} v = P_- T_+^{-1} v.$$

And hence

$$v + sv = v + P_L^\perp P_- T_+^{-1} v = P_- T_+^{-1} v \in V_-.$$

This proves the first statement in the lemma. The second statement follows from a similar computation. □

As a consequence, we see that the operator s squares to $\mathbb{1}$ and restricts to the identity on V_-. The following result tells us precisely how αs is related to S.

Lemma C.4. *For all $a \in \mathrm{Cl}(V_-)$ we have $\mathrm{k}^{-1}\,\Lambda_{i\alpha s}(a \triangleright \Omega) = a^* \triangleright \Omega = S(a \triangleright \Omega)$.*

Proof. We will prove this by induction on the degree in $\mathrm{Cl}(V_-)$ (note that while the algebra $\mathrm{Cl}(V_-)$ is not graded, it is filtered). Suppose that the claim holds for all $a \in \mathrm{Cl}(V_-)$ for a of degree n or less. We shall prove that it follows that for all $f_0, \dots, f_n \in V_-$ we have that

$$\mathrm{k}^{-1}\,\Lambda_{i\alpha s}(f_0 \dots f_n \triangleright \Omega) = S(f_0 \dots f_n \triangleright \Omega).$$

First off, we set $x := f_1 \dots f_n \triangleright \Omega$. Furthermore, there exist $y_i \in \Lambda^i L$, where $i = 0, ..., n$ such that $x = \sum_{i=0}^n y_i$. Finally, we set $v = P_L f_0$ and $w = P_L^\perp f_0$. Note that $sv = w$ and $sw = v$. Now we compute

$$\begin{aligned}\mathrm{k}^{-1}\,\Lambda_{i\alpha s}(f_0 \dots f_n \triangleright \Omega) &= \mathrm{k}^{-1}\,\Lambda_{i\alpha s}(f_0 \triangleright x)\\ &= \mathrm{k}^{-1}\,\Lambda_{i\alpha s}(v \wedge x + \iota_{\alpha w} x)\\ &= \mathrm{k}^{-1}(i\alpha(w) \wedge \Lambda_{i\alpha s} x + \Lambda_{i\alpha s}\iota_{\alpha(w)} x).\end{aligned}$$

Straightforward computations, using the induction hypothesis, then show that

$$\mathrm{k}^{-1}(i\alpha(w) \wedge \Lambda_{i\alpha s} x) = \alpha(f_n)\dots\alpha(f_0) \triangleright \Omega + \sum_{k=1}^{n} (-1)^k \langle f_k, \alpha(w)\rangle\, \mathrm{k}^{-1}\,\Lambda_{i\alpha s} f_1 \dots \widehat{f_k} \dots f_n \triangleright \Omega,$$

and that

$$\mathrm{k}^{-1}\,\Lambda_{i\alpha s}\iota_{\alpha(w)} x = -\sum_{k=1}^{n} (-1)^k \langle f_k, \alpha(w)\rangle\, \mathrm{k}^{-1}\,\Lambda_{i\alpha s} f_1 \dots \widehat{f_k} \dots f_n \triangleright \Omega.$$

Putting these results together we see that

$$\begin{aligned}\mathrm{k}^{-1}\,\Lambda_{i\alpha s}(f_0 \dots f_n \triangleright \Omega) &= \mathrm{k}^{-1}(i\alpha(w) \wedge \Lambda_{i\alpha s} x + \Lambda_{i\alpha s}\iota_{\alpha(w)} x)\\ &= \alpha(f_n)\dots\alpha(f_0) \triangleright \Omega\\ &= S(f_0 \dots f_n \triangleright \Omega),\end{aligned}$$

which completes the induction step, and hence our proof. □

Let $\alpha s = u\delta^{1/2}$ be the polar decomposition of αs. Note that the fact that αs preserves L implies that both u and $\delta^{1/2}$ preserve L. Similarly, the fact that αs commutes with α implies that both u and $\delta^{1/2}$ commute with α. From the equality $S = \mathrm{k}^{-1}\,\Lambda_{i\alpha s}$ we obtain the following.

Proposition C.5. *The polar decomposition of S is given by $S = \mathrm{k}^{-1}\,\Lambda_{iu}\Lambda_{\delta^{1/2}}$, whence $J = \mathrm{k}^{-1}\,\Lambda_{iu}$ and $\Delta^{1/2} = \Lambda_{\delta^{1/2}}$.*

We claim that $u|_L = -i\alpha\tau$, which implies that $J = \mathrm{k}^{-1}\,\Lambda_{\alpha\tau}$. The claim is proved in a sequence of lemmas in the remainder of this section.

Lemma C.6. *The equations*

$$u = \alpha \left(\frac{P_L^\perp P_- P_L}{\sqrt{T_+ T_-}} + \frac{P_L P_- P_L^\perp}{\sqrt{T_+ T_-}} \right), \qquad \text{and} \qquad \delta^{1/2} = \sqrt{\frac{T_-}{T_+}} P_L + \sqrt{\frac{T_+}{T_-}} P_L^\perp$$

hold.

Proof. The fact that $u\delta^{1/2} = \alpha s$ follows from a straightforward computation. Furthermore, the fact that u is anti-unitary can be verified directly as well. The fact that $\delta^{1/2}$ is positive will be evident from an expression that we will give later. □

We now turn to the task of simultaneously diagonalizing $T_\pm$, and proving that they have densely defined inverses. First, we identify the circle with the one-point compactification of the real line by means of the diffeomorphisms

$$\Gamma : S^1 \to \mathbb{R} \cup \infty, z \mapsto -i\frac{z+1}{z-1}, \qquad \Gamma^{-1} : \mathbb{R} \cup \infty \to S^1, x \mapsto \frac{x-i}{x+i}.$$

We note that $\Gamma(I_+) = \mathbb{R}_+$ and $\Gamma(I_-) = \mathbb{R}_-$. We define unitary transformations

$$U_\Gamma : L^2(S^1, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C}), \qquad U_\Gamma^{-1} : L^2(\mathbb{R}, \mathbb{C}) \to L^2(S^1, \mathbb{C}),$$

by

$$U_\Gamma(f)(x) = \frac{1}{\sqrt{\pi}} \frac{1}{(i+x)} f(\Gamma^{-1}x), \qquad x \in \mathbb{R},$$
$$U_\Gamma^{-1}(g)(z) = \sqrt{\pi}(i + \Gamma(z))g(\Gamma(z)), \qquad z \in S^1.$$

We recall that on $L^2(S^1, \mathbb{C})$ the maps α and τ act as follows

$$\alpha(f)(z) = \frac{1}{z}\overline{f(z)}, \quad \tau(f)(z) = \frac{1}{z} f(\overline{z}).$$

We compute how α and τ transform under U_Γ:

$$U_\Gamma \alpha U_\Gamma^{-1} g(x) = \overline{g(x)}, \qquad U_\Gamma \tau U_\Gamma^{-1} g(x) = g(-x).$$

Let us write $\mathbb{H}_\pm = \{z \in \mathbb{C} \mid \pm\Im(z) > 0\}$ for the upper and lower half plane.

Lemma C.7. *We have*

$$U_\Gamma(L \cap C^\infty(S^1, \mathbb{C})) \subseteq \{f \in L^2(\mathbb{R}, \mathbb{C}) \mid f \text{ extends to a holomorphic function } f : \overline{\mathbb{H}_-} \to \mathbb{C}\}.$$

Proof. We see that $L \cap C^\infty(S^1, \mathbb{C})$ is the span of the functions z^{-n-1} for $n \geqslant 0$. We then compute

$$(U_\Gamma z^{-n-1})(x) = \frac{1}{\sqrt{\pi}} \frac{(x+i)^n}{(x-i)^{n+1}}.$$

These functions are smooth and square integrable for all $n \in \mathbb{Z}$. Furthermore if $n \geqslant 0$, then $U_\Gamma z^{-n-1}$ extends to a holomorphic function on the lower half plane. □

Given a smooth function $g = U_\Gamma f \in C^\infty(\mathbb{R}, \mathbb{C})$ with $f \in C^\infty(S^1, \mathbb{C})$, we wish to find two holomorphic functions, say $\hat{g}_+ : \mathbb{H}_+ \to \mathbb{C}$ and $\hat{g}_- : \mathbb{H}_- \to \mathbb{C}$ which extend to square-integrable functions on the real line (denoted by the same name), such that $g = \hat{g}_+|_\mathbb{R} + \hat{g}_-|_\mathbb{R}$. This is essentially a version of the Riemann-Hilbert problem, we follow the standard solution to such problems.

For $g \in L^2(\mathbb{R},\mathbb{C})$ we define the Cauchy transform

$$\hat{g}(z) := \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(x)}{x-z}\mathrm{d}x, \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

The function $\hat{g} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$ is holomorphic. The following lemma is then a well-known consequence of the Sokhotski-Plemelj theorem, [Gak66, Section 4.2], [Mus58, Section 17], [Ple64, Chapter 14].

Lemma C.8. *Let $f \in C^\infty(S^1,\mathbb{C})$, and set $g = U_\Gamma f$. Then $g_- = U_\Gamma P_L f$ and $g_+ = U_\Gamma P_L^\perp f$ are smooth functions on $\mathbb{R}$ that extend uniquely to holomorphic functions on the lower and upper half plane, G_- and G_+, respectively. Furthermore, we have $G_- = -\hat{g}|_{\mathbb{H}_-}$ and $G_+ = \hat{g}|_{\mathbb{H}_+}$.*

For any operator X on V, let us write $X' := U_\Gamma X U_\Gamma^{-1}$. As a consequence of Lemma C.8 we obtain for $g = U_\Gamma f$ with $f \in C^\infty(S^1,\mathbb{C})$:

$$\begin{aligned}(P_L' g)(x) &= -\lim_{\varepsilon\downarrow 0}\hat{g}(x - i\varepsilon), && x \in \mathbb{R},\\ ((P_L')^\perp g)(x) &= \lim_{\varepsilon\downarrow 0}\hat{g}(x + i\varepsilon), && x \in \mathbb{R}.\end{aligned}$$

Next, we define the unitary $D : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ by

$$(Df)(t) = (f_r(t), f_l(t)) := (e^{t/2}f(e^t), e^{t/2}f(-e^t)).$$

The inverse of D is given by

$$D^{-1}(f_r, f_l)(t) = \begin{cases}\frac{1}{\sqrt{t}}f_r(\log(t)) & t > 0,\\ \frac{1}{\sqrt{-t}}f_l(\log(-t)) & t < 0.\end{cases}$$

Lemma C.9. *We have*

$$DP_+'D^{-1} = \begin{pmatrix}\mathbb{1} & 0\\ 0 & 0\end{pmatrix}, \qquad DP_-'D^{-1} = \begin{pmatrix}0 & 0\\ 0 & \mathbb{1}\end{pmatrix}.$$

Proof. Recall that Γ^{-1} carries $\mathbb{R}_\pm$ into $I_\pm$. It follows that $P_\pm'$ is the projection $L^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R}_\pm,\mathbb{C})$, from which the result follows. □

Let us write

$$D(P_L^\perp)'D^{-1} = \begin{pmatrix}P_L^{rr} & P_L^{rl}\\ P_L^{lr} & P_L^{ll}\end{pmatrix}.$$

We define

$$c_\varepsilon(u) := \frac{e^{-u/2}}{e^{-u} + 1 - i\varepsilon}, \qquad s_\varepsilon(u) := \frac{e^{-u/2}}{e^{-u} - 1 - i\varepsilon}.$$

Lemma C.10. *We have, for all $t \in \mathbb{R}$,*

$$\begin{aligned}P_L^{rr}f_r(t) &= \frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}(f_r \star s_\varepsilon)(t), & P_L^{rl}f_l(t) &= -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}(f_l \star c_{-\varepsilon})(t),\\ P_L^{lr}f_r(t) &= \frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}(f_r \star c_\varepsilon)(t), & P_L^{ll}f_l(t) &= -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}(f_l \star s_{-\varepsilon})(t),\end{aligned}$$

where $\star$ stands for the convolution product.

Proof. Straightforward, but tedious, computations. □

Next, we take the Fourier transforms of s_ε, and c_ε, where we use the following convention for the Fourier transform

$$\mathcal{F}f(k) := \int_{-\infty}^{\infty} f(u)e^{-iku}\mathrm{d}u.$$

The Fourier transforms of s_ε and c_ε can be computed using the residue theorem, alternatively, they can be found in [Bat54, Section 3.2 (15)]. For s_ε the result depends on the sign of ε, suppose that $1/2 > \varepsilon^+ > 0$ and $-1/2 < \varepsilon^- < 0$, and $-1/2 < \varepsilon < 1/2$, then we obtain

$$\mathcal{F}s_{\varepsilon^+}(k) = 2\pi i(1+i\varepsilon^+)^{ik-1/2}\frac{e^{\pi k}}{e^{-\pi k}+e^{\pi k}},$$
$$\mathcal{F}s_{\varepsilon^-}(k) = -2\pi i(1+i\varepsilon^-)^{ik-1/2}\frac{e^{-\pi k}}{e^{-\pi k}+e^{\pi k}},$$
$$\mathcal{F}c_{\varepsilon}(k) = 2\pi(1-i\varepsilon)^{ik-1/2}\frac{1}{e^{\pi k}+e^{-\pi k}}.$$

It follows that

$$(\mathcal{F}P_L^{rr}f_r)(k) = \left(\mathcal{F}\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}(f_r\star s_\varepsilon)\right)(k) = \frac{e^{\pi k}}{e^{\pi k}+e^{-\pi k}}\mathcal{F}(f_r)(k).$$

Similarly, we obtain

$$(\mathcal{F}P_L^{ll}f_l)(k) = \frac{e^{-\pi k}}{e^{\pi k}+e^{-\pi k}}\mathcal{F}(f_l)(k), \quad k\in\mathbb{R}.$$

Performing the limit $\varepsilon\downarrow 0$ and multiplying with $(2\pi i)^{-1}$ we obtain

$$(\mathcal{F}P_L^{lr}f_r)(k) = \frac{-i}{e^{\pi k}+e^{-\pi k}}\mathcal{F}(f_r)(k), \quad k\in\mathbb{R}.$$

Setting $\hat{X} := (\mathcal{F}\oplus\mathcal{F})DX'D^{-1}(\mathcal{F}^{-1}\oplus\mathcal{F}^{-1})$ we obtain

$$\hat{P}_L^\perp = \frac{1}{e^{\pi k}+e^{-\pi k}}\begin{pmatrix} e^{\pi k} & i \\ -i & e^{-\pi k}\end{pmatrix}.$$

In a similar manner one could compute $\hat{P}_L^\perp$, but it follows from the fact that $\mathbb{1} = \hat{P}_L + \hat{P}_L^\perp$ that

$$\hat{P}_L = \frac{1}{e^{\pi k}+e^{-\pi k}}\begin{pmatrix} e^{-\pi k} & -i \\ i & e^{\pi k}\end{pmatrix}.$$

We set

$$a(k) = \frac{1}{e^{\pi k}+e^{-\pi k}}.$$

We then obtain

$$\hat{T}_+ = a(k)e^{\pi k}\mathbb{1}, \qquad \hat{T}_- = a(k)e^{-\pi k}\mathbb{1}.$$

As promised, it is clear from these expression that $\hat{T}_\pm$ are injective operators with unbounded inverses; this proves Lemma C.1. We furthermore have

$$\hat{\delta}^{1/2} = \sqrt{\frac{\hat{T}_-}{\hat{T}_+}}\hat{P}_L + \sqrt{\frac{\hat{T}_+}{\hat{T}_-}}\hat{P}_L^\perp = a(k)\begin{pmatrix} a(2k)^{-1} & i(e^{\pi k}-e^{-\pi k}) \\ -i(e^{\pi k}-e^{-\pi k}) & 2\mathbb{1}\end{pmatrix}$$

which is a positive operator, hence the expression $u\delta^{1/2} = \alpha s$ really is the polar decomposition of αs, this was the missing part in the proof of Lemma C.6. Next, we compute

$$\alpha u = \left(\frac{P_L^\perp P_- P_L}{\sqrt{T_+T_-}} + \frac{P_L P_- P_L^\perp}{\sqrt{T_+T_-}}\right) = a(k)\begin{pmatrix} -2\mathbb{1} & i(e^{\pi k}-e^{-\pi k}) \\ -i(e^{\pi k}-e^{-\pi k}) & 2\mathbb{1}\end{pmatrix}.$$

On the other hand, we compute

$$i\hat{\tau}(\hat{P}_L - \hat{P}_L^{\perp}) = a(k)\begin{pmatrix} -2\mathbb{1} & i(e^{\pi k} - e^{-\pi k}) \\ -i(e^{\pi k} - e^{-\pi k}) & 2\mathbb{1} \end{pmatrix}.$$

Which allows us to conclude that $u = -i\alpha\tau(P_L - P_L^{\perp})$, and hence $u|_L = -i\alpha\tau$; from Proposition C.5 it follows that $J = \mathrm{k}^{-1}\,\Lambda(\alpha\tau)$.

D. Bundle gerbes and their modules

In this chapter we recall some of the basics on bundle gerbes and their modules. Bundle gerbes are mentioned in Chapter 7 and used more extensively in Section 9.1. Main references are [Mur96, BCM+02, Mur10].

Let M be a Fréchet manifold/topological manifold. Before giving the definition of a bundle gerbe, we recall some notation that was also used in Chapter 6. If $\pi : Y \to M$ is a surjective submersion, then we write $Y^{[n]}$ for the n-fold fibre product $Y \times_\pi .... \times_\pi Y$. We write $\pi_i : Y^{[n]} \to Y^{[n-1]}$ for the map where we omit the i-th factor.

Definition D.1. A *bundle gerbe* with band U(1) is a triple $(\mathcal{L}, Y, m)$ consisting of

- a topological manifold Y equipped with a surjective submersion $\pi : Y \to M$.
- a principal U(1)-bundle $\mathcal{L} \to Y^{[2]}$.
- an isomorphism $m : \pi_3^*\mathcal{L} \otimes \pi_1^*\mathcal{L} \to \pi_2^*\mathcal{L}$, called the multiplication of the bundle gerbe.

This data is subject to the condition that for each quadruple $(y_1, y_2, y_3, y_4) \in Y^{[4]}$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_{(y_1,y_2)} \otimes \mathcal{L}_{(y_2,y_3)} \otimes \mathcal{L}_{(y_3,y_4)} & \xrightarrow{m\otimes\mathbb{1}} & \mathcal{L}_{(y_1,y_3)} \otimes \mathcal{L}_{(y_3,y_4)} \\ \downarrow{\scriptstyle \mathbb{1}\otimes m} & & \downarrow{\scriptstyle m} \\ \mathcal{L}_{(y_1,y_2)} \otimes \mathcal{L}_{(y_2,y_4)} & \xrightarrow{m} & \mathcal{L}_{(y_1,y_4)} \end{array}$$

Our bundle gerbes will always have band U(1), so instead of "bundle gerbe with band U(1)" we shall simply say "bundle gerbe". It is customary to refer to a bundle gerbe $(\mathcal{L}, Y, m)$ by the pair $(\mathcal{L}, Y)$, a convention that we shall follow.

Definition D.2. A *trivialization* of a bundle gerbe $(\mathcal{L}, Y)$ is a principal U(1)-bundle $\mathcal{K} \to Y$ together with an isomorphism $(\pi_2^*\mathcal{K})^* \otimes \pi_1^*\mathcal{K} \to \mathcal{L}$, such that the following diagram commutes for all $(y_1, y_2, y_3) \in Y^{[3]}$

$$\begin{array}{ccc} \mathcal{L}_{(y_1,y_2)} \otimes \mathcal{L}_{(y_2,y_3)} & \xrightarrow{m} & \mathcal{L}_{(y_1,y_3)} \\ \downarrow & & \downarrow \\ \mathcal{K}_{y_1}^* \otimes \mathcal{K}_{y_2} \otimes \mathcal{K}_{y_2}^* \otimes \mathcal{K}_{y_3} & \longrightarrow & \mathcal{K}_{y_1}^* \otimes \mathcal{K}_{y_3} \end{array}$$

Let $P \to M$ be a principal G-bundle, for some Lie group G. Moreover, suppose that $\mathrm{U}(1) \to \hat{G} \to G$ is a central extension. We define a map $\Delta : P^{[2]} \to G$ through the formula $p_1 \cdot \Delta(p_1, p_2) = p_2$ for all $(p_1, p_2) \in P^{[2]}$.

Definition D.3. The lifting gerbe corresponding to the pair $(P, \hat{G})$ is the gerbe $(\Delta^*\hat{G}, P)$ equipped with the following multiplication:

$$(\Delta^*\hat{G})_{(p_1,p_2)} \otimes (\Delta^*\hat{G})_{(p_2,p_3)} \to (\Delta^*\hat{G})_{(p_1,p_3)},$$
$$((p_1, p_2), \hat{g}_{12}) \otimes ((p_2, p_3), \hat{g}_{23}) \mapsto ((p_1, p_3), \hat{g}_{12}\hat{g}_{23}).$$

Suppose that the lifting gerbe $(\Delta^*\hat{G}, P)$ has a trivialization $\mathcal{K} \to P$. Let $(p_1, p_2) \in P^{[2]}$. Then we observe that the isomorphism $\Delta^*\hat{G}_{(p_1,p_2)} \to \mathcal{K}^*_{p_1} \otimes \mathcal{K}_{p_2}$ can equally well be viewed as an isomorphism $\mathcal{K}_{p_1} \otimes (\Delta^*\hat{G})_{(p_1,p_2)} \to \mathcal{K}_{p_2}$. This allows us to equip $\mathcal{K}$ with a right $\hat{G}$-action as follows. If $\hat{g} \in \hat{G}$, then we write $g \in G$ for its basepoint. Let $p \in P$ be arbitrary, then we define the right action by

$$\mathcal{K}_p \otimes \hat{G}_g \longrightarrow \mathcal{K}_p \otimes (\Delta^*\hat{G})_{p,pg} \longrightarrow \mathcal{K}_{pg}$$

$$k \otimes \hat{g} \longmapsto k \otimes ((p, pg), \hat{g})$$

The following lemma is now a standard result in the literature, [Mur96, Mur10].

Lemma D.4. *The bundle $\mathcal{K} \to M$ with the right $\hat{G}$-action as above is a principal $\hat{G}$-bundle which lifts P.*

Let $(\mathcal{L}, Y)$ be a bundle gerbe over M.

Definition D.5. A *bundle gerbe module* for $(\mathcal{L}, Y)$ is a vector bundle $E \to Y$ equipped with an isomorphism $\phi : \mathcal{L} \otimes \pi_1^* E \to \pi_2^* E$, such that following diagram commutes for all $(y_1, y_2, y_3) \in Y^{[3]}$

$$\begin{array}{ccc} \mathcal{L}_{(y_1,y_2)} \otimes \mathcal{L}_{(y_2,y_3)} \otimes E_{y_3} & \xrightarrow{\mathbb{1}\otimes\phi} & \mathcal{L}_{(y_1,y_2)} \otimes E_{y_2} \\ \downarrow{\scriptstyle m\otimes\mathbb{1}} & & \downarrow{\scriptstyle \phi} \\ \mathcal{L}_{(y_1,y_3)} \otimes E_{y_3} & \xrightarrow{\phi} & E_{y_1} \end{array}$$

In [BCM+02, Proposition 4.2] Bouwknegt et al. prove that a trivialization of $(\mathcal{L}, Y)$ defines a bijection from the set of isomorphism classes of bundle gerbe modules for $(\mathcal{L}, Y)$ to the set of isomorphism classes of vector bundles on M. We briefly recall the definition of this bijection. Let $(\mathcal{L}, Y)$ be a trivial bundle gerbe, i.e. we assume that $\mathcal{L} = (\pi_2^*\mathcal{K})^* \otimes \pi_1^*\mathcal{K}$, for some U(1)-bundle $\mathcal{K}$ on Y. Suppose now that E is a vector bundle on M. Then $\pi^*E \otimes \mathcal{K}^*$ is a vector bundle on Y. Using the fact that $\pi\pi_1 = \pi\pi_2$, we define an action of $\mathcal{L}$ on it as follows

$$\begin{aligned} \mathcal{L} \otimes \pi_1^*(\pi^*E \otimes \mathcal{K}^*) &\xrightarrow{\sim} (\pi_2^*\mathcal{K})^* \otimes \pi_1^*\mathcal{K} \otimes \pi_1^*\pi^*E \otimes (\pi_1^*\mathcal{K})^* \xrightarrow{\sim} (\pi_2^*\mathcal{K}) \otimes \pi_2^*\pi E \\ &\xrightarrow{\sim} \pi^*(E \otimes \mathcal{K}^*) \end{aligned}$$

We omit the construction in the other direction, but simply observe that the action of a trivial bundle gerbe on a bundle gerbe module is descent data for the vector bundle.

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Glossary

$\mathcal{B}(H)$	The bounded operators on H	12
$C^\infty_{-2\pi}$	The 2π antiperiodic functions on $\mathbb{R}$	7
$\mathrm{Cl}(V)$	Clifford C*-algebra of V	10
$\mathrm{Cl}(V)^{\mathrm{s}}$	The smooth vectors in $\mathrm{Cl}(V)$	11
$\mathrm{Cl}_{\mathrm{alg}}(V)$	Algebraic Clifford algebra of V	10
$\mathrm{Cl}(f)$	The homomorphism extending f	10
$\mathrm{Cl}^{\mathrm{s}}(LM)$	The bundle of Clifford algebras on loop space	68
$\mathcal{D}(H,\phi)$	Right module maps from $L^2_\phi(\mathcal{A})$ into H	31
$\mathcal{F}_L$	The Fock space with respect to the Lagrangian L	12
$\mathcal{F}^{\mathrm{s}}(LM)$	The spinor bundle on loop space	67
$\mathrm{Fr}_{\mathrm{O}}(\mathcal{H})$	The orthogonal frame bundle of $\mathcal{H}$	74
$\mathrm{Fr}_L(\mathcal{H})$	The restricted orthogonal frame bundle of $\mathcal{H}$	74
$\mathcal{H}$	The bundle over LM with fibres $L^2(S^1, \mathbb{S}\otimes\gamma^*TM_{\mathbb{C}})$	73
$H \boxtimes_\phi K$	Connes fusion product of H with K relative to ϕ	32
$\mathrm{Imp}_L(V)$	The Banach Lie group of implementers	15
$\mathfrak{imp}(V)$	The Lie algebra of $\mathrm{Imp}_L(V)$	17
J	The modular conjugation	41
$\mathcal{J}$	A unitary structure	10
$\mathcal{J}_L$	The unitary structure corresponding to L	10
k	The Klein transformation	37
$L^2_\phi(\mathcal{A})$	The standard form of $\mathcal{A}$ determined by ϕ	31
$\mathrm{Lag}(V)$	The set of Lagrangians in V	20
$\mathrm{Lag}_L(\mathcal{H})$	The bundle of Dirac Lagrangians in $\mathcal{H}$	74
$\mathrm{Lag}_L(V)$	The set of Lagrangians in V, which are equivalent to L	20
ΛL	The exterior algebra of L	12
$\mathrm{O}(V)$	The unitary operators on V that commute with the real structure	10
$\mathrm{O}_L(V)$	The restricted orthogonal group of V with respect to the Lagrangian L	13
$\mathfrak{o}_L(V)$	The Lie algebra of $\mathrm{O}_L(V)$	15
P_L	The orthogonal projection onto L	10
$\mathbb{S}$	The odd spinor bundle on the circle	6
$\mathrm{SO}(M)$	The oriented orthonormal frame bundle of M	68
$\mathrm{Spin}(M)$	The spin frame bundle of M	60
θ_g	The Bogoliubov transformation of g	11

Hiermit erkläre ich, dass diese Arbeit bisher von mir weder an der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Greifswald noch einer anderen wissenschaftlichen Einrichtung zum Zwecke der Promotion eingereicht wurde.
Ferner erkläre ich, dass ich diese Arbeit selbstständig verfasst und keine anderen als die darin angegebenen Hilfsmittel und Hilfen benutzt und keine Textabschnitte eines Dritten ohne Kennzeichnung übernommen habe.

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Master thesis

Titel Brownian motion and topological field theories

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Preprints

1. Mai 2019 **Fusion of implementers for spinors on the circle**, *P. Kristel & K. Waldorf*, arXiv:1905.00222.

Publikationen

20. Juni 2016 **Quantum Brownian motion in a Landau level**, *E. Cobanera, P. Kristel & C. Morais Smith*, Phys. Rev. B 93, 245422, arXiv:1602.00694.

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